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REMARKS ON SOME SYSTEMS OF TWO SIMULTANEOUS FUNCTIONAL INEQUALITIES

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Abstract. We study the real (measurable and continuous at a point) functions that satisfy, almost everywhere, some systems of two simultaneous functional inequalities. In particular, we obtain generalizations and extensions of some earlier results of D. Krassowska, J. Matkowski, P. Montel, and T. Popoviciu.

1. Introduction

In what follows $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote, as usual the sets of positive integers, integers, rationals and reals, respectively. Moreover $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let $a, b \in \mathbb{R} \setminus \{0\}$, $ab^{-1} \notin \mathbb{Q}$, $ab < 0$. P. Montel [13] (see also [15] and [12, p. 228]) proved that a function $f : \mathbb{R} \to \mathbb{R}$, that is continuous at a point and satisfies the system of functional inequalities

$$f(x + a) \leq f(x), \qquad f(x + b) \leq f(x) \qquad x \in \mathbb{R}, \tag{1.1}$$

must be constant. A similar result for measurable functions has been proved in [3], where a more abstract approach is assumed.

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The result of Montel has been generalized in [8, 9, 10] by D. Krassowska and J. Matkowski in several ways (see also [11]). In particular, motivated by some problem arising in a characterization of L^p norm, they have proved in [9] (cf. [8]) that if $\alpha, \beta \in \mathbb{R}$ and $\alpha b \leq \beta a$, then the following two inequalities

$$f(x+a) \leq f(x) + \alpha, \qquad f(x+b) \leq f(x) + \beta, \qquad x \in \mathbb{R}, \tag{1.2}$$

has a solution $f : \mathbb{R} \to \mathbb{R}$, that is continuous at a point, if and only if $\alpha b = \beta a$; moreover every such a solution must be of the form $f(x) = cx + d$ for $x \in \mathbb{R}$, with some $c, d \in \mathbb{R}$. An analogous problem for functions $f : \mathbb{R}^n \to \mathbb{R}$ has been studied in [10]. Some related interesting (and proved in a quite involved way) results, for the following system of more general functional inequalities

$$f(x+a) \leq f(x) + \sum_{j=0}^{n} \alpha_j x^j, \quad f(x+b) \leq f(x) + \sum_{j=0}^{n} \beta_j x^j, \quad x \in \mathbb{R}, \tag{1.3}$$

have been obtained in [11]. More precisely, the authors have shown how to reduce by 1 the degree of the polynomials on the right hand side of (1.3), but unfortunately without a precise final description of solutions of (1.3) for arbitrary integer $n \geq 0$ (similar reductions for (1.3) with $n = 0$ have been already applied in [4, 5]). In this paper we continue this approach and present a method that allows to obtain some generalizations and extensions of those results in [9, 13, 3]. For the clarity of presentation, we consider (1.3) only with $n = 0$ and $n = 1$, but in a conditional form (on a real interval), almost everywhere, and for real functions that are Lebesgue or Baire measurable. We obtain outcomes that correspond somewhat to the results in [4, 5] and to the problem of stability of functional equations and inequalities (for some further information concerning that problem we refer to, e.g., [2, 6, 7]).

2. Preliminaries

Let us recall some definitions.

Definition 2.1. Let $E \subset \mathbb{R}$ be nonempty and $\mathcal{I} \subset 2^{\mathbb{R}}$. We say a property $p(x)$ ($x \in E$) holds $\mathcal{I}$-almost everywhere in a set E (abbreviated in the sequel to $\mathcal{I}$-a.e. in E) provided there is a set $A \in \mathcal{I}$ such that $p(x)$ holds for all $x \in E \setminus A$.

Definition 2.2. $\mathcal{I} \subset 2^{\mathbb{R}}$ is a σ-ideal provided $2^A \subset \mathcal{I}$ for $A \in \mathcal{I}$ and

$$\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{I}, \qquad \{A_n\}_{n \in \mathbb{N}} \subset \mathcal{I}.$$

If, moreover, $\mathcal{I} \neq 2^X$, then we say that $\mathcal{I}$ is proper. Next, we say that $\mathcal{I}$ is nontrivial provided $\mathcal{I} \neq \{\emptyset\}$. Finally, $\mathcal{I}$ is translation invariant (abbreviated to t.i. in the sequel) if $x + A \in \mathcal{I}$ for $A \in \mathcal{I}$ and $x \in \mathbb{R}$.

We have the following (see [4, Propositions 2.1 and 2.2]).

Proposition 2.1. *Let $\mathcal{I} \subset 2^{\mathbb{R}}$ be a t.i. σ-ideal and $U \subset \mathbb{R}$ be open and nonempty. Then*

$$\operatorname{int}\,[(U \setminus T) - V] \neq \emptyset, \qquad V \in 2^{\mathbb{R}} \setminus \mathcal{I},\ T \in \mathcal{I}, \tag{2.1}$$

where $(U \setminus T) - V = \{u - v : u \in U \setminus T, v \in V\}$.

Definition 2.3. Let $D \subset \mathbb{R}$. We say that $f : D \to \mathbb{R}$ is Lebesgue (Baire, respectively) measurable provided the set $f^{-1}(U)$ is Lebesgue measurable (has the Baire property (cf., e.g., [14]), respectively) in $\mathbb{R}$ for every open set $U \subset \mathbb{R}$.

In what follows $\mathcal{L}$ stands for the σ-ideal of all the subsets of $\mathbb{R}$ that are of the Lebesgue measure zero; $\mathcal{B}$ denotes the σ-ideal of all the subsets of $\mathbb{R}$ that are of the first category of Baire.

We need yet the following proposition. It can be derived from [4, Theorem 3.1 and Remark 3.2], but for the convenience of readers we prove it.

Proposition 2.2. *Let P be a dense subset of $\mathbb{R}$, I be a real nontrivial interval, $E \subset I$, and $v : I \to \mathbb{R}$. Then the following two statements are valid.*

(i) *If v is Lebesgue (Baire, respectively) measurable, the set $I \setminus E$ is of Lebesgue measure zero (of first category, resp.) and*

$$v(p + x) \leq v(x), \qquad x \in E, p \in P, x + p \in E, \tag{2.2}$$

then there is $d \in \mathbb{R}$ with $v(x) = d$ $\mathcal{L}$–a.e. ($\mathcal{B}$–a.e., resp.) in I.

(ii) *If $\mathcal{J} \subset 2^{\mathbb{R}}$ is a proper t.i. σ-ideal, $I \setminus E \in \mathcal{J}$, v is continuous at a point $x_0 \in I$ and (2.2) holds, then $v(x) = v(x_0)$ $\mathcal{J}$–a.e. in I.*

Proof. (i) For every $n \in \mathbb{N}$ we have

$$E \subset v^{-1}(\mathbb{R}) = \bigcup_{q \in \mathbb{Q}} v^{-1}\left(\left(q - \frac{1}{n}, q + \frac{1}{n}\right)\right).$$

Thus, for each $n \in \mathbb{N}$, there exists $q(n) \in \mathbb{Q}$ such that the set

$$E_n := v^{-1}\left(\left(q(n) - \frac{1}{n}, q(n) + \frac{1}{n}\right)\right)$$

is not of the Lebesgue measure zero (of first category, respectively).

Suppose there exists $k \in \mathbb{N}$ such that $B_k := v^{-1}\left(\left[q(k)+\frac{1}{k},\infty\right)\right)$ is not in $\mathcal{L}$ (in $\mathcal{B}$, respectively). Then, according to [1, Theorem 1] (Proposition 2.1, respectively) $\operatorname{int}(B_k - E_k) \neq \emptyset$, whence there is $p \in P$ with $p \in \operatorname{int}(B_k - E_k)$, which means that $p + e = b \in B_k \subset E$ with some $b \in B_k$ and $e \in E_k$. Hence

$$q(k) + \frac{1}{k} \le v(b) = v(p+e) \le v(e) < q(k) + \frac{1}{k}.$$

This is a contradiction.

Now suppose there exists $k \in \mathbb{N}$ such that $C_k := v^{-1}\left(\left(-\infty, q(k)-\frac{1}{k}\right]\right)$ is not in $\mathcal{L}$ (in $\mathcal{B}$, respectively). Then, analogously as before we obtain that there are $p \in P$, $c \in C_k$ and $e \in E_k$ such that $p + c = e \in E_k \subset E$, whence

$$q(k) - \frac{1}{k} < v(e) = v(p+c) \le v(c) \le q(k) - \frac{1}{k},$$

which is a contradiction.

Thus we have proved that the set $v^{-1}\left(\mathbb{R} \setminus \left(q(k)-\frac{1}{k}, q(k)+\frac{1}{k}\right)\right)$ is in $\mathcal{L}$ (in $\mathcal{B}$, respectively) for every $k \in \mathbb{N}$. Let

$$S := \bigcap_{k\in\mathbb{N}} \left(q(k) - \frac{1}{k}, q(k) + \frac{1}{k}\right).$$

It is easily seen that S has at most one point, the set

$$\begin{aligned} A &:= v^{-1}(\mathbb{R}\setminus S) = v^{-1}\left(\bigcup_{k\in\mathbb{N}}\left(\mathbb{R}\setminus\left(q(k)-\frac{1}{k}, q(k)+\frac{1}{k}\right)\right)\right) = \\ &= \bigcup_{k\in\mathbb{N}} v^{-1}\left(\mathbb{R}\setminus\left(q(k)-\frac{1}{k}, q(k)+\frac{1}{k}\right)\right) \end{aligned}$$

is in $\mathcal{L}$ (in $\mathcal{B}$, respectively), and $v(x) \in S$ for $x \in E \setminus A$.

Thus we have proved that there exists $d \in \mathbb{R}$ such that $v(x) = d$ for $x \in E\setminus A$.

(ii) Since $\mathcal{J}$ is proper and t.i., we deduce that $I \notin \mathcal{J}$, whence $E \notin \mathcal{J}$. For each $n \in \mathbb{N}$ write

$$\begin{aligned} D_n &:= v^{-1}\left(\left(v(x_0) - \frac{1}{n}, v(x_0) + \frac{1}{n}\right)\right), \\ E_n &:= v^{-1}\left(\left[v(x_0) + \frac{1}{n}, \infty\right)\right) \setminus H, \\ F_n &:= v^{-1}\left(\left(-\infty, v(x_0) - \frac{1}{n}\right]\right) \setminus H \end{aligned}$$

and $C_n := D_n \setminus H$, where $H := I \setminus E$. Since v is continuous at x_0, int $D_n \neq \emptyset$ for $n \in \mathbb{N}$.

Suppose that there is $k \in \mathbb{N}$ with $E_k \notin \mathcal{J}$. Then, on account of Proposition 2.1, there is $p \in P$ such that $p \in \operatorname{int}(E_k - C_k)$, whence $p + c = e \in E_k \subset E$ with some $c \in C_k$ and $e \in E_k$. Hence

$$v(x_0) + \frac{1}{k} \leq v(e) = v(p+c) \leq v(c) < v(x_0) + \frac{1}{k}.$$

This is a contradiction.

Next, suppose that there is $k \in \mathbb{N}$ with $F_k \notin \mathcal{J}$. Then, on account of Proposition 2.1, there is $p \in P \cap (\operatorname{int}(C_k - F_k))$, whence $p + e = c \in C_k \subset E$ with some $c \in C_k$ and $e \in F_k$. Hence

$$v(x_0) - \frac{1}{k} < v(c) = v(p+e) \leq v(e) \leq v(x_0) - \frac{1}{k}.$$

This is a contradiction, too.

In this way we have shown that $G_k := E_k \cup F_k \in \mathcal{J}$ for every $k \in \mathbb{N}$. Clearly

$$\begin{aligned} V &:= v^{-1}(\mathbb{R} \setminus \{v(x_0)\}) \\ &= v^{-1}\left(\mathbb{R} \setminus \bigcap_{n\in\mathbb{N}} \left(v(x_0) - \frac{1}{n}, v(x_0) + \frac{1}{n}\right)\right) \\ &\subset H \cup \bigcup_{k\in\mathbb{N}} G_k \in \mathcal{J} \end{aligned}$$

and $v(x) = v(x_0)$ for $x \in E \setminus V$. ■

3. The case $n = 0$

In this part of the paper we consider (1.3) for $n = 0$. The subsequent proposition contains auxiliary results for the next section.

Proposition 3.1. *Let $a_1, a_2, \alpha_1, \alpha_2 \in \mathbb{R}$, $a_1 < 0 < a_2$, $a_1 a_2^{-1} \notin \mathbb{Q}$,*

$$c_1 := \frac{\alpha_1}{a_1} \geq \frac{\alpha_2}{a_2} =: c_2 \tag{3.1}$$

and I be a real interval such that

$$|I| > a_2 - a_1, \tag{3.2}$$

where $|I|$ denotes the length of I. Then the subsequent two statements are valid.

(i) *A function* $v : I \to \mathbb{R}$ *is Lebesgue (Baire, respectively) measurable and satisfies the following two conditional inequalities*

$$\text{if } a_1 + x \in I, \quad \text{then } v(a_1 + x) - v(x) \leq \alpha_1, \tag{3.3}$$

$$\text{if } a_2 + x \in I, \quad \text{then } v(a_2 + x) - v(x) \leq \alpha_2 \tag{3.4}$$

$\mathcal{L}$*-a.e.* ($\mathcal{B}$*-a.e., resp.*) *in* I *if and only if* $c_2 = c_1$ *and there is* $d \in \mathbb{R}$ *with*

$$v(x) = c_1 x + d, \qquad \mathcal{L} - a.e. \ (\mathcal{B} - a.e., \ resp.) \ \ in \ I. \tag{3.5}$$

(ii) *Let* $\mathcal{J} \subset 2^{\mathbb{R}}$ *be a proper t.i.* σ*-ideal. Then a function* $v : I \to \mathbb{R}$ *is continuous at a point* $x_0 \in I$ *and satisfies* (3.3) *and* (3.4) $\mathcal{J}$*-a.e. in* I *if and only if* $c_2 = c_1$ *and*

$$v(x) = c_1 x + v(x_0), \qquad \mathcal{J} - a.e. \ in \ I. \tag{3.6}$$

Proof. (i) Assume that v is Lebesgue (Baire, respectively) measurable and there is $T \in \mathcal{L}$ ($T \in \mathcal{B}$, resp.) such that (3.3) and (3.4) are valid for every $x \in F := I \setminus T$. Write

$$w_i(x) = v(x) - c_i x, \qquad i = 1, 2, x \in F.$$

Clearly w_i is Lebesgue (Baire, respectively) measurable. Further, for every $i, j \in \{1, 2\}$, we have $\alpha_j \leq c_i a_j$ and consequently

$$\begin{aligned} w_i(x + a_j) &= v(x + a_j) - c_i(x + a_j) \leq \\ &\leq v(x) + \alpha_j - c_i x - c_i a_j \leq \\ &\leq w_i(x), \qquad x \in F, x + a_j \in F. \end{aligned} \tag{3.7}$$

Fix $i \in \{1, 2\}$. Let

$$H := \bigcup_{m,n \in \mathbb{Z}} (T + na_1 + ma_2) \tag{3.8}$$

and $E := I \setminus H$. Then $H \in \mathcal{L}$ ($H \in \mathcal{B}$, resp.) and it is easily seen that

$$x + na_1 + ma_2 \in E, \qquad x \in E, n, m \in \mathbb{N}_0, x + na_1 + ma_2 \in I. \tag{3.9}$$

Write $P_k := \{na_1 + ma_2 : n, m \in \mathbb{N}_0, n + m \leq k\}$ for $k \in \mathbb{N}$ and

$$P = \bigcup_{k \in \mathbb{N}} P_k.$$

Then it is well known that the set P is dense in $\mathbb{R}$ (see, e.g., [8, 9, 10]).

We show, by induction, that for every $k \in \mathbb{N}$ we have

$$w_i(x+p) \le w_i(x), \qquad x \in E, p \in P_k, x+p \in E, i = 1, 2. \tag{3.10}$$

The case $k = 1$ is just (3.7). So fix $k \in \mathbb{N}$ and assume that (3.10) holds. Take $x \in E$ and $q \in P_{k+1}$ with $x + q \in E$. Clearly, in view of (3.2), $x + q - a_j \in I$ and $p := q - a_j \in P_k$ for some $j \in \{1, 2\}$. Hence, in view of (3.9), $x + q - a_j \in E$ and consequently, by the inductive hypotheses and (3.7),

$$w_i(x+q) \le w_i(x+q-a_j) = w_i(x+p) \le w_i(x), \qquad i = 1, 2.$$

Thus we have proved that (3.10) holds for all $k \in \mathbb{N}$, which yields

$$w_i(x+p) \le w_i(x), \qquad x \in E, p \in P, x+p \in E, i = 1, 2. \tag{3.11}$$

Consequently, by Proposition 2.2, there are $d_1, d_2 \in \mathbb{R}$ and sets $A_1, A_2 \in \mathcal{L}$ ($A_1, A_2 \in \mathcal{B}$, resp.) such that $w_i(x) = d_i$ for $x \in E \setminus A_i$ and $i = 1, 2$. Clearly

$$c_1x + d_1 = v(x) = c_2x + d_2, \qquad x \in E \setminus (A_1 \cup A_2),$$

whence we have $c_1 = c_2$ and $d_1 = d_2 =: d$.

Since the converse is easy to check, this ends the proof of (i).

(ii) Let $v : I \to \mathbb{R}$ be continuous at a point $x_0 \in I$ and satisfies (3.3) and (3.4) $\mathcal{J}$-a.e. in I. Since $\mathcal{J}$ is proper and t.i., we have

$$\operatorname{int} T = \emptyset, \qquad T \in \mathcal{J}. \tag{3.12}$$

There is a set $T \in \mathcal{J}$ such that conditions (3.3) and (3.4) hold for every $x \in I \setminus T$. Analogously as in the proof of (i) we define w_i and P and obtain (3.11) for $E := I \setminus H$, where $H \in \mathcal{J}$ is given by (3.8).

Then, on account of Proposition 2.2 (ii), there are $V_1, V_2 \in \mathcal{J}$ such that

$$w_i(x) = w_i(x_0), \qquad x \in E \setminus V_i, i = 1, 2.$$

Clearly, by (3.12), we have $\operatorname{int}(V_1 \cup V_2) = \emptyset$ and

$$w_1(x_0) + c_1x = w_1(x) + c_1x = v(x) = c_2x + w_2(x_0), \qquad x \in E \setminus (V_1 \cup V_2),$$

whence $\alpha_1 a_2 = \alpha_2 a_1$ and (3.6) holds.

The converse is easy to check. ■

Remark 3.1. Let $a_1, a_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 \in (0, \infty)$. Then every function $v : I \to \mathbb{R}$ with

$$\sup_{x \in \mathbb{R}} |v(x)| \le \frac{1}{2} \min\{\alpha_1, \alpha_2\}$$

fulfils (3.3) and (3.4) for each real interval I. This shows that assumption (3.1) is necessary in Proposition 3.1.

4. The case $n = 1$

The next theorem gives results for (1.3) with $n = 1$.

Theorem 4.1. *Let $a_1, a_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{R}$, $a_1 < 0 < a_2$, $a_1 a_2^{-1} \notin \mathbb{Q}$,*

$$e := \left(\frac{\gamma_2}{a_2} - \frac{\gamma_1}{a_1}\right)\frac{1}{a_2 - a_1} \le \max\left\{\frac{\beta_1}{2a_1}, \frac{\beta_2}{2a_2}\right\}, \tag{4.1}$$

and I_0 be a real interval which contains the interval $(a_1 - a_2, a_2 - a_1)$. Then the subsequent two statements are valid.

(i) *A function $f : I_0 \to \mathbb{R}$ is Lebesgue (Baire, respectively) measurable and satisfies the following system of two inequalities*

$$\text{if } a_1 + x \in I_0, \quad \text{then } f(a_1 + x) - f(x) \le \beta_1 x + \gamma_1, \tag{4.2}$$

$$\text{if } a_2 + x \in I_0, \quad \text{then } f(a_2 + x) - f(x) \le \beta_2 x + \gamma_2 \tag{4.3}$$

$\mathcal{L}$-a.e. ($\mathcal{B}$-a.e., resp.) in I_0 if and only if

$$\frac{\gamma_1}{a_1} - \frac{\beta_1}{2} = \frac{\gamma_2}{a_2} - \frac{\beta_2}{2} \qquad \text{and} \qquad \frac{\beta_1}{a_1} = 2e = \frac{\beta_2}{a_2} \tag{4.4}$$

and there exists $d \in \mathbb{R}$ with

$$f(x) = \frac{\beta_1}{2a_1}x^2 + \left(\frac{\gamma_1}{a_1} - \frac{\beta_1}{2}\right)x + d, \tag{4.5}$$

$$\mathcal{L} - \text{a.e. } (\mathcal{B} - \text{a.e., resp.}) \text{ in } I_0.$$

(ii) *Let $\mathcal{J} \subset 2^{\mathbb{R}}$ be a t.i. σ-ideal. Then a function $f : I_0 \to \mathbb{R}$, continuous at a point $x_0 \in I_0$, satisfies conditions (4.2) and (4.3) $\mathcal{J}$-a.e. in I_0 if and only if (4.4) holds and*

$$f(x) = \frac{\beta_1}{2a_1}(x - x_0)^2 + \left(\frac{\gamma_1 + \beta_1 x_0}{a_1} - \frac{\beta_1}{2}\right)(x - x_0) + f(x_0),$$

$$\mathcal{J} - \text{a.e. in } I_0.$$

Proof. (i) It is easy to check that, in the case where (4.4) holds, the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{\beta_1}{2a_1}x^2 + \left(\frac{\gamma_1}{a_1} - \frac{\beta_1}{2}\right)x + d,$$

satisfies the equalities

$$f(a_1+x)=f(x)+\beta_1 x+\gamma_1, \qquad f(a_2+x)=f(x)+\beta_2 x+\gamma_2$$

for every $x \in \mathbb{R}$.

So assume now that a function $f : I_0 \to \mathbb{R}$ satisfies (4.2) and (4.3) for every $x \in P_0 := I_0 \setminus T$, with some $T \in \mathcal{L}$ ($T \in \mathcal{B}$, resp.). Take $\delta \in \mathbb{R}$, $\delta \geq 0$, with

$$\min\left\{\frac{\beta_1}{2a_1}\,,\,\frac{\beta_2}{2a_2}\right\} \leq e+\delta \leq \max\left\{\frac{\beta_1}{2a_1}\,,\,\frac{\beta_2}{2a_2}\right\}.$$

Write

$$v(x) := f(x)-(e+\delta)x^2, \qquad x \in I_0$$

and

$$\alpha_i := \gamma_i-(e+\delta)a_i^2, \qquad \alpha_i' := \gamma_i - e a_i^2, \qquad i=1,2.$$

Then

$$\begin{aligned}\frac{\alpha_2'}{a_2} &= \frac{\gamma_2}{a_2}-\left(\frac{\gamma_2}{a_2}-\frac{\gamma_1}{a_1}\right)\frac{a_2}{a_2-a_1} = \\ &= \frac{\gamma_2}{a_2}-\left(\frac{\gamma_2}{a_2}-\frac{\gamma_1}{a_1}\right)\left(1+\frac{a_1}{a_2-a_1}\right) = \\ &= \frac{\gamma_1}{a_1}-\left(\frac{\gamma_2}{a_2}-\frac{\gamma_1}{a_1}\right)\frac{a_1}{a_2-a_1} = \frac{\alpha_1'}{a_1},\end{aligned}$$

whence

$$\alpha_2' a_1 = \alpha_1' a_2 \tag{4.6}$$

and, according to the definitions of α_i and α_i',

$$\alpha_2 a_1 = \alpha_2' a_1 - (\delta a_2)a_2 a_1 \geq \alpha_1' a_2 - (\delta a_1)a_2 a_1 = \alpha_1 a_2. \tag{4.7}$$

First consider the case where

$$\frac{\beta_1}{2a_1} \geq e+\delta \geq \frac{\beta_2}{2a_2},$$

which means that

$$\beta_i - 2a_i(e+\delta) \leq 0, \qquad i=1,2. \tag{4.8}$$

Then, by (4.2) and (4.3), there is a set $A_0 \in \mathcal{L}$ ($A_0 \in \mathcal{B}$, resp.) such that

$$\begin{aligned}v(x+a_i) &= f(x+a_i)-(e+\delta)(x^2+2a_i x+a_i^2) \leq \\ &\leq f(x)-(e+\delta)x^2+(\beta_i-2a_i(e+\delta))x+\gamma_i-(e+\delta)a_i^2 \leq \\ &\leq v(x)+\gamma_i-(e+\delta)a_i^2 = v(x)+\alpha_i\end{aligned} \tag{4.9}$$

for $x \in E := I_0 \cap (0, \infty) \setminus A_0$ with $x + a_i \in E$, $i = 1, 2$. Consequently, on account of Proposition 3.1 (i) and (4.7), $\alpha_2 a_1 = \alpha_1 a_2$ and there exists $d \in \mathbb{R}$ such that (3.5) holds. Clearly (4.6) yields $\delta = 0$, whence $\alpha_i' = \alpha_i$ for $i = 1, 2$ and

$$(4.10)\quad f(x) = v(x) + ex^2 = d + c_1 x + ex^2, \qquad \mathcal{L} - a.e.\ (\mathcal{B} - a.e.,\ resp.)\text{ in } E.$$

Substituting this form of f in (4.2) and (4.3), we obtain

$$0 = \alpha_i' + ea_i^2 - \gamma_i = c_i a_i + ea_i^2 - \gamma_i \leq (\beta_i - 2ea_i)x$$

$\mathcal{L}$-a.e. ($\mathcal{B}$-a.e., resp.) in E, which implies $\beta_i - 2ea_i \geq 0$ for $i = 1, 2$. Note that, according to (4.8), for every $i = 1, 2$, we have $\beta_i - 2a_i e = 0$. Next, (4.6) means that

$$(\gamma_1 - ea_1^2)a_2 = (\gamma_2 - ea_2^2)a_1.$$

Consequently, (4.4) holds and from (4.10) we derive (4.5).

Now assume that

$$\frac{\beta_1}{2a_1} \leq e + \delta \leq \frac{\beta_2}{2a_2},$$

i.e., $\beta_i - 2a_i(e + \delta) \geq 0$ for $i = 1, 2$, which means that

$$(\beta_i - 2a_i(e + \delta))x \leq 0, \qquad x \in (-\infty, 0].$$

Consequently we argue analogously as in (4.9) for $x \in I_1 := I_0 \cap (-\infty, 0)$. Hence, in view of Proposition 3.1 (i) and (4.7), $\alpha_2 a_1 = \alpha_1 a_2$ and (3.5) holds with some $d \in \mathbb{R}$. We complete the proof similarly as in the previous case.

(ii) In this case we argue analogously as in the case of (i), using Proposition 3.1 (ii). ■

Remark 4.1. If in Theorem 4.1 we replace (4.2) and (4.3) by the following two conditional equalities

$$(4.11)\qquad \text{if } a_1 + x \in I_0, \text{ then } f(a_1 + x) = f(x) + \beta_1 x + \gamma_1,$$

$$(4.12)\qquad \text{if } a_2 + x \in I_0, \text{ then } f(a_2 + x) = f(x) + \beta_2 x + \gamma_2,$$

then condition (4.1) is superfluous. For, suppose that (4.1) does not hold. Then

$$e > \max\left\{\frac{\beta_1}{2a_1}, \frac{\beta_2}{2a_2}\right\} = -\min\left\{\frac{-\beta_1}{2a_1}, \frac{-\beta_2}{2a_2}\right\}$$

and consequently

$$\left(\frac{-\gamma_2}{a_2} - \frac{-\gamma_1}{a_1}\right)\frac{1}{a_2 - a_1} < \min\left\{\frac{-\beta_1}{2a_1}, \frac{-\beta_2}{2a_2}\right\}.$$

Now it is enough to observe yet that, in view of (4.11) and (4.12), (4.1)–(4.3) hold with $f, \beta_1, \beta_2, \gamma_1, \gamma_2$ replaced by $-f, -\beta_1, -\beta_2, -\gamma_1, -\gamma_2$, respectively.

Remark 4.2. It is easily seen in the proof of Theorem 4.1 (i) that if we assume that (4.2) and (4.3) are fulfilled for every $x \in I_0$, then there exists $d \in \mathbb{R}$ with

$$f(x) = \frac{\beta_1}{2a_1}x^2 + \left(\frac{\gamma_1}{a_1} - \frac{\beta_1}{2}\right)x + d, \qquad x \in I_0. \tag{4.13}$$

If $I_0 = \mathbb{R}$ in Theorem 4.1, then condition (4.1) can be relaxed to some extent as in the subsequent corollary.

Corollary 4.1. *Let $a'_1, a'_2, \beta'_1, \beta'_2, \gamma'_1, \gamma'_2 \in \mathbb{R}$, $a'_1 < 0 < a'_2$, $a'_1/a'_2 \notin \mathbb{Q}$, $\beta'_2 < \beta'_1$ and $\beta'_2\beta'_1 \geq 0$. Then the subsequent two statements are valid.*

(i) *A function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue (Baire, respectively) measurable and satisfies the following system of two inequalities*

$$f(a'_i + x) - f(x) \leq \beta'_i x + \gamma'_i, \qquad x \in \mathbb{R}, i = 1, 2, \tag{4.14}$$

if and only if

$$\frac{\gamma'_1}{a'_1} - \frac{\beta'_1}{2} = \frac{\gamma'_2}{a'_2} - \frac{\beta'_2}{2} \qquad \text{and} \qquad \frac{\beta'_1}{a'_1} = \frac{\beta'_2}{a'_2} \tag{4.15}$$

and there exists $d \in \mathbb{R}$ with

$$f(x) = \frac{\beta'_1}{2a'_1}x^2 + \left(\frac{\gamma'_1}{a'_1} - \frac{\beta'_1}{2}\right)x + d, \qquad x \in \mathbb{R}. \tag{4.16}$$

(ii) *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $x_0 \in \mathbb{R}$ and satisfies* (4.14) *if and only if* (4.15) *and* (4.16) *hold with some $d \in \mathbb{R}$.*

Proof. (i) It is easy to prove by induction that (4.14) yields

$$f(na'_i + x) - f(x) \leq n\beta'_i x + n\gamma'_i + \sum_{j=0}^{n-1} j\beta'_i a'_i, \qquad x \in \mathbb{R}, i = 1, 2, \tag{4.17}$$

for each $n \in \mathbb{N}$. Further, $\beta'_2\beta'_1 \geq 0$ and $a'_1a'_2 < 0$, whence

$$\max\left\{\frac{\beta'_1}{2a'_1}, \frac{\beta'_2}{2a'_2}\right\} \geq 0.$$

Since $\beta'_2 - \beta'_1 < 0$ and $a'_2 - a'_1 > 0$, this means that there is $m \in \mathbb{N}$ with

$$\frac{1}{m}\left(\frac{\gamma'_2}{a'_2} - \frac{\gamma'_1}{a'_1}\right)\frac{1}{a'_2 - a'_1} + \frac{(m-1)(\beta'_2 - \beta'_1)}{2m(a'_2 - a'_1)} < \max\left\{\frac{\beta'_1}{2a'_1}, \frac{\beta'_2}{2a'_2}\right\}. \tag{4.18}$$

Write

$$a_i = ma'_i, \qquad \beta_i = m\beta'_i, \qquad \gamma_i = m\gamma'_i + \sum_{j=0}^{m-1} j\beta'_i a'_i, \qquad i = 1, 2.$$

Then, in view of (4.17), (4.2) and (4.3) are fulfilled for all $x \in \mathbb{R}$, with $I_0 = \mathbb{R}$. Moreover, (4.18) actually means that (4.1) holds. Now using Theorem 4.1(i) and Remark 4.2 we obtain (4.13) and (4.4). It is easy to check that those two conditions imply (4.15) and (4.16).

The converse is easy to verify.

(ii) We argue analogously as in the case of (i), using Theorem 4.1 (ii) with $\mathcal{J} = \{\emptyset\}$. ■

Remark 4.3. There arises a natural question whether similar results can be obtained for (1.3) with $n > 1$.

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