

Introduction to Algebraic Topology, Part – II
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Lecture – 30
Application

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Module-30 Applications

As a simple consequence of Mayer–Vietoris, we shall prove:

Theorem 3.10

(Homology suspension theorem) Let X be a topological space. Then there is a canonical isomorphism

$$S: \tilde{H}_n(X) \rightarrow H_{n+1}(SX), \quad n \geq 0,$$

where SX denotes the suspension of X ($SX = S^0 * X$).

Having established excision and Mayer Vietoris sequence, let us now give some immediate applications of Mayer Vietoris sequence. The very first one itself is an important result and is called homology suspension theorem. There are similar results called homotopy suspension theorem and so on. The homology suspension theorem is much powerful than the corresponding homotopy suspension theorem which will need more hypotheses, and yields lesser less conclusion. Also homology suspension theorem is very easily obtained and there is no hypothesis at all you will see.

Let X be a topological space. There is a canonical isomorphism S , (that is the standard notation so I have to use that) from $\tilde{H}_n(X)$ to $\tilde{H}_{n+1}(SX)$, for $n \geq 0$. Here this SX denote the suspension of X . The suspension of X can be written as the join of S_0 and X . Or you can think of it as a double cone, take two copies of cone over X and glue them together along X on their bases.

So, that is the suspension isomorphism on the domain I have put a twiddle here and on the codomain whether you put a twiddle or not it is immaterial because $n \geq 0$ and so $n + 1 \geq 1$. Then we know that that reduce homology coincides with unreduced homology. So, you can put a twiddle here or not it does not matter. But here the twiddle is necessary for $n = 0$. If you do not put that then this will not be correct for $n = 0$.

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Proof: Note that the suspension SX is always path connected and hence $H_*(SX) \approx \tilde{H}_*(SX)$ in positive dimensions. So, we shall establish isomorphisms $S : \tilde{H}_n(X) \rightarrow \tilde{H}_{n+1}(SX), n \geq 0$. We have:

$$\mathbb{S}^0 * X = \{-1, 1\} * X$$

is the union of two cones over X , viz. $\{N\} * X$ and $\{S\} * X$ identified along their common base X . Also for any cone $\{v\} * X$, we have $\{v\} * X \setminus \{v\}$ is homeomorphic to $(0, 1] \times X$, which can be deformed to $1 \times X$. It follows that both $\mathbb{S}^0 * X \setminus \{S\} = A_{-1}$ and $\mathbb{S}^0 * X \setminus \{N\} = A_1$ are contractible and $X \hookrightarrow A_{-1} \cap A_1$ is a deformation retract. Since A_1 and A_{-1} are both open in $\mathbb{S}^0 * X$, we have a Mayer-Vietoris sequence

$$\tilde{H}_{i+1}(A_1) \oplus \tilde{H}_{i+1}(A_{-1}) \rightarrow \tilde{H}_{i+1}(\mathbb{S}^0 * X) \xrightarrow{\Delta} \tilde{H}_i(A_1 \cap A_{-1}) \rightarrow \tilde{H}_i(A_1) \oplus \tilde{H}_i(A_{-1})$$

The slide also features a video inset of Anant R. Shastri and navigation controls at the bottom.

So, this is the first remark here. The suspension SX is always path connected and hence $H_*(SX)$ is $\tilde{H}_*(SX)$ in positive dimensions okay? Whereas if X is not path connected then for $H_0(X)$, you know, there will be problems. So, you have to take $\tilde{H}_0(X)$ which will be different from $H_0(X)$. Only $\tilde{H}_0(X)$ will be isomorphic to $H_1(SX)$, okay? So, putting twiddle is a must because I want have the statement for all non negative integers, 0 included, okay?

So, SX is the double cone you can treat it as S_0 join X okay? Instead of $\{-1, 1\}$, which is a dangerous notation, it is better to write S^0 all the time, just like you are taking any other sphere. Also note that a sphere can be treated as a double cone over a sphere of 1 dimension lower, which is its equator, two copies of the cone, namely, upper one $N * \mathbb{S}^{n-1}$ and lower one $\{S\} * \mathbb{S}^{n-1}$, the hemispheres identified along their common boundary.

The first thing that you must notice is that for any cone over X , this is the notation, $\{v\} * X$ is nothing but the join of the singleton space and X . If we throw away the point v from it the space

is homeomorphic to $(0, 1] \times X$. where the class of $0 \times X$ corresponds to the single point v called the tip or the vertex of the cone. (Equivalently, you could have the other definition wherein $1 \times X$ is identified to a single point. But stick to any one of the conventions.)

So that is the notation here. The subspace $(0, 1] \times X$ can be deformed to the base subspace $1 \times X$, because $(0, 1]$ can be deformed to singleton $\{1\}$. So, it follows that the suspension minus the south pole, let denote it by A_{-1} and suspension minus the north pole denoted by A_1 , these are both contractible okay?

Similarly, $X \times 0$ sitting inside $A_{-1} \cap A_1$ is a deformation retract. You collapse both the sides to the equator. So, the general picture is exactly similar the case of a 2-sphere. Here $X \times 0$ represents the equator. So, $A_{-1} \cap A_1$ can completely deformed to $X \times 0$, okay?

So, these are the few things that you have to observe which are topological aspects. here both A_1 and A_{-1} are open subsets, because each time you have thrown away a single point namely, the respective vertices. So, what we have is a Mayer Vietoris sequence now because these are both open so therefore form an excessive couple. So, I can take the direct sum $\tilde{H}_{i+1}(A_1) \oplus \tilde{H}_{i+1}(A_{-1})$ to the reduced homology of the union and then the connecting morphism δ to \tilde{H}_i of the intersection and then again the direct sum and so on. Now these two end groups here are both identically 0, why? Because the spaces are contractible. So, the reduced homology is 0 even in dimension 0. This is a part of the long exact sequence, the scene repeating for every i .

So, the exactness therefore means this delta is an isomorphism. So from the homology of the suspension to the homology of intersection of these two spaces.

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Cell Complexes Categories and Functors Homology Groups Other Homology groups Assorted Topics	Module-22: Basic Homological Algebra Module-24: Homology of Chain complexes Module-25: Euler Characteristics Module-26: Singular Homology Groups Mayer-Vietoris
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which yields an exact sequence

$$0 \rightarrow \tilde{H}_{i+1}(SX) \xrightarrow{\delta} \tilde{H}_i(A_1 \cap A_{-1}) \rightarrow 0$$

for each i and hence the connecting homomorphism δ_i is an isomorphism. Being a deformation retract, $\eta: X \hookrightarrow A_1 \cap A_{-1}$ also induces isomorphisms in homology. The composite $S = \delta^{-1} \circ \eta_*$:

$$S: \tilde{H}_i(X) \xrightarrow{\eta_*} \tilde{H}_i(A_1 \cap A_{-1}) \xrightarrow{\delta_i^{-1}} \tilde{H}_{i+1}(SX)$$

is therefore, an isomorphism. ♣

Now being a deformation retract, the inclusion map η from X to $A_1 \cap A_{-1}$ induces an isomorphism η_* in reduced homology. Therefore, you may replace this one by X . That is, start with η_* from $\tilde{H}_i(X)$ to $\tilde{H}_i(A_1 \cap A_{-1})$ then go back by δ^{-1} . So that is the isomorphism S okay? η_* is an isomorphism, δ is also an isomorphism and I am taking inverse of that one and asking the composite.

So, you see that is why the Mayer-Vietoris sequence is called a ready to use tool. So, all basic things you have done already. It gives you the proof immediately okay? There is this word canonical. Why is it canonical? The inclusion map here is canonical okay? Also taking suspension is a functor, and from the snake lemma the connecting homomorphism is canonical.

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is therefore, an isomorphism.

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Cell Complexes	Module-22: Basic Homological Algebra
Categories and Functors	Module-23: Homology of Chain complexes
Homology Groups	Module-24: Euler Characteristic
Other Homology groups	Module-25: Singular Homology Groups
Associated Topics	Mayer-Vietoris

Remark 3.16

Whenever a pair of closed subspaces $\{X_1, X_2\}$ is given, introducing intermediate pairs of open sets $\{U_1, U_2\}$ such that X_i are deformation retract of U_i is a typical way excision Theorem 3.7 is used in practice. This is the role played by subsets A_{\pm} in the proof of the above theorem. Alternatively, we could have used Theorem 3.9 as follows: Note that the inclusion $X \hookrightarrow CX$ is a cofibration which means that the inclusion $\{N\} * X \hookrightarrow SX$ is a also cofibration. Therefore the pair $\{\{N\} * X, \{S\} * X\}$ is an excisive couple.

A remark. Whenever a pair of closed subspaces X_1, X_2 is given, introducing intermediate pairs of open sets U_1 and U_2 such that X_i a deformation retract of U_i is a typical way excision theorem will be used in practice. Convert the sets into slightly larger open subsets which will deform to the given closed sets. So that is what we have done in the proof of this suspension theorem. If you take the two cones forming the suspension, they intersect along the equator X . But neither of the cone is an open subset. So, what do you do? You take a larger subset which is open. Instead of these cones, I am just looking at the two poles, and removing one of them each time, okay? So those are much larger open sets. Now excision theorem can be applied. But as far as homology is concerned it is the same thing as homology of this cone because the open sets deform to the cones. And then in this special case, the cones are contractible and so their reduced homology is zero.

So, this is a standard way in which we can apply the excision theorem, whenever you have to deal with closed sets. But this may not be always possible. For that we have other techniques such as cofibrations and so on, which you have seen earlier.

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Remark 3.17

Here is an explicit description of the homology suspension homomorphism S . Recall that the cone construction is functorial, viz., given any $f : X \rightarrow Y$, there is a map $C(f) : C(X) \rightarrow C(Y)$ given by

$$C(f)[x, t] = [f(x), t].$$

Let us now use the fact that $\Delta_{n+1} = C(\Delta_n)$, with the apex of the cone corresponding to the vertex e_{n+1} . Given any singular n -simplex $\tau : \Delta_n \rightarrow X$, we can take

$$\tau[N] := C(\tau)_+ : \Delta_{n+1} \rightarrow \{N\} * X; \quad \tau[S] := C(\tau)_- : \Delta_{n+1} \rightarrow \{S\} * X$$

where $C(\tau)_\pm$ sends e_{n+1} to N or S respectively, in the cone construction.



However the simplicity of this proof brings some kind of mystery. Where and how does the final isomorphism come out. Topologically, what is the role of the inclusion map from X to SX in this situation? Can one, in principle at least, directly in some sense, starting with an n -cycle in X , 'suspend' it and get an $(n + 1)$ -cycle or the other way round? I would like to have such an explicit way, a direct way and see how this suspension isomorphism looks like at the homology level, if not at the chain level. We know there is an isomorphism we have no doubt about that. But how does it look like at least in some special cases?

So, this is here I am going to do it in much more generality but then it will be explicit okay. So, for this you have to really know how the cones and suspensions etc. are defined. The first thing is to recall that the cone construction is functorial, namely, when you form a cone on a topological space, along with that given a function f from X to Y , you associate a map from CX to CY which you can call cone of f , okay?


$C(f)$ from CX to CY . How is this defined? $C(f)[x, t]$, remember CX is nothing but the quotient of $X \times [0, 1]$, so, I am writing these notation $[x, t]$ for the element which is represented by $(x, t) \in X \times [0, 1]$. So, take $C(f)([x, t])$ equal to $[f(x), t]$ the class of $(f(x), t)$. This gives you the cone construction which you notice is canonical, namely, if you have another map g from Y to Z , then $C(g \circ f)$ equal to $C(g) \circ C(f)$, okay and $C(Id) = Id$. So that is the meaning of saying that cone construction is functorial.

Now one more fact we want to use, namely, look at the standard $(n + 1)$ -simplex Δ_{n+1} . You can think of it as a cone over the subspace Δ_n , okay? Note Δ_0 which is a single point, but Δ_1 can be thought of as a cone over Δ_0 , this one is a line segment and then Δ_2 can be thought of as the cone over Δ_1 , this one is a triangle and so on. In this convention, one the apex of the cone is the last vertex e_{n+1} . Alternatively, you take the apex to be e_0 for all of them. Both conventions are possible.

So, I want to attack this suspension map directly at the chain level. So, chains are after all generated by singular and simplex, take a singular and simplex that means combines function to Δ_n to X this is $\tau(N)$ and $\tau(S)$, I am going to define 2 such things namely if you look at the upper cone namely X and star with N the North Pole that will give you a cone construction of this τ itself $C(\tau)$.

So, I want to attack this suspension map directly at the chain level okay? So, chains are after all generated by singular simplexes. Take a singular n -simplex that means a continuous function τ from Δ_n into X . I am going to define two copies of the cone of τ , namely, if you look at the upper over X viz., send e_{n+1} to N , the North Pole as the apex that will give you a cone construction of this tau, denote it by $\tau[N]$. That will give you a map from Δ_{n+1} to $N * X$, that is the upper part. Similarly another cone over τ in which the last point e_{n+1} is going to S . Denote it by $\tau(S)$. So, there are two such cone constructions out of the singular simplex τ , okay? Both of them are inside the suspension SX .

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Categories and Functors Homology Groups Other Homology groups Related Topics	Module-26 Homology of Chain complexes Module-25 Euler Characteristic Module-29 Singular Homology Groups Mayer-Vietoris	
<p>Now, let us define</p> $\Sigma(\tau) := (-1)^{n+1}(\tau[N] - \tau[S]).$ <p>It is a matter of straight forward verification that Σ gets linearly extended to give a functorial chain map</p> $\Sigma : S(X) \rightarrow S(SX)$ <p>of degree +1. Therefore, we get an induced graded homomorphism</p> $\Sigma_* : H_n(X) \rightarrow H_n(SX)$ <p>which is of degree 1. The claim is that this homomorphism is nothing but the one we have got in the homology suspension theorem:</p> $\Sigma_* = \delta^{-1} \circ \eta_*.$		
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Cell Complexes Categories and Functors Homology Groups	Module-22 Basic Homological Algebra Module-24 Homology of Chain complexes Module-25 Euler Characteristic	

Now what we do is to take the sum of these two singular simplexes with a correct sign, viz., define $\Sigma(\tau) = (-1)^{n+1}(\tau[N] - \tau[S])$. Clearly, $\Sigma(\tau)$ is a cycle okay? Starting with $n = 0$, you can explicitly write down a full formula for Σ of the identity of n -simplex later on.

I have defined Σ only on a singular n -simplex, but if you take a general n -chain $c = \sum n_i \tau_i$, $\Sigma(c) = \sum n_i \Sigma(\tau_i)$ gives you the unique linear extension of Σ to a chain map $S.(X)$ to $S.(SX)$ of degree +1. Be careful about that. Unlike the situation so far, where we have come across with chain maps of degree 0 only. So, an n -simplex in X has gone to $(n + 1)$ -chain in SX , okay? Once you verify that it is a chain map, then you get a homomorphism at the homology level okay? But out a tilde and take the reduced homology.

We claim that this Σ_* is nothing but the morphism $\delta^{-1} \circ \eta_*$, which we have already proved, is an isomorphism. So, we want to prove that Σ_* is the suspension isomorphism S . That is the same thing as showing that $\delta \circ \Sigma$ is η_* . Remember that η is the inclusion induced map under this deformation. So, it is like identity map as far as the homology is concerned.

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In order to show that $\delta \circ \Sigma_* = \eta_*$, we need to go through the definition of the connecting homomorphism in the Snake lemma:


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- Cell Complexes
- Categories and Functors
- Homology Groups**
- Other Homology groups
- Acyclic Types
- Module 22: Basic Homological Algebra
- Module 24: Homology of Chain complexes
- Module 25: Euler Characteristics
- Module 26: Singular Homology Groups
- Mayer-Vietoris

So, in order to show that this delta component sigma star is eta star we need to go back, okay, to the snake lemma. Go through the steps how δ was defined. And then you will get it. There is no shorter way here to see how this is equal to this one okay, other than going through the steps of the definition of δ . No theorems will give you this. So, you have to go back to the snake lemma okay.

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$$\begin{array}{ccccccc}
 0 & & 0 & & 0 & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \text{Ker } \partial' & \xrightarrow{i} & \text{Ker } \partial & \xrightarrow{j} & \text{Ker } \partial'' & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 S_n(A_+ \cap A_-) & \xrightarrow{i} & S_n(A_+) \oplus S_n(A_-) & \xrightarrow{j} & S_n(SX) & \rightarrow & 0 \\
 \downarrow \partial' & & \downarrow \partial & & \downarrow \partial'' & & \\
 0 \rightarrow S_{n-1}(A_+ \cap A_-) & \xrightarrow{i} & S_{n-1}(A_+) \oplus S_{n-1}(A_-) & \xrightarrow{j} & S_{n-1}(SX) & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \text{Coker } \partial' & \xrightarrow{i} & \text{Coker } \partial & \xrightarrow{j} & \text{Coker } \partial'' & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & & 0 & & 0 & &
 \end{array}$$

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- Cell Complexes
- Categories and Functors
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So, let me do that. So I have copied that diagram slightly differently to suit the present situation, viz., the spaces involved are A_+ , A_- are the two cone over X , their intersection being X and union being SX . So, therefore the entire thing becomes very easy here. Now how was the connecting homomorphism defined from this kernel you come here this inclusion map then you

pick up something here then you push it down then pick up something here and push it down right?

What is the element here I am taking? So, I start with Σ_* of something n -cycle okay and then apply δ and see what is it. Whether I get back the same element that is what I have to check okay?

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Cell Complexes
Categories and Functors
Homology Groups
Other Homology groups
Assorted Topics

Module-22 Basic Homological Algebra
Module-24 Homology of Chain complexes
Module-25 Euler Characteristic
Module-26 Singular Homology Groups
Mayer-Vietoris

Starting with an element $[\tau] \in H_n(X)$, represented by an n -cycle τ , we have $\Sigma_*([\tau])$ is represented by $(-1)^{n+1}(\tau[N] - \tau[S]) \in S_{n+1}(SX)$. Since $j(\tau[N], -\tau[S]) = \tau[N] - \tau[S]$, we look at

$$\begin{aligned} & \partial(\tau[N], -\tau[S]) \\ &= (\partial\tau)[N] + (-1)^{n+1}\tau - \partial(\tau)[S] - (-1)^{n+1}\tau \\ &= (-1)^{n+1}(\tau, -\tau) = (-1)^{n+1}j(\tau). \end{aligned}$$

This proves the claim. It also justifies the word 'canonical' in the statement of the theorem 3.10.

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Starting with an element $[\tau]$ in $H_n(x)$ instead of a singular simplex, represented by a cycle τ , τ is a cycle means $\partial(\tau) = 0$, right? so that is in the kernel okay? And then we have $\Sigma_*(\tau)$ okay? Then I want to show that δ of that is again equal to $\eta_*(\tau)$.

So, $\Sigma_*(\tau)$ is equal to $(-1)^n(\tau[N] - \tau[S])$, right? But $(\tau[N] - \tau[S])$ is equal to $j(\tau[N], -\tau[S])$, from the direct sum. For the time being ignore the sign $(-1)^{n+1}$, we shall adjust it at the end. Therefore in the definition of δ , I do not have to pick up something else. It is already there, namely, $(\tau[N], -\tau[S])$. Remember that $j(a_1, a_2) = a_1 + a_2$.

So, now look at the boundary of this element. The boundary of the direct sum is nothing but a direct sum of the boundaries. So, I have to apply boundary here, boundary here to get an element here. So that is what is boundary of $(\tau[N], -\tau[S])$. It is nothing but boundary of this, minus boundary of that. Now you must know how to compute boundary of this one, $\tau[N]$.

This τ itself is a summation of singular simplexes $\tau = \sum n_j \tau_j$. For each fixed j , what is the boundary of $\tau_j[N]$. Since $\tau_j[N]$ is the linear extension of τ_j and the map which sends e_{n+1} to the vertex N , it follows $\tau_j[N] \circ F^i = \tau_j \circ F^i[N]$ for $i \leq n$ and $\tau_j[N] \circ F^{n+1} = \tau_j$. Similar conclusion applies to $\tau_j[S]$ also. Since $\partial(\tau[N]) = \sum n_j \partial(\tau_j[N])$ etc., and since τ is a cycle, we get, $\partial(\tau[N], -\tau[S]) = (-1)^{n+1}(\tau, -\tau) = (-1)^{n+1}i(\tau)$. We can now bring in the other factor $(-1)^{n+1}$ which we had ignored and the conclusion follows.

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Cell Complexes, Categories and Functors, Homology Groups, Other Homology groups, Assorted Topics, Module-22 Basic Homological Algebra, Module-24 Homology of Chain complexes, Module-25 Euler Characteristic, Module-26 Singular Homology Groups, Mayer-Vietoris

Example 3.6

Computation of $H_*(\mathbb{S}^n)$ We know that \mathbb{S}^n is homeomorphic to the suspension of \mathbb{S}^{n-1} . Therefore, starting with the homology of \mathbb{S}^0 , one can directly apply the homology suspension theorem above inductively, and compute the homology of all spheres. At each step the missing information is about $\tilde{H}_0(\mathbb{S}^n)$, $n \geq 1$ which clearly vanishes since \mathbb{S}^n , $n \geq 1$ are path connected. Therefore

$$\tilde{H}_k(\mathbb{S}^n) = \begin{cases} 0, & k \neq n; \\ \mathbb{Z}, & k = n. \end{cases} \quad (19)$$

Starting with a generator for $\tilde{H}_0(\mathbb{S}^0)$ we can successively take its suspension to get an explicit generator for $H_n(\mathbb{S}^n)$ for all $n \geq 1$.

Next let us do computation of the homology of \mathbb{S}^n itself. Here, for the first time, we are doing something non trivial now okay? Again I have been telling you that \mathbb{S}^n is nothing but the suspension of \mathbb{S}^{n-1} , etc., finally \mathbb{S}^n is the iterated suspension of \mathbb{S}^0 . And \mathbb{S}^0 is what? Just the discrete space with two points, for which the entire homology modules we know okay? All the higher homology modules are 0. And because it has 2 components H_0 is $\mathbb{Z} \oplus \mathbb{Z}$.

But when you take the reduced homology, it will be infinite cyclic okay? Therefore, immediately, you see that $H_1(\mathbb{S}^1)$ is infinite cyclic and $H_2(\mathbb{S}^2)$ is infinite cyclic and so on, you would immediately get that $H_n(\mathbb{S}^n)$ is infinite cyclic. Here whether you put tilde or not it does not matter. Only thing is that twiddle is necessary when you take $n = 0$. When $n = 0$, $\tilde{H}_0(\mathbb{S}^0)$ is \mathbb{Z} . But if you do not put twiddle then it is $\mathbb{Z} \oplus \mathbb{Z}$. Okay? For that reason you have take reduced homology, okay? So, what happens to other homology groups. Let us look at $\tilde{H}_0(\mathbb{S}^1)$. For H_0 ,

there is no information coming from this Mayer-Vietoris sequence or the homology suspension theorem.

But we already know that H_0 depends upon number of path connected components of the space. \mathbb{S}^1 is path connected. Therefore, H_0 is infinite cycle and \tilde{H}_0 is 0. This is true for all sphere \mathbb{S}^n for $n > 0$. To compute $H_i(\mathbb{S}^n)$, for $n > i > 0$, we can use the homology suspension theorem iteratively and conclude that it is isomorphic to $\tilde{H}_0(\mathbb{S}^{n-i}) = 0$ and for $i > n$, it is equal to $\tilde{H}_{i-n}(\mathbb{S}^0) = 0$.

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Let the two 0-simplices \mathbb{S}^0 be denoted by u, v where u takes the value -1 and v takes the value 1 . It follows that $g_0 = u - v$ is a generator of $\tilde{H}_0(\mathbb{S}^0)$. Now

$$\Sigma_*(g) = -(g[N] - g[S]) = [u, S] - [v, S] - [u, N] + [v, N]$$

gives us a generator for $H_1(\mathbb{S}^1)$.

Figure 17: Generator of $H_1(\mathbb{S}^1)$

Once you have this you can actually write down a generator for $\tilde{H}_n(\mathbb{S}^n)$, as mentioned earlier in this picture. So, this u, v represent $-1, 1$ respectively of \mathbb{S}^0 , Under the augmentation $u - v$ goes to 0 and so it is 0-cycle. The element $[u - v]$ in H_0 represented by $u - v$ is a generator of $\tilde{H}_0(\mathbb{S}^0)$. Denote this by g_0 and inductively define $g_n = \Sigma_*(g_{n-1})$ to get a generator of $\tilde{H}_n(\mathbb{S}^n)$.

How does g_1 look like? It is equal to $\Sigma_*(g_0) = u[N] - u[S] - v[N] - v[S]$. If you put arrows indicate the direction of edges then you will see that it is like a quadrilateral traced in the clockwise direction. So, this is the picture in dimension 1, okay?

Like this now you can go on to make a cone over double cone over this one to get the picture of \mathbb{S}^2 and the 2-cycle will be the sum of various triangles there, oriented triangles, correctly oriented

okay? So, when you study simplicial homology, you will come again to have a look at this picture.

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Example 3.7


Homology of the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$

Since \mathbb{D}^n is contractible, its reduced homology vanishes. On the other hand, from the previous example we know $H_i(\mathbb{S}^{n-1})$. Piecing them together via the long exact sequence

$$\cdots \rightarrow H_i(\mathbb{D}^n) \rightarrow H_i(\mathbb{D}^n, \mathbb{S}^{n-1}) \rightarrow H_i(\mathbb{S}^{n-1}) \rightarrow \cdots$$

we get

$$H_i(\mathbb{D}^n, \mathbb{S}^{n-1}) = \begin{cases} 0, & i \neq n; \\ \mathbb{Z}, & i = n. \end{cases} \quad (20)$$



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<ul style="list-style-type: none"> Cell Complexes Categories and Functors Homology Groups Other Homology groups Assorted Topics 	<ul style="list-style-type: none"> Module 22 Basic Homological Algebra Module 24 Homology of Chain complexes Module 25 Euler Characteristic Module 26 Singular Homology Groups Mayer-Vietoris
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NPTEL

Example 3.8

Next thing is we can now compute the homology of the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$. This time we do not need Mayer Vietoris sequence. We just need the long homology sequence of the pair okay? \mathbb{D}^n is contractible, therefore if you take the long homology exact sequence, what we have is $\tilde{H}_i(\mathbb{D}^n)$ to $\tilde{H}_i(\mathbb{D}^n, \mathbb{S}^{n-1})$ to $\tilde{H}_{i-1}(\mathbb{S}^{n-1})$ to $\tilde{H}_{i-1}(\mathbb{D}^n)$ and so on, the two end terms in this part are 0 and hence this middle arrow will be an isomorphism. Since we have computed this thing now we know $H_i(\mathbb{D}^n, \mathbb{S}^{n-1})$ is 0 if $i \neq n$ and is equal to \mathbb{Z} if $i = n$, okay? So when you are discussing relative homology, there is no need to put the twiddle here.

So, next example, I think we cannot do that example now. It is time. So let us to stop here.