**Introduction to Algebraic Topology, Part – II**
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**Lecture – 30**
**Application**

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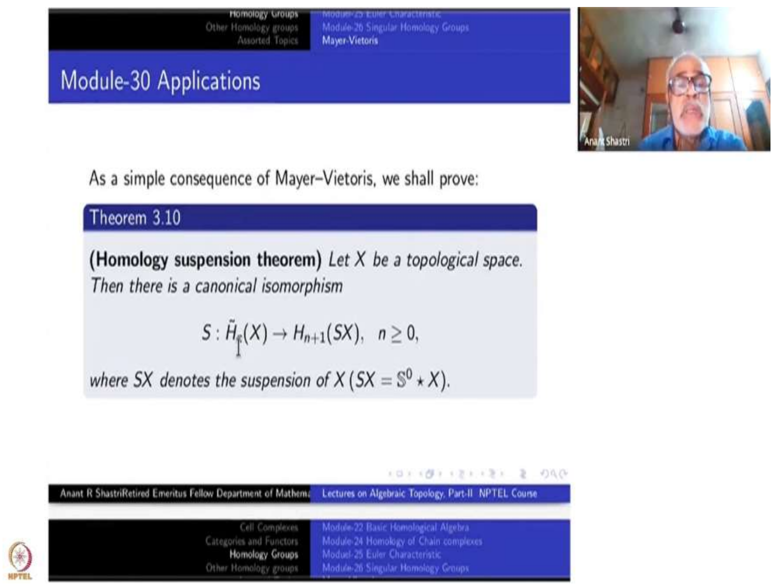


Having established excision and Mayer Vietoris sequence, let us now give some immediate applications of Mayer Vietoris sequence. The very first one itself is an important result and is called homology suspension theorem. There are similar results called homotopy suspension theorem and so on. The homology suspension theorem is much powerful than the corresponding homotopy suspension theorem which will need more hypotheses, and yields lesser less conclusion. Also homology suspension theorem is very easily obtained and there is no hypothesis at all you will see.

Let $X$ be a topological space. There is a canonical isomorphism $S$, (that is the standard notation so I have to use that) from $\tilde{H}_n(X)$ to $\tilde{H}_{n+1}(SX)$, for $n \geq 0$. Here this $SX$ denote the suspension of $X$. The suspension of $X$ can be written as the join of $S_0$ and $X$. Or you can think of it as a double cone, take two copies of cone over $X$ and glue them together along $X$ on their bases.

So, that is the suspension isomorphism on the domain I have put a twiddle here and on the codomain whether you put a twiddle or not it is immaterial because $n \geq 0$ and so $n+1 \geq 1$. Then we know that that reduce homology coincides with unreduced homology. So, you can put a twiddle here or not it does not matter. But here the twiddle is necessary for $n = 0$. If you do not put that then this will not be correct for $n = 0$.

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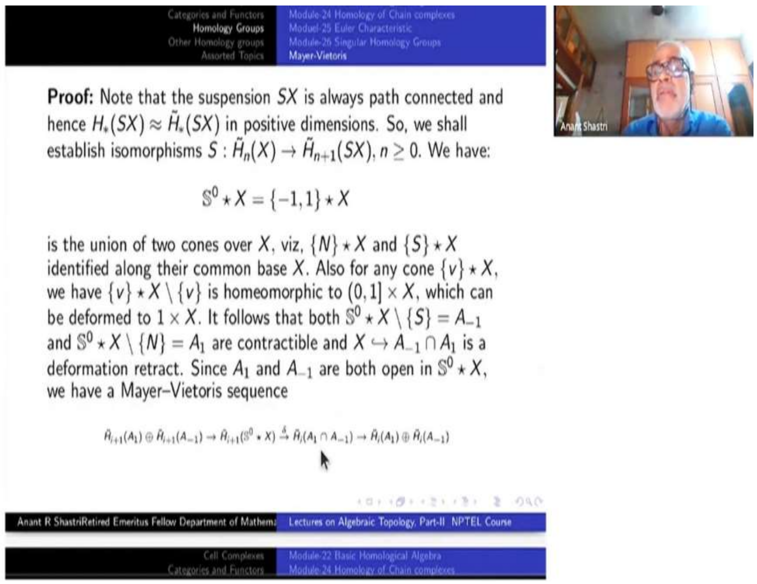


So, this is the first remark here. The suspension $SX$ is always path connected and hence $H_*(SX)$ is $\tilde{H}_*(SX)$ in positive dimensions okay? Whereas if $X$ is not path connected then for $H_0(X)$, you know, there will be problems. So, you have to take $\tilde{H}_0(X)$ which will be different from $H_0(X)$. Only $\tilde{H}_0(X)$ will be isomorphic to $H_1(SX)$, okay? So, putting twiddle is a must because I want have the statement for all non negative integers, 0 included, okay?

So, $SX$ is the double cone you can treat it as $S_0$ join $X$ okay? Instead of $\{-1, 1\}$, which is a dangerous notation, it is better to write $S^0$ all the time, just like you are taking any other sphere. Also note that a sphere can be treated as a double cone over a sphere of 1 dimension lower, which is its equator, two copies of the cone, namely, upper one $N * \mathbb{S}^{n-1}$ and lower one $\{S\} * \mathbb{S}^{n-1}$, the hemispheres identified along their common boundary.

The first thing that you must notice is that for any cone over $X$, this is the notation, $\{v\} * X$ is nothing but the join of the singleton space and $X$. If we throw away the point $v$ from it the space is homeomorphic to $(0, 1] \times X$. where the class of $0 \times X$ corresponds to the single point $v$ called the tip or the vertex of the cone. (Equivalently, you could have the other definition wherein $1 \times X$ is identified to a single point. But stick to any one of the conventions.)

So that is the notation here. The subspace $(0, 1] \times X$ can be deformed to the base subspace $1 \times X$, because $(0, 1]$ can be deformed to singleton $\{1\}$. So, it follows that the suspension minus the south pole, let denote it by $A_{-1}$ and suspension minus the north pole denoted by $A_1$, these are both contractible okay?

Similarly, $X \times 0$ sitting inside $A_{-1} \cap A_1$ is a deformation retract. You collapse both the sides to the equator. So, the general picture is exactly similar the case of a 2-sphere. Here $X \times 0$ represents the equator. So, $A_{-1} \cap A_1$ can completely deformed to $X \times 0$, okay?

So, these are the few things that you have to observe which are topological aspects. here both $A_1$ and $A_{-1}$ are open subsets, because each time you have thrown away a single point namely, the respective vertices. So, what we have is a Mayer Vietoris sequence now because these are both open so therefore form an excessive couple. So, I can take the direct sum $\tilde{H}_{i+1}(A_1) \oplus \tilde{H}_{i+1}(A_{-1})$ to the reduced homology of the union and then the connecting morphism $\delta$ to $\tilde{H}_i$ of the intersection and then again the direct sum and so on. Now these two end groups here are both identically $0$, why? Because the spaces are contractible. So, the reduced homology is 0 even in dimension $0$. This is a part of the long exact sequence, the scene repeating for every $i$.

So, the exactness therefore means this delta is an isomorphism. So from the homology of the suspension to the homology of intersection of these two spaces.

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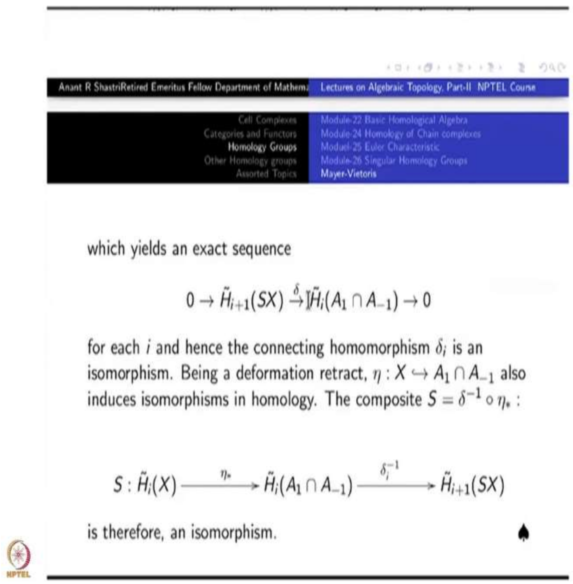


Now being a deformation retract, the inclusion map $\eta$ from $X$ to $A_1 \cap A_{-1}$ induces an isomorphism $\eta_*$ in reduced homology. Therefore, you may replace this one by $X$. That is, start with $\eta_*$ from $\tilde{H}_i(X)$ to $\tilde{H}_i(A_1 \cap A_{-1})$ then go back by $\delta^{-1}$. So that is the isomomorphism $S$ okay? $\eta_*$ is an isomorphism, $\delta$ is also an isomorphism and I am taking inverse of that one and asking the composite.

So, you see that is why the Mayer-Vietoris sequence is called a ready to use tool. So, all basic things you have done already. It gives you the proof immediately okay? There is this word canonical. Why is it canonical? The inclusion map here is canonical okay? Also taking suspension is a functor, and from the snake lemma the connecting homomorphism is canonical.

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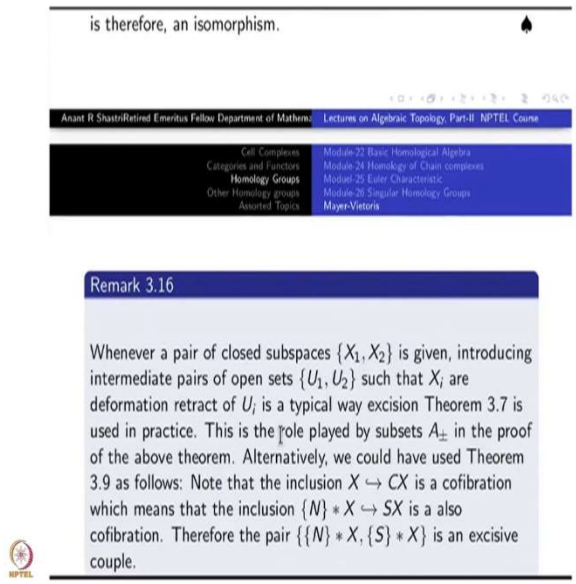


A remark. Whenever a pair of closed subspaces $X_1, X_2$ is given, introducing intermediate pairs of open sets $U_1$ and $U_2$ such that $X_i$ a deformation retract of $U_i$ is a typical way excision theorem will be used in practice. Convert the sets into slightly larger open subsets which will deform to the given closed sets. So that is what we have done in the proof of this suspension theorem. If you take the two cones forming the suspension, they intersect along the equator $X$. But neither of the cone is an open subset. So, what do you do? You take a larger subset which is open. Instead of these cones, I am just looking at the two poles, and removing one of them each time, okay? So those are much larger open sets. Now excision theorem can be applied. But as far as homology is concerned it is the same thing as homology of this cone because the open sets deform to the cones. And then in this special case, the cones are contractible and so their reduced homology is zero.

So, this is a standard way in which we can apply the excision theorem, whenever you have to deal with closed sets. But this may not be always possible. For that we have other techniques such as cofibrations and so on, which you have seen earlier.

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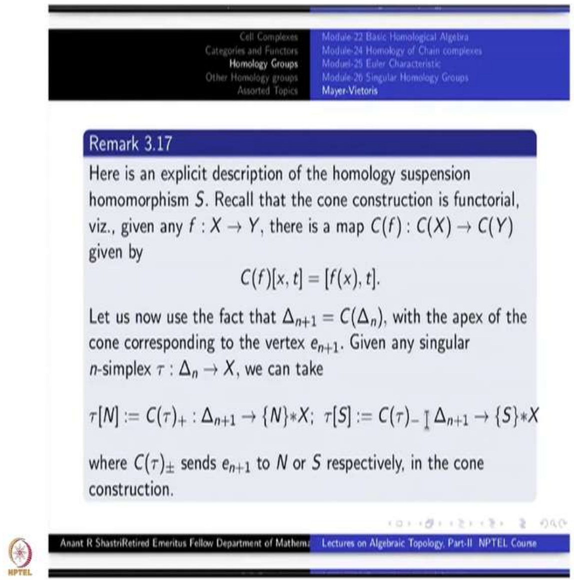


However the simplicity of this proof brings some kind of mystery. Where and how does the final isomorphism come out. Topologically, what is the role of the inclusion map from $X$ to $SX$ in this situation? Can one, in principle at least, directly in some sense, starting with an $n$-cycle in $X$, `suspend' it and get an $(n+1)$-cycle or the other way round? I would like to have such an explicit way, a direct way and see how this suspension isomorphism looks like at the homology level, if not at the chain level. We know there is an isomorphism we have no doubt about that. But how does it look like at least in some special cases?

So, this is here I am going to do it in much more generality but then it will be explicit okay. So, for this you have to really know how the cones and suspensions etc. are defined. The first thing is to recall that the cone construction is functorial, namely, when you form a cone on a topological space, along with that given a function $f$ from $X$ to $Y$, you associate a map from $CX$ to $CY$ which you can call cone of $f$, okay?

$C(f)$ from $CX$ to $CY$. How is this defined? $C(f)[x, t]$, remember $CX$ is nothing but the quotient of $X \times [0, 1]$, so, I am writing these notation $[x, t]$ for the element which is represented by $(x, t) \in X \times [0, 1]$. So, take $C(f)([x, t])$ equal to $[f(x), t]$ the class of $(f(x), t)$. This gives you the cone construction which you notice is canonical, namely, if you have another map $g$ from $Y$ to $Z$, then $C(g \circ f)$ equal to $C(g) \circ C(f)$, okay and $C(Id) = Id$. So that is the meaning of saying that cone construction is functorial.

Now one more fact we want to use, namely, look at the standard $(n+1)$-simplex $\Delta_{n+1}$. You can think of it as a cone over the subspace $\Delta_n$, okay? Note $\Delta_0$ which is a single point, but $\Delta_1$ can be thought of as a cone over $\Delta_0$, this one is a line segment and then $\Delta_2$ can be thought of as the cone over $\Delta_1$, this one is a triangle and so on. In this convention, one the apex of the cone is the last vertex $e_{n+1}$. Alternatively, you take the apex to be $e_0$ for all of them. Both conventions are possible.

So, I want to attack this suspension map directly at the chain level. So, chains are after all generated by singular and simplex, take a singular and simplex that means combines function to $\Delta_n$ to $X$ this is $\tau(N)$ and $\tau(S)$, I am going to define 2 such things namely if you look at the upper cone namely $X$ and star with $N$ the North Pole that will give you a cone construction of this $\tau$ itself $C(\tau)$.

So, I want to attack this suspension map directly at the chain level okay? So, chains are after all generated by singular simplexes. Take a singular $n$-simplex that means a continuous function $\tau$ from $\Delta_n$ into $X$. I am going to define two copies of the cone of $\tau$, namely, if you look at the upper over $X$ viz., send $e_{n+1}$ to $N$, the North Pole as the apex that will give you a cone construction of this tau, denote it by $\tau[N]$. That will give you a map from $\Delta_{n+1}$ to $N * X$, that is the upper part. Similarly another cone over $\tau$ in which the last point $e_{n+1}$ is going to $S$. Denote it by $\tau(S)$. So, there are two such cone constructions out of the singular simplex $\tau$, okay? Both of them are inside the suspension $SX$.

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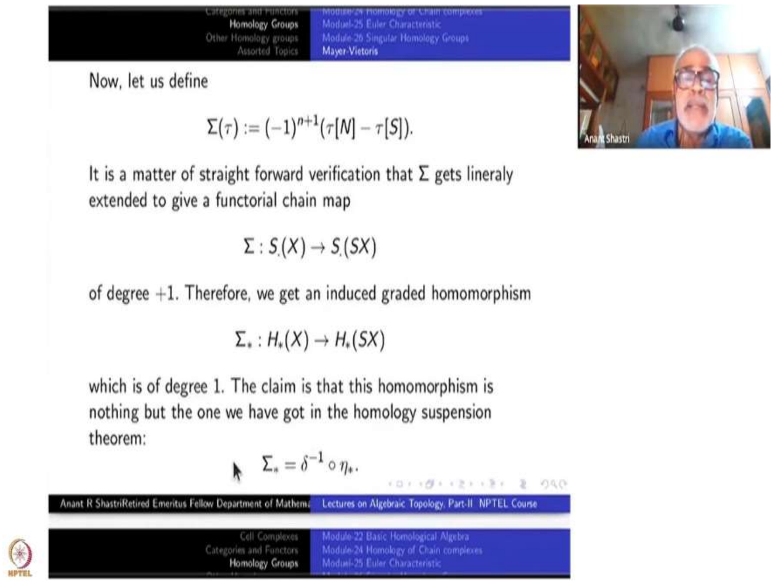


Now what we do is to take the sum of these two singular simplexes with a correct sign, viz., define $\Sigma(\tau) = (-1)^{n+1}(\tau[N] - \tau[S])$. Clearly, $\Sigma(\tau)$ is a cycle okay? Starting with $n = 0$, you can explicitly write down a full formula for $\Sigma$ of the identity of $n$-simplex later on.

I have defined $\Sigma$ only on a singular $n$-simplex, but if you take a general $n$-chain $c = \sum n_i \tau_i, \Sigma(c) = \sum n_i \Sigma(\tau_i)$ gives you the unique linear extension of $\Sigma$ to a chain map $S.(X)$ to $S.(SX)$ of degree $+1$. Be careful about that. Unlike the situation so far, where we have come across with chain maps of degree $0$ only. So, an $n$-simplex in $X$ has gone to $(n+1)$-chain in $SX$, okay? Once you verify that it is a chain map, then you get a homomorphism at the homology level okay? But out a tilde and take the reduced homology.

We claim that this $\Sigma_*$ is nothing but the morphism $\delta^{-1} \circ \eta_*$, which we have already proved, is an isomorphism. So, we want to prove that $\Sigma_*$ is the suspension isomorphism $S$. That is the same thing as showing that $\delta \circ \Sigma$ is $\eta_*$. Remember that $\eta$ is the inclusion induced map under this deformation. So, it is like identity map as far as the homology is concerned.

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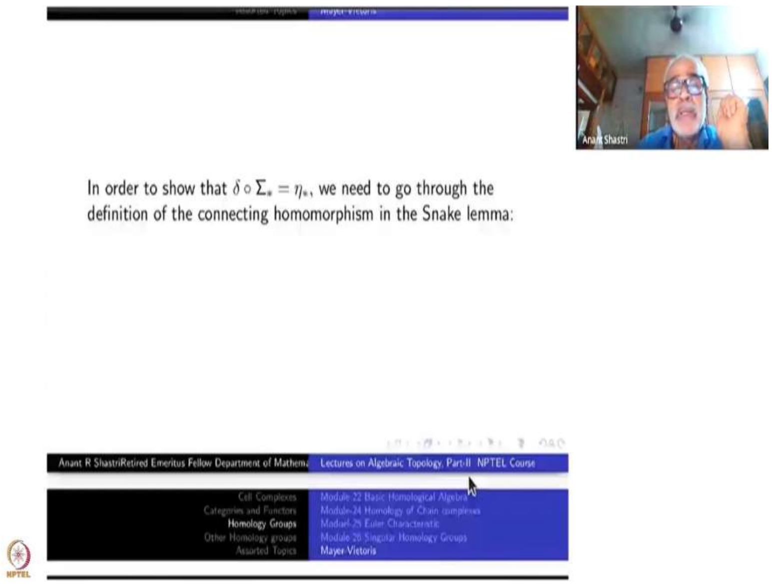


So, in order to show that this delta component sigma star is eta star we need to go back, okay, to the snake lemma. Go through the steps how $\delta$ was defined. And then you will get it. There is no shorter way here to see how this is equal to this one okay, other than going through the steps of the definition of $\delta$. No theorems will give you this. So, you have to go back to the snake lemma okay.

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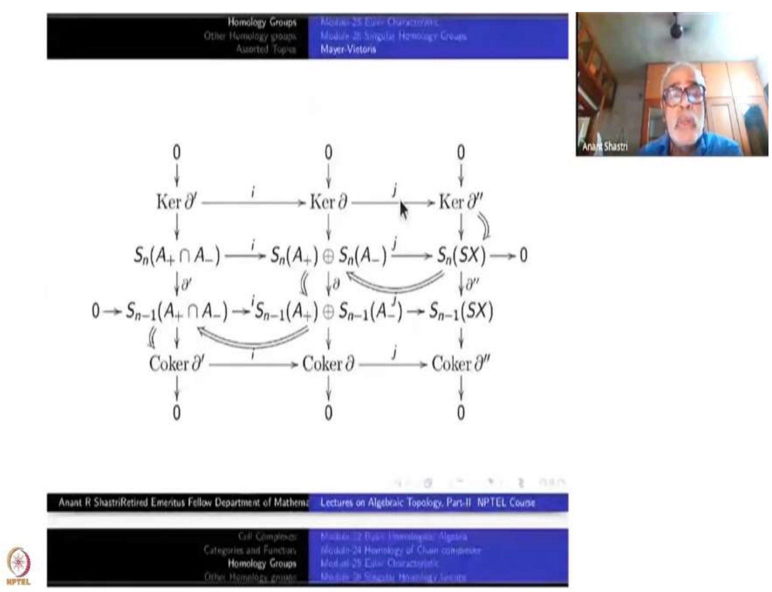


So, let me do that. So I have copied that diagram slightly differently to suit the present situation, viz., the spaces involved are $A_+, A_-$ are the two cone over $X$, their intersection being $X$ and union being $SX$. So, therefore the entire thing becomes very easy here. Now how was the connecting homomorphism defined from this kernel you come here this inclusion map then you pick up something here then you push it down then pick up something here and push it down right?

What is the element here I am taking? So, I start with $\Sigma_*$ of something $n$-cycle okay and then apply $\delta$ and see what is it. Whether I get back the same element that is what I have to check okay?

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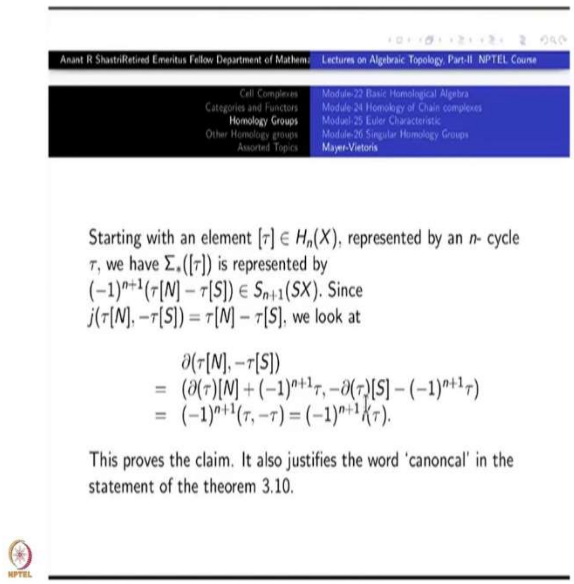


Starting with an element $[\tau]$ in $H_n(x)$ instead of a singular simplex, represented by a cycle $\tau$, $\tau$ is a cycle means $\partial(\tau) = 0$, right? so that is in the kernel okay? And then we have $\Sigma_*(\tau)$ okay? Then I want to show that $\delta$ of that is again equal to $\eta_*(\tau)$.

So, $\Sigma_*(\tau)$ is equal to $(-1)^n(\tau[N] - \tau[S])$, right? But $(\tau[N] - \tau[S])$ is equal to $j(\tau[N], -\tau[S])$, from the direct sum. For the time being ignore the sign $(-1)^{n+1}$, we shall adjust it at the end. Therefore in the definition of $\delta$, I do not have to pick up something else. It is already there, namely, $(\tau[N], -\tau[S])$. Remember that $j(a_1, a_2) = a_1 + a_2$.

So, now look at the boundary of this element. The boundary of the direct sum is nothing but a direct sum of the boundaries. So, I have to apply boundary here, boundary here to get an element here. So that is what is boundary of $(\tau[N], -\tau[S])$. It is nothing but boundary of this, minus boundary of that. Now you must know how to compute boundary of this one, $\tau[N]$.

This $\tau$ itself is a summation of singular simplexes $\tau = \sum n_j \tau_j$. For each fixed $j$, what is the boundary of $\tau_j[N]$. Since $\tau_j[N]$ is the linear extension of $\tau_j$ and the map which sends $e_{n+1}$ to the vertex $N$, it follows $\tau_j[N] \circ F^i = \tau_j \circ F^i[N]$ for $i \leq n$ and $\tau_j[N \circ F^{n+1} = \tau_j$. Similar conclusion applies to $\tau_j[S]$ also. Since $\partial(\tau[N]) = \sum n_j \partial(\tau_j[N])$ etc., and since $\tau$ is a cycle, we get, $\partial(\tau[N], -\tau[S]) = (-1)^{n+1}(\tau, -\tau) = (-1)^{n+1} i(\tau)$. We can now bring tin the other factor $(-1)^{n+1}$ which we had ignored and the conclusion follows.

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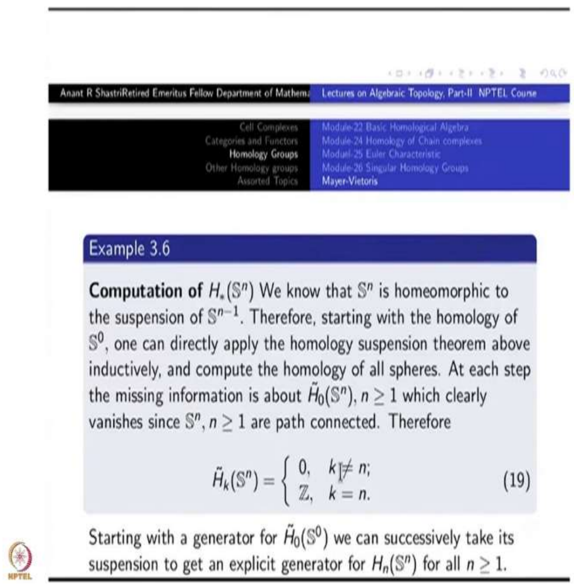


Next let us do computation of the homology of $\mathbb{S}^n$ itself. Here, for the first time, we are doing something non trivial now okay? Again I have been telling you that $\mathbb{S}^n$ is nothing but the suspension of $\mathbb{S}^{n-1}$, etc., finally $\mathbb{S}^n$ is the iterated suspension of $\mathbb{S}^0$. And $\mathbb{S}^0$ is what? Just the discrete space with two points, for which the entire homology modules we know okay? All the higher homology modules are 0. And because it has 2 components $H_0$ is $\mathbb{Z} \oplus \mathbb{Z}$.

But when you take the reduced homology, it will be infinite cyclic okay? Therefore, immediately, you see that $H_1(\mathbb{S}^1)$ is infinite cyclic and $H_2(\mathbb{S}^2)$ is infinite cyclic and so on, you would immediately get that $H_n(\mathbb{S}^n)$ is infinite cyclic. Here whether you put tilde or not it does not matter. Only thing is that twiddle is necessary when you take $n = 0$. When $n = 0$, $\tilde{H}_0(\mathbb{S}^0)$ is $\mathbb{Z}$. But if you do not put twiddle then it is $\mathbb{Z} \oplus \mathbb{Z}$. Okay? For that reason you have take reduced homology, okay? So, what happens to other homology groups. Let us look at $\tilde{H}_0(\mathbb{S}^1)$. For $H_0$, there is no information coming from this Mayer-Vietoris sequence or the homology suspension theorem.

But we already know that $H_0$ depends upon number of path connected components of the space. $\mathbb{S}^1$ is path connected. Therefore, $H_0$ is infinite cycle and $\tilde{H}_0$ is $0$. This is true for all sphere $\mathbb{S}^n$ for $n > 0$. To compute $H_i(\mathbb{S}^n)$, for $n > i > 0$, we can use the homology suspension theorem iteratively and conclude that it is isomorphic to $\tilde{H}_0(\mathbb{S}^{n-i}) = 0$ and for $i > n$, it is equal to $\tilde{H}_{i-n}(\mathbb{S}^0) = 0$.

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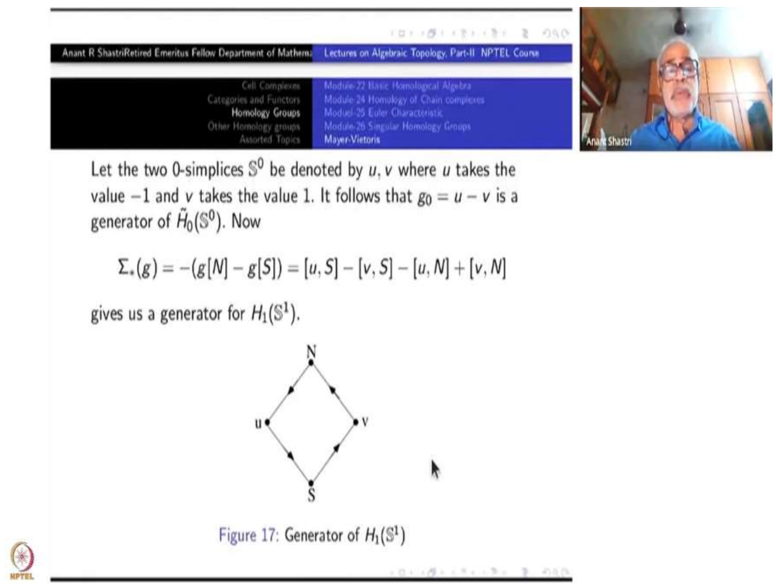


Once you have this you can actually write down a generator for $\tilde{H}_n(\mathbb{S}^n)$, as mentioned earlier in this picture. So, this $u, v$ represent $-1, 1$ respectively of $\mathbb{S}^0$, Under the augmentation $u - v$ goes to $0$ and so it is $0$-cycle. The element $[u - v]$ in $H_0$ represented by $u - v$ is a generator of $\tilde{H}_0(\mathbb{S}^0)$. Denote this by $g_0$ and inductively define $g_n = \Sigma_*(g_{n-1})$ to get a generator of $\tilde{H}_n(\mathbb{S}^n)$.

How does $g_1$ look like? It is equal to $\Sigma_*(g_0) = u[N] - u[S] - v[N] - v[S]$. If you put arrows indicate the direction of edges then you will see that it is like a quadrilateral traced in the clockwise direction. So, this is the picture in dimension $1$, okay?

Like this now you can go on to make a cone over double cone over this one to get the picture of $\mathbb{S}^2$ and the 2-cycle will be the sum of various triangles there, oriented triangles, correctly oriented okay? So, when you study simplicial homology, you will come again to have a look at this picture.

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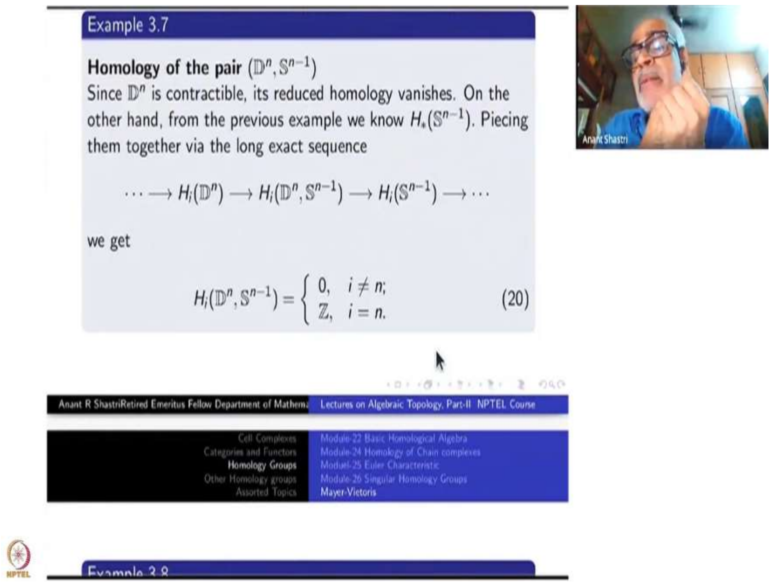


Next thing is we can now compute the homology of the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$. This time we do not need Mayer Vietoris sequence. We just need the long homology sequence of the pair okay? $\mathbb{D}^n$ is contractible, therefore if you take the long homology exact sequence, what we have is $\tilde{H}_i(\mathbb{D}^n)$ to $\tilde{H}_i(\mathbb{D}^n, \mathbb{S}^{n-1})$ to $\tilde{H}_{i-1}(\mathbb{S}^{n-1})$ to $\tilde{H}_{i-1}(\mathbb{D}^n)$ and so on, the two end terms in this part are 0 and hence this middle arrow will be an isomorphism. Since we have computed this thing now we know $H_i(\mathbb{D}^n, \mathbb{S}^{n-1})$ is 0 if $i \neq n$ and is equal to $\mathbb{Z}$ if $i = n$, okay? So when you are discussing relative homology, there is no need to put the twiddle here.

So, next example, I think we cannot do that example now. It is time. So let us to stop here.