LABORATORY MANUAL FOR MATHEMATICS PRACTICALS (WITH FOSS TOOLS)

For 3rd Semester

B. Sc. (CBCS)

1 Lagrange's theorem

binary operation: Let A be a non-empty set. Then a binary operation $*$ on A is a function $* : A \times A \longrightarrow A$ defined by $*(a, b) = a * b$.

In other words $*$ is binary operation on a set A if $a * b \in A, \forall a, b \in A$. In this case we say that A is closed under $*$ or closure property holds in A w.r.t $*$. Binary operations are, usually, denoted by $+, -, *, \div$ etc.

Group: A non-empty set G with a binary operation $*$, denoted by $(G, *)$, is said to be a group if the following properties (or axioms) are satisfied.

1. **Closure property:** $a * b \in G$ for any $a, b \in G$.
2. **Associate property:** $(a * b) * c = a * (b * c)$ for any $a, b, c \in G$.
3. **Existence of identity element:** There exists e in G such that $a * e = e * a = a$ for all $a \in G$.
4. **Existence of inverse element:** For any a in G there exists a^{-1} in G such that $a * a^{-1} = a^{-1} * a = e$.

Further the group $(G, *)$ is called abelian (or commutative) group if along with above properties $a * b = b * a$, $\forall a, b \in G$, also holds.A group $(G, *)$ is simply denoted by G.

Lagrange's theorem: If G is any finite group and H is any subgroup of G, then $O(H)$ divides $O(G)$.

Maxima program to verify Lagrange's theorem for the group G = {1,-1, i ,-i } and its subset H = {1, -1}

```
G : set(1,-1,%i,-%i);

H : set(1,-1);

HXH : cartesian_product(H,H);

HHinv : makeset(a*b^-1,[a,b],HXH);

if  HHinv = H  then  disp("H is a subgroup of G")

else disp("H is NOT a subgroup of G") $

n : length(G);

m : length(HHinv);

if  mod(n,m) = 0  then  disp("Lagranges theorem is satisfied")

else disp("Lagranges theorem is NOT satisfied")$
```

OUTPUT

{-1,1,-%i,%i}

{-1,1}

{[-1,-1],[-1,1],[1,-1],[1,1]}

{-1,1}

"H is a subgroup of G"

4

2

"Lagranges theorem is satisfied"

Exercise

Verify Lagranges theorem for the following

(i) Let $G = \{1, -1, i, -i\}$ is a group and $H = \{1, i\}$.

(ii) Let $G = \{1, -1, i, -i\}$ is a group and $H = \{1, -i\}$.

(iii) Let $G = \{1, -1, i, -i\}$ is a group and $H = \{-1, 1\}$.

(Note : In order to verify Lagranges theorem first we have to prove H is subgroup of G).

(The following problems to be entered in the record : Q. No. (i) , (ii) and the worked problem)

Left and right coset and finding the index of a group

Definition: Let H be any subgroup of a group G and a be any element of G. Then the set,
$Ha = \{ha : h \in H\}$ is called a right coset of H in G generated by a and the set
$aH = \{ah : h \in H\}$ is called a left coset of H in G generated by a with respect to multiplicative binary operation. Similarly $H + a = \{h + a : h \in H\}$ is right coset and $a + H = \{a + h : h \in H\}$ is left coset of H with respect to additive binary operation. The cosets are also called residue classes modulo the subgroup. If e is the identity element of G, then $He = H = eH$. Hence H itself is a right as well as left coset. Since $ea \in Ha$, we have $a \in Ha$ and therefore $Ha \neq \phi$. Consequently no coset can be empty.

To find the 'index' and the distinct cosets of the subgroup H={0,4,8} of the group (Z_{12} , $+_{12}$)

```
(%i8)   kill(all)$
        Z:set(0,1,2,3,4,5,6,7,8,9,10,11);
        n:length(Z);
        H:set(0,4,8);
        m:length(H);
        Index:n/m;
        disp("Number of distinct cosets of H is",Index)$
        disp("Distinct cosets of H are :")$
        for i:1 thru Index do(HXi:cartesian_product(H,{i}),
                              Hi:makeset(mod(i+a,n),[i,a],HXi),
                              disp(Hi))$
(Z)     {0,1,2,3,4,5,6,7,8,9,10,11}
 (n)     12
 (H)     {0,4,8}
 (m)     3
 (Index) 4
 Number of distinct cosets of H is
 4
 Distinct cosets of H are :
 {1,5,9}
 {2,6,10}
 {3,7,11}
 {0,4,8}
```

Exercise

1. Find all the right cosets of the subgroup $H = \{0, 3\}$ in the group $(Z_6, +_6)$.
2. Find all the distinct cosets of $H = \{0, 3, 6\}$ in $(Z_9, +_9)$.

(The following problems to be entered in the record : Q. No. (1.) , (2.) and the worked problem)

LAB--2

Convergent, divergent and oscillatory sequences

Definitions:

1. A sequence $\{x_n\}$ is said to be convergent if the sequence tends to a finite quantity, say l.
2. A sequence $\{x_n\}$ is said to be divergent if the limit of the sequence is infinite (positive or negative).
3. A sequence $\{x_n\}$ is said to be oscillatory if the the sequence neither tends to a unique finite limit nor to $+\infty$ or $-\infty$.

Maxima programme to test the nature of the sequence $\{ x_n \}$

```
Xn : 1+1/n;

lim : limit(xn,n,inf);

if abs(lim) = inf then

print("sequence is divergent")

elseif  abs(lim)#inf  and abs(lim) # ind then print("sequence is convergent")

else

print("sequence is oscillatory")$
```

OUTPUT

(%o4) $\frac{1}{n}+1$

(%o5) 1

sequence is convergent

(The following problems to be entered in the record : Q. No. (i) , (iii) , (ix) , (xiii))

Discuss the convergence of the following sequences

(i) $\frac{(2n+3)}{(3n+4)}$ (ii) n (iii) $-n^2$ (iv) $(-1)^n$

(v) $\frac{1}{n}$ (vi) $1-\frac{1}{n}$ (vii) $\left(1+\frac{1}{n}\right)^n$ (viii) $n^{\frac{1}{n}}$

(ix) $1+(-1)^n$ (x) $\frac{(-1)^{(n-1)}}{n}$ (xi) $\frac{(n+1)^{(n+1)}}{n^n}$ (xii) $\frac{(n+(-1)^n)}{n}$

(xiii) $n[\log(n+1)-\log(n)]$ (xiv) $\frac{\log(n+1)-\log(n)}{\sin(\frac{1}{n})}$

LAB—3

Convergent, divergent and oscillatory series

Definition: Let $\sum u_n$ be a series and $\{s_n\}$ be the corresponding "sequence of partial sums". Then

(i) *the series $\sum u_n$ is convergent, if the sequence $\{s_n\}$ is convergent* i.e.,

$$\lim_{n\to\infty} s_n = l$$

.

(ii) *the series $\sum u_n$ is divergent, if the sequence $\{s_n\}$ is divergent* i.e.,

$$\lim_{n\to\infty} s_n = +\infty \quad \text{or} \quad -\infty$$

.

(iii) (a) *the series $\sum u_n$ is said to oscillate finitely, if the sequence $\{s_n\}$ oscillates finitely.*

(iii) (b) *the series $\sum u_n$ is said to oscillate infinitely, if the sequence $\{s_n\}$ oscillates infinitely.*

Maxima code to test for the convergence of the series ∑ 1/n(n+1)

```
kill(all)$

un : 1/(i*(i+1))$

u : partfrac(un,i)$

load("simplify_sum")$
```

```
s : sum (u, i, 1, n);

simp : simplify_sum(s);

seq : partfrac(simp,n);

sequence : limit(seq,n,inf);

if abs(sequence)=inf  then

disp(" The series is divergent")

elseif  abs(sequence)#inf  and  abs(sequence)#ind  then

disp(" The series is convergent")

else

disp(" The series is oscillatory")$
```

OUTPUT

(%o4) $\sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1}$

(%o5) $\frac{n}{n+1}$

(%o6) $1 - \frac{1}{n+1}$

(%o7) 1

The series is convergent

Exercise : Discuss the convergence of the following series : (i) $\sum(-1/2)^{n-1}$ (ii) $\sum (-1)^n$ (iii) $\sum n^3$

(these problems must to be entered in the record)

Tests for the convergence

D'Alembert's ratio test

Theorem: *Let $\sum u_n$ be a series of positive terms, and*

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l.$$

Then if $l < 1$, the series $\sum u_n$ is convergent and if $l > 1$, the series $\sum u_n$ is divergent.

If,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1,$$

then the ratio test fails.

Raabe's test

Theorem: *Let $\sum u_n$ be a series of positive terms and*

$$\lim_{n\to\infty} \left(\frac{u_n}{u_{n+1}} - 1\right) n = l.$$

Then if $l > 1$, the series $\sum u_n$ is convergent and if $l < 1$, the series $\sum u_n$ is divergent.

Maxima code to test for the convergence of the series $\sum 1/n^2$

(D'Alembert's ratio and Raabe's tests)

```
kill(all)$
/*D'Alembert's ratio test*/
u(n):=1/n^2;
D:limit(u(n+1)/u(n), n,inf);
if D<1 then
disp("By D'Alembert's ratio test the series is convergent")
elseif D>1 then
disp("By D'Alembert's ratio test the series is divergent")
else
disp("D'Alembert's ratio test fails and we use
Raabe's test to verify the convergence")$

/* Raabe's test*/
if D=1 then
R:limit(n*((u(n)/u(n+1))-1),n,inf);
if R>1 then
```

```
disp("By Rabee's test series is convergent")
elseif R<1 then
disp("By Rabee's test series is divergent")
else
disp("Both tests fail")$
```

OUTPUT

(%o2) 1
"D'Alembert's ratio test fails and we use Raabe's test to verify the convergence"
(%o4) 2
"By Rabee's test series is convergent"

Exercise

Discuss the convergence of the following series

(i) $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \ldots$

(ii) $\frac{2^{(n+1)}}{3^n + 1}$

(iii) $\frac{1^2.2^2}{1!} + \frac{2^2.3^2}{2!} + \frac{4^2.5^2}{4!} + \ldots$

(iv) $\frac{(n+1)!}{3^n}$

(v) $\frac{x^n}{n^2+1}$

(vi) $\frac{1}{5} + \frac{2!}{5^2} + \frac{3!}{5^3} + \ldots$

(vii) $\frac{5^n}{2^n+5}$

(viii) $1 + \frac{2p}{2!} + \frac{3p}{3!} + \frac{4p}{4!} + \ldots$

(ix) $\frac{2^n}{n^3}$

(x) $\frac{2^n.n!}{n^n}$

(xi) $\frac{n^2}{2^n}$

(The following problems to be entered in the record : Q. No. (iii) , (vi) , (x) , (xi))

LAB—4 : The sum of the series

Maxima code to find sum to infinity of the series

$$\sum \frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + \cdots$$

Maxima code::

```
kill(all)$
load("simplify_sum")$
u(k):=k^3/factorial(k)$
S:sum(u(k),k,1,inf)$
print("The given series is:",S)$
S1:simplify_sum(S)$
print("Sum to infinity of the series is:",S1)$
```

OUTPUT

The given series is: $\displaystyle\sum_{k=1}^{\infty} \frac{k^3}{k!}$

Sum to infinity of the series is: 5 %e

Exercise

Find sum to infinity of the following series

(i) $1 + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \ldots$

(ii) $\sum_{n=1}^{n=\infty} \frac{(n+1)^3}{n!} x^n$

(iii) $\frac{5}{1!} + \frac{7}{3!} + \frac{9}{5!} + \ldots$

(iv) $\sum_{n=1}^{n=\infty} \left(\frac{n^2+n+1}{n!} \right) x^n$

(v) $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots$

(The following problems to be entered in the record : Q. No. (i) , (iii) , (v) and the worked example)

LAB—5 Continuity of a function

Definition:

1. A function $f(x)$ defined in a neighbourhood of a point $'a'$ and also at $'a'$ is said to be continuous at $x = a$, if $\lim_{x \to a} f(x) = f(a)$.

2. A function $f(x)$ is said to be continuous at $x = a$, if for every $\epsilon > 0$, there exists a real number $\delta > 0$, such that,

$$|f(x) - f(a)| < \epsilon, \text{ whenever } |x - a| < \delta.$$

(Note : Here ϵ is very small positive number which quantifies accuracy. It can be taken as $10^{-2}, 10^{-3}$ etc.)

3. The continuity of a function $f(x)$ at the end points of the closed interval $[a, b]$ is defined as

   (i) $f(x)$ is continuous at $x = a$, if $\lim_{x \to a^+} f(x) = f(a)$.

   (ii) $f(x)$ is continuous at $x = b$, if $\lim_{x \to b^-} f(x) = f(b)$.

1. Discuss continuity of the function : $f(x) = \dfrac{x^2 - 9}{x - 3}$ at $x = 3.$ **given that f(3) = 6**

kill(all)$

a : 3$

fa : 6$

f(x) := (x^2-9)/(x-3);

LHL : limit(f(x),x,a,plus);

RHL : limit(f(x),x,a,minus);

If LHL=RHL and LHL=fa then

print("Given function is continuous at","x=",a)

else

print("One of the conditions of continuity fails

hence the given function is not continuous at","x=",a)$

wxplot2d(f,[x,-5,5]);

OUTPUT

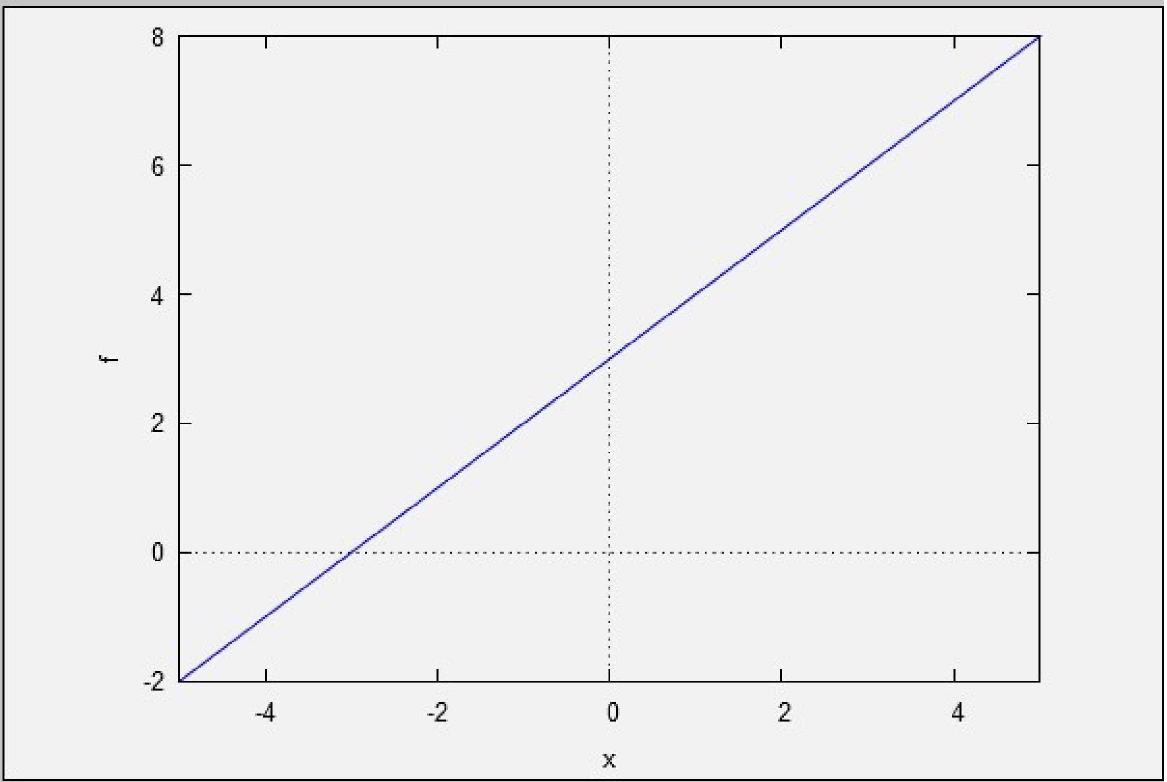
(%o3) $f(x) := \dfrac{x^2-9}{x-3}$

(%o4) 6

(%o5) 6

Given function is continuous at $x = 3$

(%t7)



(%o7)

2. Discuss continuity of the function : $f(x) = \begin{cases} 3 - x^2 & x < -2 \\ 0 & x = -2 \\ 11 - x^2 & x > 2 \end{cases}$ **at x = - 2**

kill(all)$

a:-2$

fa:0$

```
f(x):=3-x^2;

g(x):=11-x^2;

LHL:limit(f(x),x,a,plus);

RHL:limit(g(x),x,a,minus);

if LHL=RHL and LHL=fa then

print("Given function is continuous at","x=",a)

else

print("One of the conditions of continuity fails

hence the given function is not continuous at","x=",a)$

wxplot2d([f,g],[x,-5,5],[y,-10,10]);
```

OUTPUT

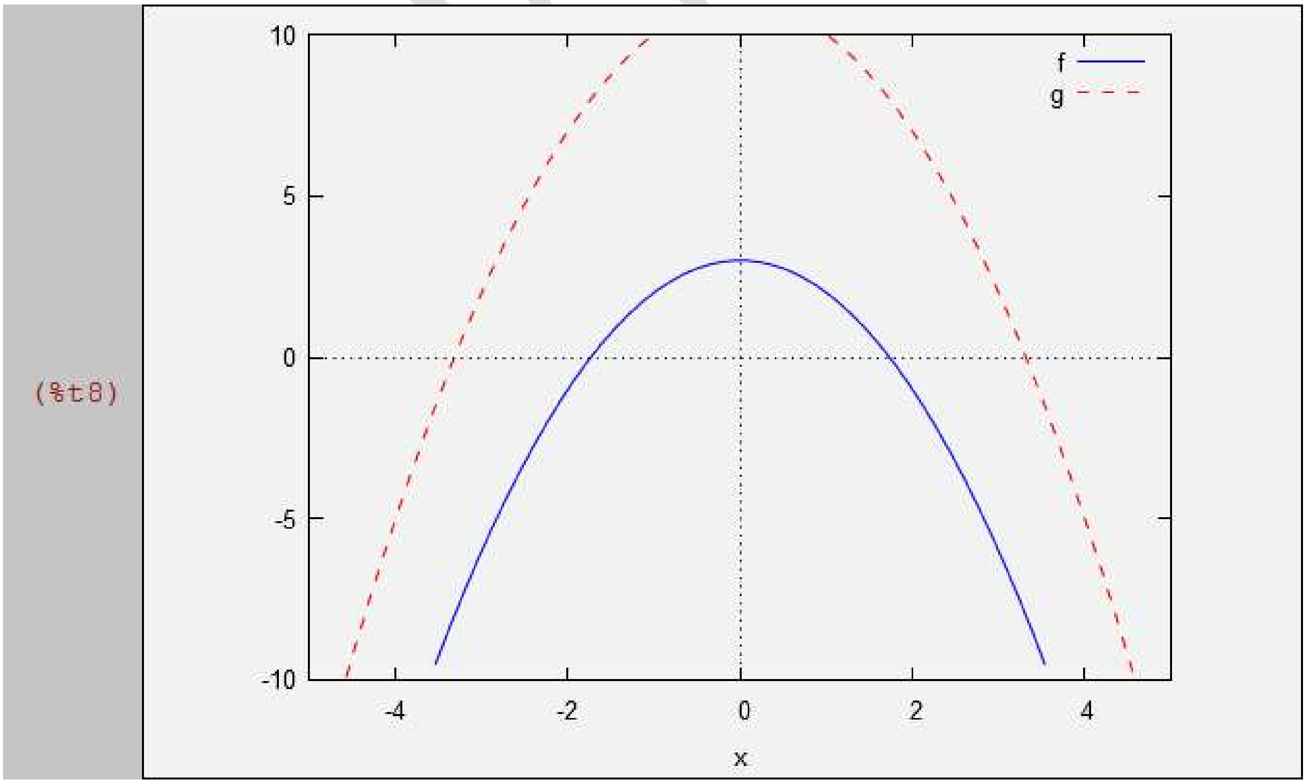
(%o3) $f(x):=3-x^2$

(%o4) $g(x):=11-x^2$

(%o5) -1

(%o6) 7

One of the conditions of continuity fails hence the given function is not continuous at $x=-2$



Note that the two curves do not meet at x = -2 and hence the given function is not continuous at x = -2.

Exercise

Discuss the continuity of the following functions

(iii) $f(x) = \begin{cases} x^2+3 & x \le 1 \\ x+1 & x > 1 \end{cases}$ at $x = 1$.   (iv) $f(x) = \begin{cases} 3x-2 & x < 1 \\ 4x^2-3x & x > 1 \end{cases}$ at $x = 1, 2, -3$.

(The following problems to be entered in the record : Q. No. (iii) and the worked problems (totally three))

LAB—6 : Differentiability of a function

Definition: Let $f(x)$ be a function defined in a domain $D \subset R$ and 'x_0' be any point in D. Then

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

if it exists, is called the derivative of $f(x)$ at $x = x_0$. The derivative of $f(x)$ at $x = x_0$ is denoted by $f'(x_0)$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

if the latter limit exists.

1. Examine the differentiability at x = 1 for the function

$$f(x) = \begin{cases} x^2-1 & x \ge 1 \\ 1-x & x < 1 \end{cases}$$

Maxima code

```
kill(all)$

x0 : 1$

f1(x) := 1-x;

f2(x) := x^2-1;

LHL : limit(ratsimp((f1(x)-f1(x0))/(x-x0)), x, x0, minus);

RHL : limit(ratsimp((f2(x)-f2(x0))/(x-x0)), x, x0, plus);
```

if LHL = RHL then

print("Given function is differentiable at", "x=", a)

else

print("Given function is not differentiable at", "x=",a)$

OUTPUT

```
(%o2) f1(x):=1-x
(%o3) f2(x):=x^2-1
(%o4) -1
(%o5) 2
Given function is not differentiable at x= a
```

Exercise

Examine the differentiability of the following functions using Maxima

(i) $f(x)=\begin{cases} x^2+3 & x\geq 1 \\ 1-x & x<1 \end{cases}$ at $x=1$. (ii) $f(x)=\begin{cases} 6x-9 & x>3 \\ x^2 & x\leq 3 \end{cases}$ at $x=3$.

(iii) $f(x)=\begin{cases} x^2\sin\dfrac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$ at $x=0$ (iv) $f(x)=\begin{cases} 1-2x & -1\leq x\leq 0 \\ 1-3x & 0<x\leq 1 \\ x-3 & 1\leq x\leq 2 \end{cases}$ at $x=0,1$.

(v) $f(x)=\begin{cases} 1+x & x<2 \\ 0 & x\geq 2 \end{cases}$ at $x=2$ (vi) $f(x)=\begin{cases} 1-a & x>a \\ a-x & x<2 \\ 0 & x=0 \end{cases}$ at $x=0$.

(The following problems to be entered in the record : Q. No. (ii) , (v) and the worked problem

(totally three))

LAB—7 Rolle's theorem

If $f(x)$ is a function defined on $[a, b]$, such that,

(i) $f(x)$ is continuous on $[a, b]$,

(ii) $f(x)$ is differentiable on (a, b),

(iii) $f(a) = f(b)$,

then there exists at least one value of $x = c$, such that $a < c < b$, for which $f'(c) = 0$.

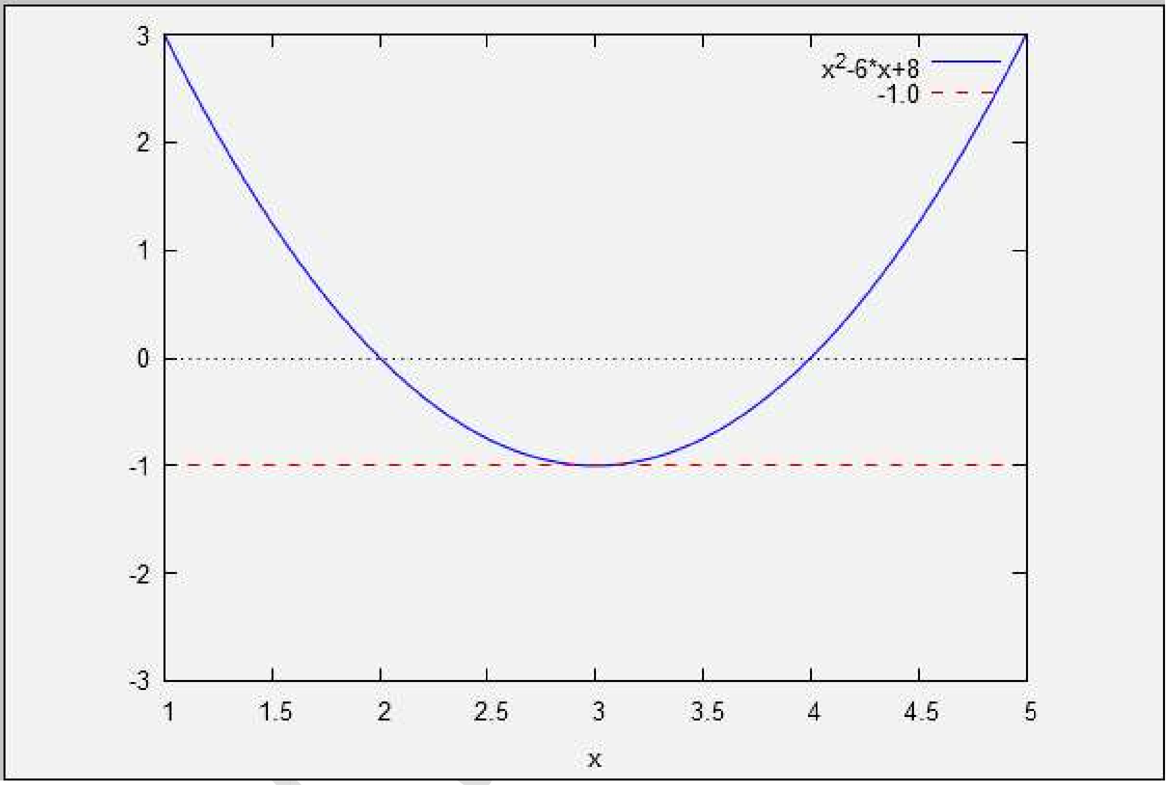
Maxima code to verify the Rolle's theorem for the function $f(x) = x^2 - 6x + 8$ in the interval $[2, 4]$

```
kill(all)$

a:2 ;  b:4 ;

f(x):=x^2-6*x+8;

l:f(a);  m:f(b);

df:diff(f(x),x)$

c:find_root(df,x,a,b);

t(x):=f(c)$           (Equation of tangent at  (c,f(c) )

wxplot2d([f(x),t(x)],[x,a-1,b+1],[y,-3,3]);
```

Note : If f(x) satisfies all conditions of Rolle's theorem then the tangent at (c, f(c)) to the curve y=f(x) is parallel to the X-axis . Red line is theTangent .

OUTPUT

```
(%o1) 2
(%o2) 4
(%o3) f(x):=8-6 x+x^2
(%o4) 0
(%o5) 0
(%o7) 3.0
(%t9)
```



2. Verfiy Rolle's theorem for e^x in the interval [0,π]

```
  kill(all)$
a:0 ; b:%pi ;
f(x):=%e^x;
l:f(a); m:f(b);
df:diff(f(x),x);
c:find_root(df,x,a,b);
t(x):=f(c);
wxplot2d([f(x),t(x)],[x,a-1,b+1],[y,-3,3]);
```

OUTPUT

```
(%i7)

(%o1) 0
 (%o2) π
 (%o3) f(x):=%e^x
 (%o4) 1
 (%o5) %e^π
 (%o6) %e^x
find_root: function has same sign at endpoints: f(0.0)=1.0, f(3.141592653589793)
23.14069263277926
 -- an error. To debug this try: debugmode(true);
```

Since f(0) ≠ f(π) , Rolle's theorem <u>is not satisfied</u> by e^x in [0,π]

3. Verfiy Rolle's theorem for log((x^2+3)/4x) in the interval [1,3]

kill(all)$

a:1 ; b:3 ;

f(x):=log((x^2+3)/(4*x));

l:f(a) ; m:f(b);

df:diff(f(x),x)$

c:find_root(df,x,a,b);

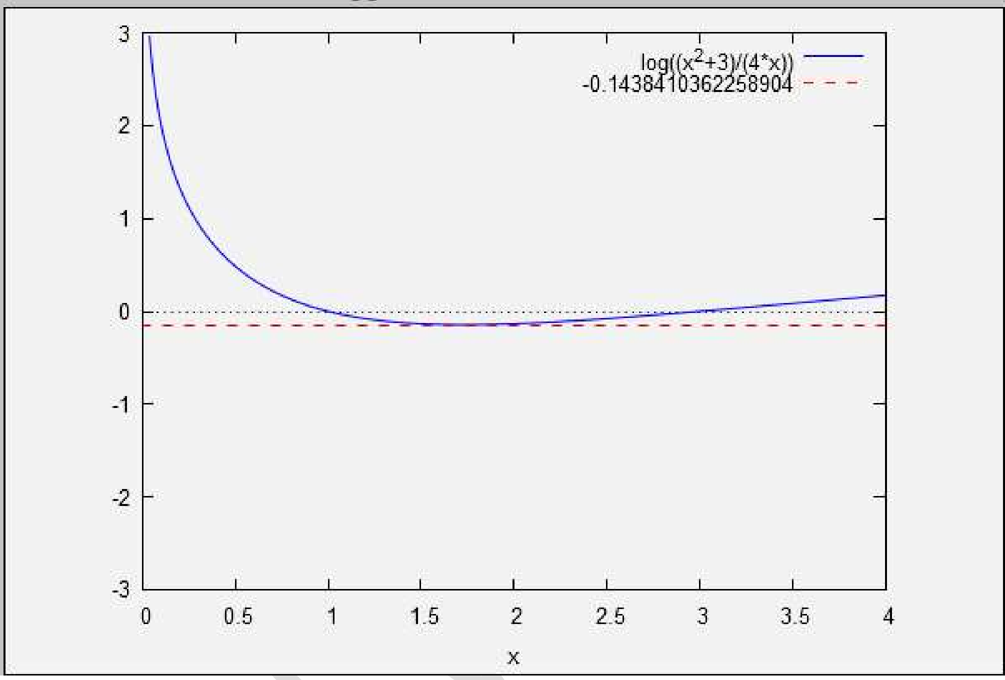
t(x):=f(c);

wxplot2d([f(x),t(x)],[x,a-1,b+1],[y,-3,3]);

OUTPUT

```
(%o1) 1
(%o2) 3
(%o3) f(x):=log((3+x^2)/(4 x))
(%o4) 0
(%o5) 0
(%o7) 1.732050807568877
(%o8) t(x):=f(c)
plot2d: expression evaluates to non-numeric value somewhere in plotting range.
plot2d: some values were clipped.
```

(%t9)



4. Verfiy Rolle's theorem for sin(x)/e^x in the interval [0,π].

kill(all)$

a:0 ; b:%pi ;

f(x):=sin(x)/%e^x;

l:f(a); m:f(b);

df:diff(f(x),x)$

c:find_root(df,x,a,b);

t(x):=f(c);

wxplot2d([f(x),t(x)],[x,a-1,b+1],[y,-3,3]);

OUTPUT

(%o1) 0

(%o2) π

(%o3) $f(x) := \frac{\sin(x)}{\%e^{x}}$

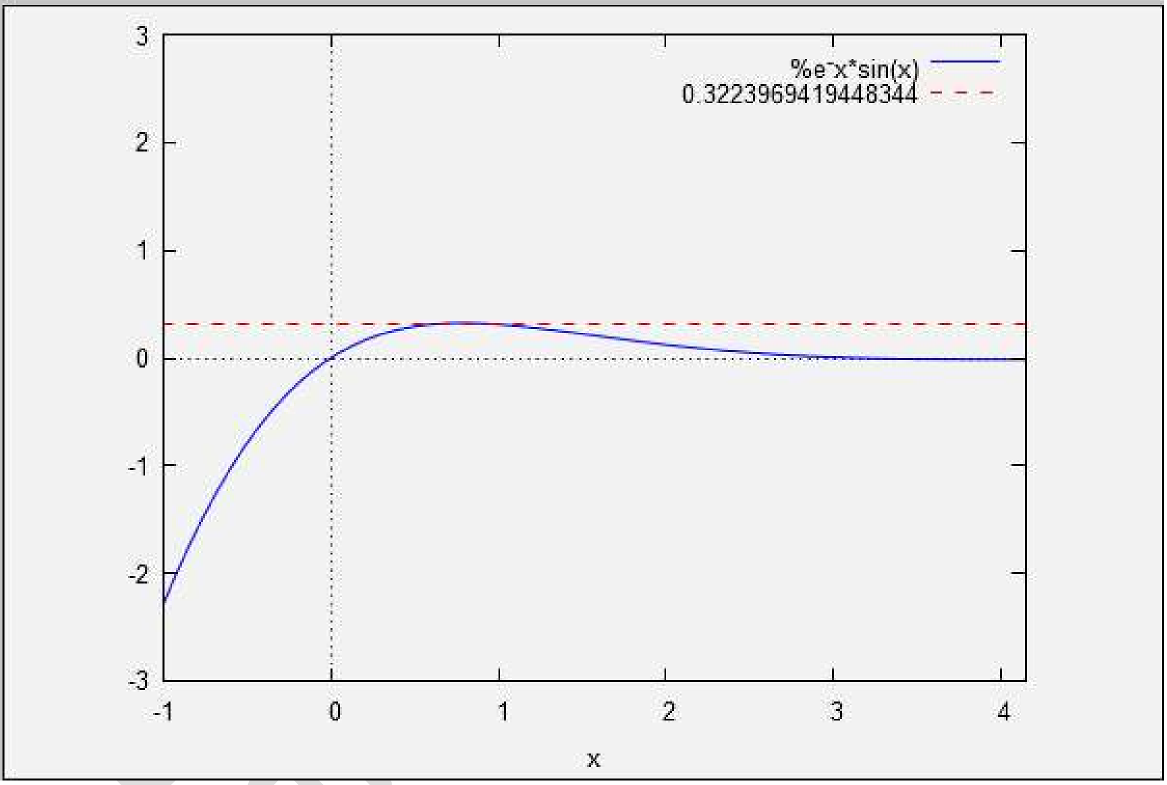
(%o4) 0

(%o5) 0

(%o7) 0.7853981633974483

(%o8) t(x):=f(c)

(%t9)



Exercise

Verify Rolle's theorem for the following functions

(i) $f(x) = e^x$ in the interval $[0, \pi]$

(ii) $f(x) = 8x - x^2$ in the interval $[2, 6]$

(iii) $f(x) = \log\left(\frac{x^2+3}{4x}\right)$ in the interval $[1, 3]$

(iv) $f(x) = x(x-3)^2$ in the interval $[0, 3]$

(v) $f(x) = \frac{\sin(x)}{e^x}$ in the interval $[0, \pi]$

(vi) $e^x(\sin(x) - \cos(x))$ in the interval $\left[\frac{pi}{4}, \frac{5\pi}{4}\right]$

(vii) $x^3 - 3x^2 - x + 3$ in the interval $[1, 3]$

(The following problems to be entered in the record : The 4 worked problems)

LAB—8 Lagrange's mean value theorem

If a function $f(x)$

(i) is continuous on $[a, b]$,

(ii) is differentiable on (a, b),

then there exists atleast one value $c \in (a, b)$ such that,

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

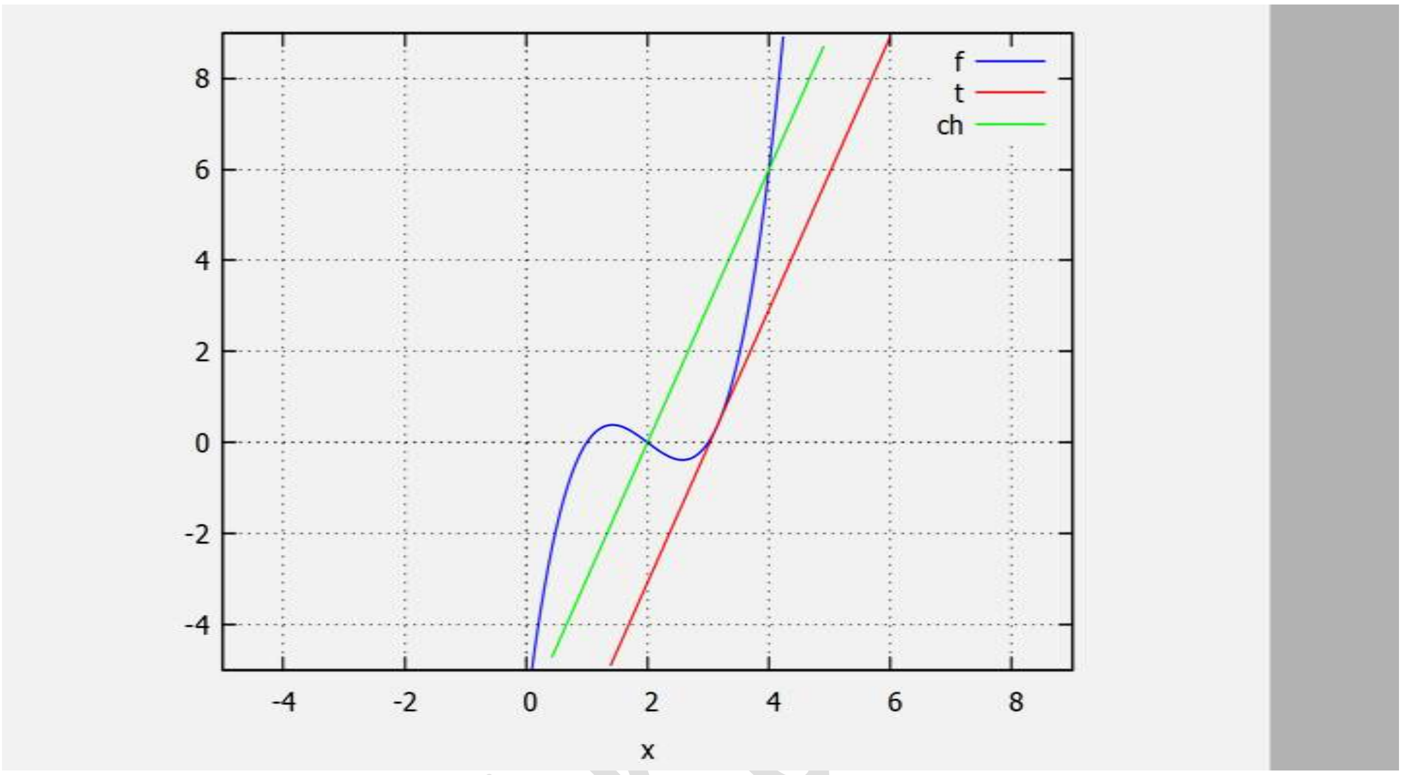
1. Verify Lagrange's Mean value theorem for f(x)= (x-1)(x-2)(x-3) in interval [0,4]

```
kill(all);

a:0;  b:4;

f(x):=(x-1)*(x-2)*(x-3);

p:(f(b)-f(a))/(b-a);          > slope of the chord

df:diff(f(x),x);

c:find_root(df-p, a+1,b);

c1:ev(df , x=c);              .>  slope of the tangent

ch(x):=p*(x-a)+f(a);        > Equation of chord joining   (a , f(a))  and  (b , f(b))

t(x):=c1*(x-c)+f(c);        >Equation of tangent at (c, f(c))

if a<c and c<b then print("Lagrange's Mean value theorem is satisfied by f(x)")

else Print("Lagrange's Mean value theorem is NOT satisfied by f(x)")$

wxplot2d([f,t,ch],[x, a-5, b+5],[y, a-5, b+5],[z, a-5, b+5]);    ( or we can also use  plot2d  )
```

OUTPUT

```
(%o1) 0

(%o2) 4

(%o3) f(x):=(x-1)*(x-2)*(x-3)

(%o4) 3

(%o5) (x-2)*(x-1)+(x-3)*(x-1)+(x-3)*(x-2)

(%o6) 3.154700538379251

(%o7) 3.0

(%o8) ch(x):=f(a)+p*(x-a)

(%o9) t(x):=f(c)+c1*(x-c)

"Lagrange's Mean value theorem is satisfied by f(x)"
```

(Note:: If f(x) satisfies Lagrange's Mean value theorem in the interval [a,b] then there exists a point c such that the tangent to the curve y=f(x) at (c,f(c)) is parallel to the chord joining the points (a,f(a)) and (b,f(b)) on the curve)



2. Verify Lagrange's Mean value theorem for f(x) = x(x-1)(x-2) in the interval [0, 0.5]

```
kill(all);

a:0;  b:0.5;

f(x):=x*(x-1)*(x-2);

p:(f(b)-f(a))/(b-a);            > slope of the chord

df:diff(f(x),x);

c:find_root(df-p, a ,b);

c1:ev(df , x=c);              .>  slope of the tangent

ch(x):=p*(x-a)+f(a);       > Equation of chord joininig   (a , f(a))  and  (b , f(b))

t(x):=c1*(x-c)+f(c);       >Equation of tangent at (c,f(c))

if a<c and c<b then print("Lagrange's Mean value theorem is satisfied by f(x)")

else Print("Lagrange's Mean value theorem is NOT satisfied by f(x)")$

wxplot2d([f,t,ch],[x, a, b],[y, a,b],[z, a,b]);     ( or we can also use  plot2d )
```

OUTPUT

(%o1) 0

(%o2) 0.5

(%o3) f(x):=x*(x-1)*(x-2)

(%o4) 0.75

(%o5) (x-1)*x+(x-2)*x+(x-2)*(x-1)

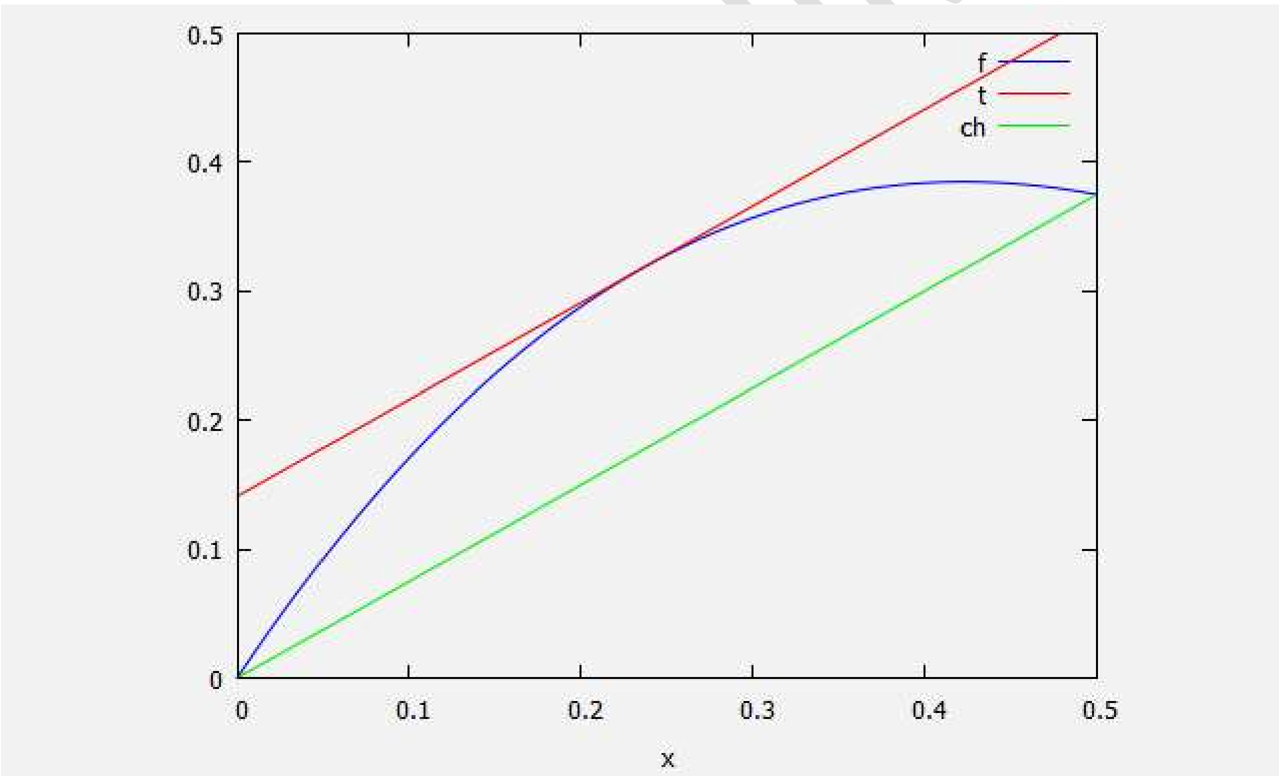
(%o6) 0.2362373841740267

(%o7) 0.7499999999999999

(%o8) ch(x):=f(a)+p*(x-a)

(%o9) t(x):=f(c)+c1*(x-c)

"Lagrange's Mean value theorem is satisfied by f(x)"



3. . Verify Lagrange's Mean value theorem for f(x) = log(x) in the interval [1, e]

kill(all);

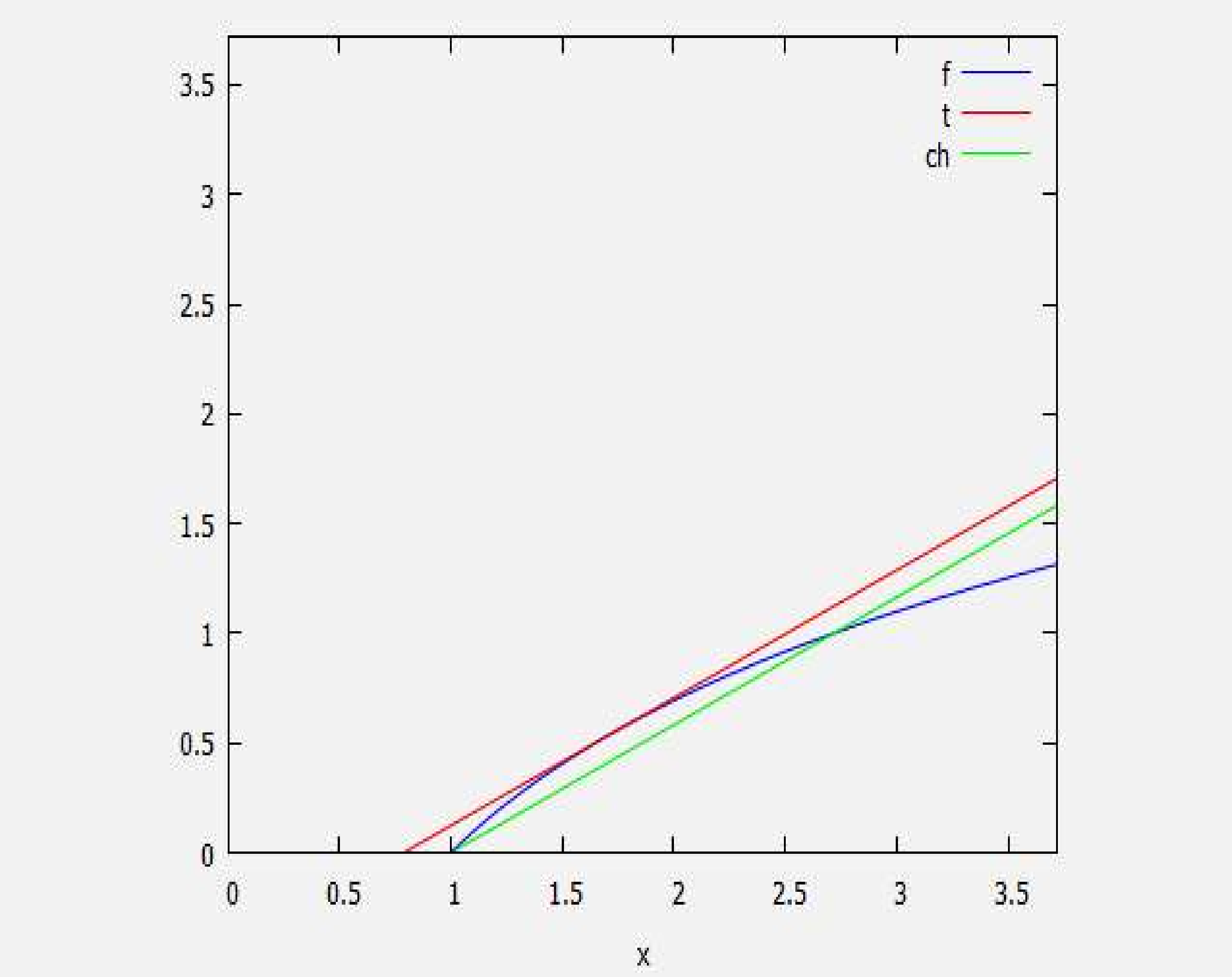
a:1;   b: %e ;

f(x):=log(x) ;

```
p:(f(b)-f(a))/(b-a);            > slope of the chord

df:diff(f(x),x);

c:find_root(df-p, a ,b);

c1:ev(df , x=c);               .>  slope of the tangent

ch(x):=p*(x-a)+f(a);        > Equation of chord joininig   (a , f(a))  and  (b , f(b))

t(x):=c1*(x-c)+f(c);        >Equation of tangent at (c,f(c))

if a<c and c<b then print("Lagrange's Mean value theorem is satisfied by f(x)")

else Print("Lagrange's Mean value theorem is NOT satisfied by f(x)")$

wxplot2d([f,t,ch],[x, a-1, b+1],[y, a-1,b+1],[z, a-1,b+1]);     ( or we can also use  plot2d )
```

OUT PUT

```
 (%o1) 1

(%o2) %e

(%o3) f(x):=log(x)

(%o4) 1/(%e-1)

(%o5) 1/x

(%o6) 1.718281828459045

(%o7) 0.5819767068693265

(%o8) ch(x):=f(a)+p*(x-a)

(%o9) t(x):=f(c)+c1*(x-c)

"Lagrange's Mean value theorem is satisfied by f(x)"
```



Exercise

Verify Lagrange's mean value theorem for the following functions

(i) $f(x) = x(x-1)(x-2)$ in $\left[0, \frac{1}{2}\right]$

(ii) $f(x) = x^2 - 3 * x - 1$ in $\left[\frac{-1}{7}, \frac{13}{7}\right]$

(iii) $f(x) = sqrt(25 - x^2)$ in $[-3, 4]$

(iv) $f(x) = \log x$ in $[1, e]$

(The following problems to be entered in the record : The 3 worked problems and Q.No. (ii) above.)

LAB—9 Cauchy’s mean value theorem

Definition: If two functions $f(x)$ and $g(x)$ are such that

(i) both $f(x)$ and $g(x)$ are continuous in $[a, b]$

(ii) both $f(x)$ and $g(x)$ are differentiable in (a, b)

(iii) $g'(x) \neq 0$ any where in (a, b), then there exists at least one point $c \in (a, b)$, such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

1) Verify Cauchy’s Mean value theorem for f(x)=log(x) , g(x)=1/x in [1 , e]

```
kill(all)$
a:1; b: %e;
f(x):=log(x);
g(x):=1/x;
p:(f(b)-f(a))/(g(b)-g(a));
df:diff(f(x),x);
dg:diff(g(x),x);
c:find_root((df/dg)-p,a,b)$
disp("value of c = ",c)$
if a<c and c<b then print("Cauchy's Mean value theorem is satisfied by f(x)& g(x)")
else Print("Cauchy's Mean value theorem is NOT satisfied by f(x)& g(x)")$
```

OUTPUT

(%o1) 1

(%o2) %e

(%o3) $f(x):=\log(x)$

(%o4) $g(x):=\frac{1}{x}$

(%o5) $\frac{1}{\%e^{-1}-1}$

(%o6) $\frac{1}{x}$

(%o7) $-\frac{1}{x^2}$

value of c =

1.581976706869326

Cauchy's Mean value theorem is satisfied by f(x)& g(x)

2) Verify Cauchy's Mean value theorem for f(x)=$\sqrt{x}$, g(x)=1/ $\sqrt{x}$ in [1 , 2]

```
kill(all)$
a:1;
b:2;
f(x):=sqrt(x);
g(x):=1/sqrt(x);
p:(f(b)-f(a))/(g(b)-g(a));
df:diff(f(x),x);
dg:diff(g(x),x);
c:find_root((df/dg)-p,a,b)$
disp("value of c = ",c)$
if a<c and c<b then print("Cauchy's Mean value theorem is satisfied by f(x)& g(x)")
else Print("Cauchy's Mean value theorem is NOT satisfied by f(x)& g(x)")$
```

OUTPUT

(%o1) 1

(%o2) 2

(%o3) $f(x):=\sqrt{x}$

(%o4) $g(x):=\frac{1}{\sqrt{x}}$

(%o5) $\frac{\sqrt{2}-1}{\frac{1}{\sqrt{2}}-1}$

(%o6) $\frac{1}{2\sqrt{x}}$

(%o7) $-\frac{1}{2\,x^{3/2}}$

value of c =

1.414213562373095

Cauchy's Mean value theorem is satisfied by f(x)& g(x)

3) Verify Cauchy's Mean value theorem for $f(x)=x^3$, $g(x)=x^2$ in [1 , 3]

```
kill(all)$

a:1;

b:3;

f(x):=x^3;

g(x):=x^2;

p:(f(b)-f(a))/(g(b)-g(a));

df:diff(f(x),x);

dg:diff(g(x),x);

c:find_root((df/dg)-p,a,b)$

disp("value of c = ",c)$

if a<c and c<b then print("Cauchy's Mean value theorem is satisfied by f(x)& g(x)")

else Print("Cauchy's Mean value theorem is NOT satisfied by f(x)& g(x)")
```

OUTPUT

```
(%o1) 1
(%o2) 3
(%o3) f(x):=x^3
(%o4) g(x):=x^2
(%o5) 13/4
(%o6) 3 x^2
(%o7) 2 x
value of c =
2.166666666666667
Cauchy's Mean value theorem is satisfied by f(x)& g(x)
```

(The following problems to be entered in the record : The 3 problems worked above.)

LAB—10 Taylor's theorem

Definition Let $f(x)$ be a function defined on $[a,b]$, such

(i) $f^{n-1}(x)$ is continuous on $[a,b]$

(ii) $f^{n-1}(x)$ is derivable on (a,b)

Then there exists a real number $c \in (a,b)$, such that

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\cdots+\frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(x)+\frac{(b-a)^n}{n!}f^n(c).$$

1.Expand the function f(x) = $\log_e$(1+x) around x=1 up to the term containing x^4 by Taylor's Expansion.

Maxima code:: **taylor(log(1+x),x,1,4);**

OUTPUT

(%i10)

(%o10)/T/ $\log(2)+\frac{x-1}{2}-\frac{(x-1)^2}{8}+\frac{(x-1)^3}{24}-\frac{(x-1)^4}{64}+...$

2. Expand the function f(x) = e^x around x=1 up to the term containing x^5 by Taylor's Expansion.

Maxima code:: taylor(%e^x,x,1,5);

OUTPUT

$$(\%o15)/T/\ \%e+\%e\,(x-1)+\frac{\%e\,(x-1)^2}{2}+\frac{\%e\,(x-1)^3}{6}+\frac{\%e\,(x-1)^4}{24}+\frac{\%e\,(x-1)^5}{120}+...$$

3. Expand the function f(x) = $e^{x.\cos(x)}$ up to the term containing x^4 by Maclaurin's expansion.

Maxima code:: taylor(%e^x*cos(x),x,0,4);

OUTPUT

$$(\%o16)/T/\ 1+x-\frac{x^3}{3}-\frac{x^4}{6}+...$$

4. Expand the function f(x) = tan(x) up to the term containing x^5 by Maclaurin's expansion.

Maxima code:: taylor(tan(x),x,0,5);

OUTPUT

$$(\%o18)/T/\ x+\frac{x^3}{3}+\frac{2\,x^5}{15}+...$$

(The following problems to be entered in the record : The 4 problems worked above.)

LAB—11 Evaluation of limits by L'Hospital's rule

Let $f(x)$ and $g(x)$ be two functions defined on $[a, b]$ and satisfy the Cauchy's theorem. We know that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

provided both the limits exist and

$$\lim_{x \to a} g(x) \neq 0.$$

L'Hospital's rule:

Suppose $f(x)$ and $g(x)$ are functions satisfying the conditions

(i)

$$\lim_{x \to a} f(x) = 0$$

and

$$\lim_{x \to a} g(x) = 0,$$

(ii)

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right hand side exists. The expression $\frac{f(x)}{g(x)}$ in this case is said to assume the indeterminate form $\frac{0}{0}$ as $x \to a$.

The other simple indeterminate forms are $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^∞, 1^∞ and ∞^0.

1. Evaluate

$$\lim_{x \to 0} \frac{(x - \sin x)}{x^3}$$

(0/0 form)

Maxima code:: limit((x-sin(x))/x^3,x,0);

OUTPUT :: 1/6

2. Evaluate $$\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\tan 3x}$$ (∞/∞ form)

Maxima code:: limit(tan(x)/tan(3*x),x,%pi/2);

OUTPUT :: 3

3. Evaluate $\lim_{x \to 0} x \log \tan x$ (0x∞ form)

Maxima code :: limit(x*log(tan(x)),x,0);

OUTPUT :: 0

Exercise:: Evaluate the limits of the following functions.

(i) $(\cos x)^{\frac{1}{x^2}}$ as $x \to 0$

(ii) $(1 - x^2)^{\frac{1}{\log(1-x)}}$ as $x \to 1$

(iii) $x \tan \frac{1}{x}$ as $x \to \infty$

(iv) $\left(\frac{1}{x} \cot x\right)$ as $x \to 0$

(v) $\frac{1 - \cos x}{x \log(1+x)}$ as $x \to 0$

(vi) $\frac{x^x - x}{1 - x + \log x}$ as $x \to 0$

(vii) $\frac{\log(\theta - \pi/2)}{\tan \theta}$ as $x \to \pi/2$

(viii) $\frac{\log \tan 2x}{\log \tan x}$ as $x \to 0$

(ix) $(\sin x)^{\tan x}$ as $x \to \pi/2$

(x) $\left(\frac{1 + \cos 2x}{2}\right)^{\frac{1}{x^2}}$ as $x \to 0$

(The following problems to be entered in the record : The 3 problems worked above and Q.No.(i),(iii),(iv)).