Dynamic Algebras as a well-behaved fragment of Relation Algebras

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Abstract

The varieties **RA** of relation algebras and **DA** of dynamic algebras are similar with regard to definitional capacity, admitting essentially the same equational definitions of converse and star. They differ with regard to completeness and decidability. The **RA** definitions that are incomplete with respect to representable relation algebras, when expressed in their **DA** form are complete with respect to representable dynamic algebras. Moreover, whereas the theory of **RA** is undecidable, that of **DA** is decidable in exponential time. These results follow from representability of the free intensional dynamic algebras.

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1 Introduction

1.1 Overview

Binary relations have proved a fruitful framework in both logic and computer science. In logic they have served as the eliminator of variables [TG87], and in computer science as the illuminator of software [dBdR72, Pra76]. One finds the algebraic versions of these topics today under the respective rubrics of relation algebra [TG87] and dynamic algebra [Koz79b, Pra79a]. The nonalgebraic origins of the former lie in the subject of foundations of mathematics, and of the latter in that of logics of programs [KT89].

When the organizers of this conference very kindly asked me to talk on a subject of my choice it seemed a foregone conclusion that a conference organized by relation algebraists would expect a talk on dynamic algebras. Not having worked in this area since 1981 however, I would have preferred to talk about my more recent work on concurrent behavior. With dynamic algebras already well covered at the conference by Dexter Kozen, I hoped at the start of the meeting that this might be possible. The day before my talk I took an informal poll, whose outcome determined the topic of my talk and ultimately the unexpected results of this paper.

At the time I knew little of either the results or the history of relation algebra. I was vaguely aware that Tarski had shown the equational theory of representable relation algebras to be undecidable in the 1940's, and that the equational theory of that class was not finitely based, but I was unaware of the larger finitely axiomatized variety **RA**.

On learning more of the background of relation algebras from various helpful sources, especially George McNulty and Roger Maddux, it occurred to me that it would be a nice idea to organize this paper as a comparison of the merits of relation and dynamic algebras. It also seemed a good idea to build up these notions from Boolean monoids and Boolean modules [Bri81] respectively, with the former mingling logical and relative notions in a single sort and the latter keeping them segregated.

My initial impression was that modules improved on monoids in the areas of definitional force, completeness, representability, and decidability, while monoids had the advantages of homogeneity of sort and expressive power. Both form finitely based varieties.

In the course of making the case for these claims I learned to my surprise that the dynamic algebra definitions of converse and star were equally effective, suitably modified, in a Boolean monoid, that is, an ordered monoid whose partial order is a Boolean algebra. What had misled me about the suitability of Boolean monoids as a medium for defining converse was that every published equational axiomatization of **RA** had four equations mentioning converse, and that the equational theory of the representable relation algebras was not finitely axiomatizable, making the axiomatization appear an ad hoc attempt to deal with an impossible situation. Moreover since none of these axiomatizations mentioned star I assumed that a satisfactory axiomatization of star must be similarly out of the question. I only recently learned of the Ng-Tarski equations for star [NT77, Ng84].

As it turns out, in a Boolean monoid converse and star can each be defined with a single equation. Each equation abstracts the essence of the dynamic algebra definition of that operation. The equation for star does not mention converse (unlike the three-equation Ng-Tarski axiomatization), and vice versa.

In this account of **RA**, converse and star become siblings, as they are in **DA**. In fact, although converse appeared in the first dynamic logic paper four pages ahead of star [Pra76], the relative importance of star to programmers had caused converse to temporarily disappear from dynamic logic by the time of Segerberg's axiomatization of propositional dynamic logic [Seg77]. It was restored and straightforwardly axiomatized the following year by Parikh [Par78], whose axioms have here become our single-equation definition of **RA** converse.

The surviving advantages of **DA** remain those of decidability of the equational theory, and representability, as per Kozen [Koz80] and Brink [Bri81] but extended to star at least for the free algebras [Pra79a, Pra80a], from which follows equational completeness of **DA** with respect to the Kripke model.

I attribute these advantages to the "maintenance of a suitable distance" between the Boolean and monoidal sorts. Too far apart (complete independence of the two sorts) and one is left with a Boolean algebra and a monoid, in neither of which can either converse or star be defined equationally. **RA** represents the other extreme, in which the two sorts are identified. The Boolean module organization of **DA** keeps the sorts distinct but lets them communicate via operations diamond $\diamond : K \times B \to B$ and test $? : B \to K$. This is close enough to permit the equational definitions of star and converse, yet not so close as to compromise representability, completeness, or decidability of the associated equational theory.

For chronological completeness let me mention here the work of my group during the past several years on concurrent processes. The direction this work has taken has been heavily influenced by the insights of both dynamic logic and dynamic algebra. The passage from relation algebras to process algebras may be described as inverse abstraction (back to relations as sets of pairs), two generalizations (from pairs as labeled linearly ordered doubletons to labeled partial orders [Pra86], and from partial orders to generalized metrics [CCMP91]), and abstraction back to algebra, with the resulting logic having models far removed from binary relations, yet remaining remarkably like **RA**. The Boolean algebra is relaxed to a lattice and the monoid becomes commutative (it now represents collision instead of composition). Converse and star remain meaningful but lack some of the vitality they show in **RA** and **DA**.

1.2 Conventions

A *variety* is the class of models of an equational theory; equivalently, a class closed under homomorphisms, subalgebras, and direct products (HSP closure). A *universal Horn* formula has the form $s_1{=}t_1 \;\&\; \ldots \;\&\; s_n{=}t_n \to s{=}t$. A *quasivariety* is the class of models of a universal Horn theory; equivalently the closure $\mathrm{ISP}(K)$ of a pseudoelementary class K under isomorphisms, subalgebras, and direct products. We supply more details about "pseudoelementary" in conjunction with the examples of Boolean monoids.

Given a class C of similar algebras we denote by *type* C the similarity type to which it belongs, by Θ_C its equational theory, and by Φ_C its universal Horn theory. All theories are presumed to have infinitely many variables, for definiteness countably many.

We denote reducts to type A by superscript A. For example a dynamic algebra $\mathcal{D} = (\mathcal{B}, \mathcal{K}, \diamond, ?)$, of type **DA**, has a reduct $\mathcal{B} = (B, \vee, 0, ^-)$ of type $\mathbf{DA}^B$, and another $\mathcal{K} = (K, 0, +, ;, ^{\smile}, ^*)$, of type $\mathbf{DA}^K$. The equational theory Θ_{RA} of relation algebras has a reduct Θ_{RA}^{BM} to Boolean monoids, defined as those theorems of **RA** that are in the language of **BM**, i.e. that omit converse. Thus Θ_A must by definition be a conservative extension of Θ_A^B, whence to say that Θ_{RA}, as an extension of Θ_{BM}, is not conservative over Θ_{BM} is to say that Θ_{RA}^{BM} strictly includes Θ_{BM}.

The equational theory Θ_{DA} of the two-sorted class **DA** of dynamic algebras partitions as $\Theta_{DA} = \Theta_{DA}^B + \Theta_{DA}^K$ consisting respectively of equations between Boolean terms and between Kleenean terms. On the other hand, although Φ_{DA} has reducts Φ_{DA}^B and Φ_{DA}^K, these do not exhaust Φ_{DA}, which may have equations of both sorts in the one formula. The sets V^B and V^K of variables of each sort are disjoint. We reserve the sort names themselves for the underlying sets of each sort, assumed disjoint; thus the set of individuals of a dynamic algebra $\mathcal{D}$ partitions as $D = B + K$.

1.3 Logical and Relative

The two theories of this paper, those of relation algebras and dynamic algebras, arise out of the following dilemma.

An n-ary relation on a set X is a subset of X^n. If p is a unary relation we write $x \in p$ as $p(x)$, while for a binary relation a we write $(x, y) \in a$ as xay.

That a relation is a set imbues it with a *logical* character: for any n the set 2^{X^n} of all n-ary relations on X forms a Boolean algebra in the usual way. That pairs can be linked, xay and ybz forming $xa;bz$, and reversed, xay becoming $ya^{\smile}x$, confers on binary relations their *relative* character: the binary relations on X form a monoid under composition or relative product $a;b$ with identity 1', with the additional operations of converse, $a^{\smile}$, and star, a^*.

A calculus with both a logical and a relative character would seem very useful for both the foundations of mathematics [TG87] and logics of programs [KT89]. Since relations exhibit both characteristics they should form an excellent basis for such a calculus.

It is clear that unary relations cannot supply the relative part. However either unary or binary relations can supply the logical part. Depending on which way we resolve this dilemma we obtain one of two finitely axiomatized varieties, Boolean monoids or Boolean modules [Bri81]. Each may be equipped independently with converse and star, each definable with a single equation in each case. A Boolean monoid equipped with converse is called a relation algebra, and one equipped with converse and star is called an **RAT** [NT77, Ng84], a relation algebra with transitive closure. Brink [Bri81] defines Boolean modules to be equipped with converse. A Boolean module with converse and star is a dynamic algebra [Koz79a, Pra79a].

2 Boolean monoids

As pointed out by Brink [Bri81], Boolean monoids predate Boolean modules by a decade, 1860 (De Morgan) vs. 1870 (Peirce), so it is appropriate that we consider them first. They achieve economy of concept through homogeneity of data: a single sort as opposed to the two sorts of Boolean modules. Our purpose in treating Boolean monoids in this paper is two-fold: to prime the reader already acquainted with **RA** for the perspective from which we shall view **DA**; and to embellish the standard **RA** story with some details, some imported from **DA**, some resulting from extensive discussions with Roger Maddux and George McNulty, and some just filling gaps.

An *ordered monoid* $\mathcal{A} = (A, \leq, ;)$ is a set A which is both a partial order $(A, \leq)$ and a monoid $(A, ;)$ whose *composition* or *relative product* $a;b$ is monotone with respect to $\leq$ in each argument. We denote the unit of the monoid by 1'. A *Boolean monoid* is an ordered monoid whose partial order is a Boolean algebra. We denote join and complement by $a+b$ and a^- respectively. To make the class of Boolean monoids a variety we treat a Boolean monoid as an algebra $(A, +, ^-, ;, 1')$. The class **BM** of Boolean monoids is then definable by finitely many equations.

We take as logical abbreviations $a \leq b$ for $a+b=b$, $a \leq b \leq c$ for $a+b=bc$, ab for $(a^- + b^-)^-$, 1 for $1' + 1'^-$, 0 for 1^-, and $a \to b$ for $a^- + b$. For relative abbreviations we take $a \mathbin{+\!\!\!\!\jmath}\, b$ for $(a^-;b^-)^-$ (relative sum), 0' for $1'^-$, $a \hookrightarrow b$ for $a^- \mathbin{+\!\!\!\!\jmath}\, b$ (relative implication), and $a \hookleftarrow b$ for $a \mathbin{+\!\!\!\!\jmath}\, b^-$ (relative coimplication).[1]

Example 1. The motivating example of a Boolean monoid is the set of all binary relations on a *base set* X, under the usual composition of binary relations. We refer to this algebra as **BM** X. **BM** X is a *simple* algebra, one with only two congruences, the identity and the clique. We refer to those Boolean monoids isomorphic to a subalgebra of **BM** X for some X as the *simple representable Boolean monoids*, forming the class **SRBM**.

A *pseudoelementary class* is a reduct of an elementary class [Mak64, Ekl77, 4.3]. That is, it is obtainable by omitting some of the sorts and operations of some class definable with a first order theory.

Proposition 1 **SRBM** *is pseudoelementary.*

Proof: Let X and R be the sorts of the following two-sorted first-order theory of the set R of all binary relations over a set X. The language of this theory has all the Boolean monoid operations, as operations on R, together with a ternary relation (x, a, y) expressing that elements x and y of X are related by the binary relation a, an element of R. We require that if $a \neq b$ then there exist x, y such that exactly one of (x, a, y) and (x, b, y) hold; this ensures that each element of R acts

[1]De Morgan [DM60] writes ab, ab', and $a{,}b$ for $a;b$, $a \hookleftarrow b$, and $a \hookrightarrow b$ respectively, construing them as "an a of a b of", "an a of every b of", and "an a of none but b's of," and asserting their sufficiency. Peirce [Pei33, 3.242,1880] observed their interdefinability, notating them ab, a^b, ${}^a b$ and adjoining a fourth connective for relative sum, which he subsequently notated $a \dagger b$; the notation $a \mathbin{+\!\!\!\!\jmath}\, b$ we use here is due to Schröder [Sch95], subsequently (1897) adopted in modified form (the "scorpion tail") by Peirce. For $0, 1, 0', 1'$ Peirce writes $0, \infty, \mathbf{n}, 1$ respectively, and calls relations $a \leq 1'$, $a \geq 0'$, $a \geq 1'$, and $a \leq 0'$ respectively *concurrent*, *opponent*, *self-relative*, and *alio-relative*.

like a distinct binary relation, i.e. a distinct subset of X^2. It is easy to write down axioms that say that R includes the identity relation (there exists a such that for all x, y, (x, a, y) if and only if $x = y$) and the empty relation (there exists a such that for all x, y, $\neg(x, a, y)$), and is closed under the Boolean operations (with complement being relative to X^2) as well as composition. Now by "forgetting" the set X and the ternary relation (x, a, y) we are left with a one-sorted relation algebra isomorphic to a subalgebra of $\mathbf{BM}\ X$, that is, a simple representable Boolean monoid. ∎

Corollary 2 *ISP(***SRBM***) is a quasivariety.*

Proof: Every pseudoelementary class closed under isomorphisms, subalgebras, and direct products is a quasivariety, and these operations preserve the property of being pseudoelementary. ∎

I am indebted to George McNulty for the idea of using pseudoelementary classes in this argument.

We call the members of ISP(**SRBM**) *representable Boolean monoids*, forming the class **RBM**. Thus up to isomorphism the representable Boolean monoids are formed as subalgebras of direct products of simple representable ones. The top element 1 of the direct product of simple representable Boolean monoids amounts to an equivalence relation on the disjoint union of the base sets of those Boolean monoids, namely the relation $x \equiv y$ which holds just for those x, y coming from the same base set.

Example 2. In any Boolean algebra, take composition to be meet, and hence relative sum to be join, called a *cartesian* Boolean monoid since it is also a cartesian closed category [Mac71, IV.6-1(b)]. The same concept is termed "Boolean" in the relation algebra literature, e.g. Jónsson [Jón82], which presents the obvious conflict with the present terminology.

A Boolean monoid is called *normal* when it satisfies $a; 0 = 0 = 0; a$, and *additive* when it satisfies $a; (b + c) = a; b + a; c$ and $(a + b); c = a; c + b; c$. We shall call it *Peircean* when it satisfies $(a \mathbin{\underset{\smile}{+}} b); c \leq a \mathbin{\underset{\smile}{+}} (b; c)$ and $a; (b \mathbin{\underset{\smile}{+}} c) \leq (a; b) \mathbin{\underset{\smile}{+}} c$ [Pei33, 3.334]. Examples 1 and 2 enjoy all three of these properties.

Example 3. The *dual* of a Boolean monoid is obtained by exchanging relative product and relative sum, itself a Boolean monoid.

The duals of examples 1 and 2 are not in general normal, additive, or Peircean.

Example 4. The set of all subsets of any equivalence relation E on a set X is closed under the usual composition of binary relations and hence forms a Boolean monoid. We have already encountered these above as the representable Boolean monoids up to isomorphism.

When E is the identity relation on X the resulting representable Boolean monoid can be seen to be cartesian. Conversely every cartesian Boolean monoid is so representable via the Stone embedding of a Boolean algebra into a power set and the embedding of X in X^2 as the latter's diagonal.

We now treat the operations of converse and star in turn.

2.1 Residuation

An ordered monoid is called a *residuated order* when it has operations $a\backslash c$ and c/b defined as

$$a;b \leq c \quad \leftrightarrow \quad b \leq a\backslash c \tag{RKR}$$

$$a;b \leq c \quad \leftrightarrow \quad a \leq c/b \tag{LKL}$$

These operations are called respectively the right and left *residuals* of c *over* a, b respectively [WD39, Dil39, Fuc63, Bir67, Jón82]. They may also be viewed as yet another pair of implications, bringing the number of implications we have now encountered to five, namely $a{\rightarrow}b$, $a{\hookrightarrow}b$, $a{\hookleftarrow}b$, $a\backslash b$, and a/b. In the case of commutative monoids, where $a;b = b;a$, this reduces via $a{\hookrightarrow}b = b{\hookleftarrow}a$ and $a\backslash b = b/a$ to just three implications.

We will find it convenient later on to break down (RKR) and (LKL) into four universal Horn formulas,

$$a;b \leq c \quad \rightarrow \quad b \leq a\backslash c \tag{KR}$$

$$b \leq a\backslash c \quad \rightarrow \quad a;b \leq c \tag{RK}$$

$$a;b \leq c \quad \rightarrow \quad a \leq c/b \tag{KL}$$

$$a \leq c/b \quad \rightarrow \quad a;b \leq c. \tag{LK}$$

The letter K in the names of these formulas connotes the theorem De Morgan refers to as Theorem K [DM60]. This theorem, which was brought to my attention by Roger Maddux, amounts to the assertion of KR and KL; RK and LK can be derived from these given $a^{\smile\smile} = a$. The L and R refer to the left and right residuals respectively, and their placement relative to K indicates the direction of the implication.

Residuated orders enjoy a number of useful properties [Bir67, Theorem XIV-4], all of which are easily proved. In particular $a;b$ preserves arbitrary sups in each argument, e.g. if the empty sup or least element 0 exists then $a;0 = 0 = 0;a$, and if the sup $a+b$ of a and b exists then $(a+b);c = a;c+b;c$ and $c;(a+b) = c;a+c;b$. Residuation of a over b (on either side) is monotone in a and antimonotone in b. Furthermore $a\backslash b$ (and likewise b/a) preserves arbitrary infs in the b argument, e.g. $a\backslash 1 = 1$ and $a\backslash(bc) = (a\backslash b)(a\backslash c)$. It also "antipreserves" arbitrary sups in the a argument in that it maps them to the corresponding infs, e.g. $0\backslash a = 1$ and $(a+b)\backslash c = (a\backslash c)(b\backslash c)$. And residuation is axiomatizable inequationally, namely via

$$a;(a\backslash b) \quad \leq \quad b \tag{rK}$$

$$b \quad \leq \quad a\backslash(a;b) \tag{Kr}$$

$$(b/a);a \quad \leq \quad b \tag{lK}$$

$$b \quad \leq \quad (b;a)/a. \tag{Kl}$$

In the theory of ordered monoids each of these inequalities is equivalent to the universal Horn formula with the corresponding upper case identifier: (rK) to (RK), etc.

We say that a is *reflexive* when $1' \leq a$, and *transitive* when $a;a \leq a$. These properties are of interest in their own right, but are of particular interest in the following section on star.

Proposition 3 *In a residuated order, $a\backslash a$ and a/a are each both reflexive and transitive.*

Proof: We show this just for $a\backslash a$. Evidently $a;1' \leq a$, whence $1' \leq a\backslash a$, showing reflexivity. Now

$$\begin{array}{rcl} a\backslash a & \leq & a\backslash a \\ \text{Hence}\quad a;(a\backslash a) & \leq & a \\ \text{so}\quad a;(a\backslash a);(a\backslash a) & \leq & a;(a\backslash a) \\ & \leq & a \\ \text{Thus}\quad (a\backslash a);(a\backslash a) & \leq & a\backslash a \end{array}$$

∎

When an ordered monoid has the structure of a lattice or a Boolean algebra, the corresponding residuated order is called a residuated lattice or residuated Boolean algebra respectively. A residuated cartesian lattice ($a;b = ab$), often assumed also to have a least element 0, is called a *Heyting algebra.*

3 Converse

A *relation algebra* (RA) is a residuated Boolean algebra with a unary operation called *converse*, notated $a^{\smile}$, satisfying $a\backslash b = (a^{\smile};b^{-})^{-}$ and $a/b = (a^{-};b^{\smile})^{-}$. This definition of the class **RA** is the content of Theorem 2.2 of Chin and Tarski [CT51], which together with the axioms for a Boolean monoid suffice to axiomatize **RA**.[2]

It is customary in giving equational axiomatizations of **RA** to mention converse in at least four equations, which invariably include

$$a^{\smile\smile} = a$$

$$(a;b)^{\smile} = b^{\smile};a^{\smile}$$

$$(a+b)^{\smile} = a^{\smile} + b^{\smile}$$

along with (any) one of (Kr), (Kl), (rK), or (lK). For a change of pace here is a one-equation definition of converse.

Proposition 4 *The equations for a Boolean monoid, together with the equation*

$$((b^{-};a^{\smile})^{-};a) + b = b(a^{\smile};(a;b)^{-})^{-},$$

constitute a complete equational axiomatization of **RA**.

[2]Monotonicity of $a;b$ in a and b is implied by (LKL) and (RKR) respectively. Thus a complete axiomatization of **RA** need consist just of these two plus the axioms for a Boolean algebra and for a monoid, omitting monotonicity. The inequational versions do not imply monotonicity and hence must accompany the full theory of **BM**.

This equation can thus be regarded as an equational definition of converse.

Proof: Examination reveals this equation just to be (lK) and (Kr) combined via the equivalence $x \leq y \leq z$ iff $x + y = yz$. The essential novelty here is that just two of the four inequalities (Kr), (rK), (Kl), and (lK) suffice to completely axiomatize converse and hence to prove the other two inequalities.

From (lK) and (Kr) we can obtain $a^{\smile\smile} = a$, which we decompose as follows.

$$a^{\smile\smile} \leq a \tag{I}$$

$$a \leq a^{\smile\smile} \tag{C}$$

We obtain (I) from (Kr) thus.

$$\begin{array}{rcll} b & \leq & (a^{\smile};(a;b)^{-})^{-} & \text{(Kr)} \\ \text{Hence } 1' & \leq & (a^{\smile};a^{-})^{-} & \text{(setting } b = 1'\text{)} \\ \text{so } a^{\smile\smile} & \leq & a^{\smile\smile};(a^{\smile};a^{-})^{-} & \\ & \leq & a & \text{(Kr)} \end{array}$$

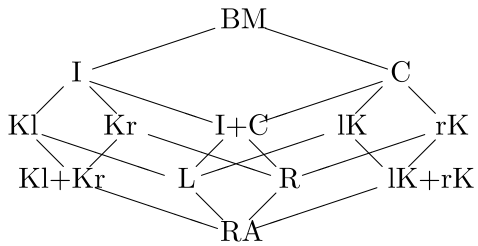
We obtain (C) from (lK) as follows.

$$\begin{array}{rcll} (b^{-};a^{\smile})^{-};a & \leq & b & \text{(lK)} \\ \text{Hence } a^{\smile -};a & \leq & 0' & (b \to 0') \\ \text{so } a^{\smile\smile -};a^{\smile} & \leq & 0' & (a \to a^{\smile}) \\ \text{whence } 1' & \leq & (a^{\smile\smile -};a^{\smile})^{-} & \text{(contrapos.)} \\ \text{Therefore } a & \leq & (a^{\smile\smile -};a^{\smile})^{-};a & \\ & \leq & a^{\smile\smile} & \text{(lK)} \end{array}$$

Substituting $a^{\smile}$ for a in Kr, taking the contrapositive, and using $a^{\smile\smile} = a$ then yields rK. Kl is obtained similarly from lK.

Finally we obtain KR from Kr. Assume $a;b \leq c$. Then $b \leq a\backslash(a;b) \leq a\backslash c$. Similarly KL follows from Kl, while RK follows from rK, and LK from lK. But by Theorem 2.2 of Chin and Tarski [CT51] this and the equations for Boolean monoids constitute a complete axiomatization of **RA**. ∎

The relationships we have obtained are summarized by the following lattice of inclusions between varieties of Boolean monoids. Each variety is labeled with the list of equations defining it, with BM implicit (so I denotes BM together with $a^{\smile\smile} \leq a$). Kr+rK is abbreviated to R (the theory of "right-handed" relation algebras) and Kl+lK to L (ditto for left handed), and Kl+Kr+lK+rK to RA. (Hence Theorem 2.2 [CT51] amounts to the statement that a relation algebra is a right-handed relation algebra that is also left handed.) Meet in this lattice corresponds to intersection of the corresponding varieties. I have not verified whether all 13 varieties are distinct, but I conjecture that they are.



The equivalences (RKR) and (LKL) have from the very beginning been a staple of writings on the algebra of binary relations. Roger Maddux brought to my attention that these are given as a property of converse by all three of the major 19th century writers on the algebra of binary relations, De Morgan [DM60, Theorem K][3], Peirce [Pei33, 3.147(170)], and Schröder [Sch95, §17,2)]. In the form

$$(c; b^{\smile})a = 0 \quad \equiv \quad (a; b)c = 0 \quad \equiv \quad (a^{\smile}; c)b = 0$$

it is taken as the defining characteristic of converse and hence of **RA** in Theorem 2.2 of Chin and Tarski and Definition 4.1 of Part II of Jónsson and Tarski [JT52], foreshadowed by Tarski [Tar41, XVI]. (An advantage of the latter form is that it is defined in any lower semilattice with a least element, rendering it applicable to considerably more general structures than Boolean algebras, e.g. Heyting algebras and relevant logics.)

We write **Rel** X for the result of equipping **BM** X with the standard operation of converse for binary relations. We write **Eqv** E for the relation algebra of all subsets of an equivalence relation E.

A *representable relation algebra* (RRA) is a representable Boolean monoid equipped with converse having its usual meaning for binary relations. Equivalently it is a subalgebra of an algebra **Eqv** E. The class **RRA** of such forms a variety [Tar55].

3.1 Star

The operation star, or ancestral, or reflexive transitive closure, which we shall notate a^*, resembles converse in some respects. It is a unary operation definable in any ordered monoid. Whereas $a^{\smile}$ is a reversed, a^* is a iterated indefinitely. Under progressively stronger assumptions about the ordered monoid progressively more may be said about star. For residuated Boolean algebras there are a number of equivalent definitions of star, including more than one equational definition.

An *ordered monoid with star* is an ordered monoid $(A, \leq, ;, 1')$ such that for each $a \in A$ there exists an element of A, denoted a^*, such that

$$1' \leq a^* \tag{S1}$$

$$a; a^* \leq a^* \tag{S2}$$

[3] De Morgan gives only KR and KL, but also was well aware of the involutary nature of converse, which as we have seen entails KR→RK and KL→LK. In nominating Peirce rather than De Morgan as the "creator of the modern theory of relations" Tarski [Tar41] appears not to have taken Theorem K into account. Peirce gave an equivalent equational characterization of converse in 1870, [Pei33], but was no better equipped than De Morgan to appreciate its completeness.

$$a;b \leq b \ \rightarrow\ a^*;b \leq b \tag{S3}$$

We say that a *converges at* b when $a;b \leq b$. Thus (S2) asserts that a converges at a^*, and (S3) asserts that a^* converges wherever a does.

Proposition 5 *The following hold in any ordered monoid with star.*

$$a^*;a^* \leq a^*; \tag{Tr}$$

$$a \leq a^*; \tag{Gr}$$

$$1\text{'} \leq b \ \ \& \ \ b;b \leq b \ \ \& \ \ a \leq b \quad \rightarrow \quad a^* \leq b \tag{Cl}$$

Proof: Setting b to a^* in (S3) and using (S2) gives (Tr). (That is, a^* converges at itself.) By "multiplying" (S1) on the left by a we obtain $a \leq a;a^*$, which with (S2) yields (Gr). For (Cl), if $a \leq b$ and $b;b \leq b$ then $a;b \leq b$, whence by (S3) $a^*;b \leq b$. If moreover 1' $\leq b$ then $a^* = a^*;1\text{'} \leq a^*;b \leq b$, giving (Cl). ∎

We define the *reflexive transitive closure* of a to be the least reflexive transitive element greater or equal to a, when it exists.

Proposition 6 *In an ordered monoid with star, a^* is implicitly defined as the reflexive transitive closure of a.*

Proof: (S1) asserts that a^* is reflexive, (Tr) that it is transitive, and (Gr) that it is greater or equal to a. (Cl) asserts that it is the least such. There can be at most one least such, whence a^* is uniquely defined. ∎

Star, ancestral, or reflexive transitive closure, is variously notated a^* [Kle56], ${}^*a$, and a_0 [Ded01, Sch95], while transitive closure has been written a^+, a^ω [NT77], and a_{00} [Sch95]. The first-mentioned in each of these lists is the notation universally used in computer science and is adopted here. In any ordered monoid a^+ is definable in terms of a^* via the equation $a^+ = a;a^*$. In an ordered monoid in which the operation $a + 1\text{'}$ of reflexive closure is defined, and hence in a semilattice monoid, a^* is definable via either of the equations $a^* = a^+ + 1\text{'}$ or $a^* = (a + 1\text{'})^+$.

Star was first studied in detail by Schröder [Sch95], who notated it a_0 following Dedekind's 1888 notation [Ded01] for the "chain" of a "transformation" a. There Dedekind gave three axioms for chains, in paragraphs numbered respectively 45-47. Schröder translated Dedekind's axioms into the language of binary relations, numbering them (D45)-(D47), as follows.

$$b \leq a^*;b \tag{D45}$$

$$a;a^*;b \leq a^*;b \tag{D46}$$

$$a;c + b \leq c \quad \rightarrow \quad a^*;b \leq c \tag{D47}$$

Proposition 7 (S1) *is equipollent with* (D45), (S2) *with* (D46), *and* (S3) *with* (D47).

Proof: We may recover (S1)-(S3) from (D45)-(D47) respectively by setting b to 1' in (D45) and (D46), and setting c to b in (D47) and simplifying. In the other direction, (D45) and (D46) can be obtained by multiplying (S1) and (S2) respectively on the right by b. Substituting c for b in (S3) yields $a;c \leq c \rightarrow a^*;c \leq c$, which in turn implies $a;c + b \leq c \rightarrow a^*;c \leq c$. But $b \leq c$ implies $a^*;b \leq a^*;c$, yielding (D47). ∎

Our definition of star is not equational because (S3) is not an equation, only a universal Horn formula. Ng and Tarski [NT77, Ng84] define the class **RAT** of relation algebras with star[4] and give a finite equational axiomatization of **RAT**. Their proof appeals to the transitivity of a/a and thus relies on converse. Here we show that the larger class of Boolean monoids with star also constitutes a variety. In the absence of converse we shall depend on complement, reformulating Segerberg's induction axiom for star in dynamic logic [Seg77] as an equational property of Boolean monoids.

Recall that normal means $a;0 = 0 = 0;a$ and additive that $a;(b+c) = a;b + a;c$ and $(a+b);c = a;c + b;c$.

Proposition 8 *The class of normal additive Boolean monoids with star is a finitely axiomatized variety, with star defined by the equations* (S1), (S2), *and*

$$a^*;b \;\leq\; b + a^*;((a;b)b^-). \tag{Ind}$$

Proof: (Ind→S3) This is where we use normality. Given $a;b \leq b$ we wish to show $a^*;b \leq b$. From the hypothesis we infer $(a;b)b^- = 0$, whence

$$\begin{aligned} a^*;b &\leq b + a^*;((a;b)b^-) \\ &= b + a^*;0 \\ &= b. \end{aligned}$$

(S3→Ind) Here we depend on additivity. It suffices to show that substituting the right hand side of (Ind) for b in (S3) satisfies the hypothesis $a;b \leq b$ of (S3), assured by the following calculation.

$$\begin{aligned} a;(b + a^*;((a;b)b^-)) &\leq b + (((a;b)b^-) + a;a^*;((a;b)b^-)) \\ &\leq b + a^*;((a;b)b^-) \qquad ((\text{S1}),(\text{S2})) \end{aligned}$$

∎

Equation (Ind) viewed suitably is relational induction. To see this take its contrapositive

$$b \wedge a^* \hookrightarrow (b \rightarrow a \hookrightarrow b) \leq a^* \hookrightarrow b, \tag{Ind'}$$

[4]Ng and Tarski treat transitive closure rather than reflexive transitive closure, whence the T in **RAT**, but as we have already noted these are interdefinable equationally, and thus the same variety is obtained whether the operation is taken to be star or transitive closure.

writing $a \to b$ for $\neg a + b$ ("static" or material implication as opposed to "dynamic" implication $a \hookrightarrow b$). Read b as an induction hypothesis, $a^* \hookrightarrow$ as "after any number of a's", and $a \hookrightarrow$ as "after one a". Then (Ind') in English is "if b holds and after any number of a's the truth of b implies that b still holds after one more a, then b holds after any number of a's."

We may relate this to mathematical induction as follows. Take X to be the set N of natural numbers and take $a \subseteq N^2$ to be the successor relation, whence a^* is $\leq_N$. Restrict attention to those b satisfying $b \leq 1'$, allowing us to view b as an arbitrary predicate $b(x)$ on X defined as holding at x just when $(x, x) \in b$. (Ind') then asserts that if $b(x)$, and if for all $y \geq x$ $b(y)$ implies $b(y+1)$, then $b(y)$ for all $y \geq x$. Hence (Ind') holds just for "upwardly closed" b, namely those predicates b such that $b(x)$ and $x \leq y$ implies $b(y)$. In particular $b(0)$ implies that b holds everywhere. This is precisely the content of mathematical induction.

Here we wrote $a \hookrightarrow$ and $a^* \hookrightarrow$ everywhere in preference to $a\backslash$ and $a^*\backslash$ so as to define star independently of converse. We could however have used $a\backslash$ in place of $a \hookrightarrow$: all occurrences of $a \hookrightarrow$ and $a^* \hookrightarrow$ in (Ind') can be made $a\backslash$ and $a^*\backslash$. This has the effect of taking the converse of a, computing its star, then taking the converse of the result, but the answer still comes out in the end to a^*. This observation may prove useful for nonclassical logics like **RA** but in which $a\backslash b$ is given as a primitive implication without a separate notion of converse, e.g. linear entailment in Girard's linear logic [Gir87].

As with converse a one-equation definition of star is possible: the reader may verify that

$$(1' + a^* + a^+); b \;=\; (a^*; b)(b + a^*; ((a; b)b^-))$$

is equipollent with S1-S3, where a^+ abbreviates $a; a^*$.

We have already remarked on the Ng-Tarski variety **RAT** of relation algebras with star (equivalently, with transitive closure). Relation algebras offer more opportunities to axiomatize star equationally than do Boolean monoids, as the five equivalences of the following proposition indicate.

Proposition 9 *For relation algebras with an operation a^* satisfying* (S1) *and* (S2)*, the following five formulas are equivalent.*

$$a; b \leq b \;\to\; a^*; b \leq b \tag{S3}$$

$$a^*; b \;\leq\; b + a^*; ((a; b)b^-) \tag{Ind}$$

$$1' \leq b \;\;\&\;\; b; b \leq b \;\;\&\;\; a \leq b \quad\to\quad a^* \leq b \tag{Cl}$$

$$a^* \;\leq\; (a+b)^* \;\;\&\;\; (a/a)^* \;=\; a/a \tag{Ta}$$

$$a^* \;\leq\; (a+b)^* \;\;\&\;\; (a(a/a))^+ \;=\; a(a/a) \tag{Ng}$$

where a^+ abbreviates $a; a^*$.

Proof: (Ta) appears in [NT77] (in the equivalent form $a; (a\backslash a)^+ = a$), and is attributed in [Ng84] to Tarski. (Ng) appears in [Ng84], where the equivalence of (Cl), (Ta), and (Ng) are treated. We showed the equivalence of (S3) and (Ind) above, and that (S3) implied (Cl), assuming (S1) and (S2).

It suffices therefore to show that (Ta) implies (S3). Assume $a;b \leq b$. Then $a \leq b/b$, whence $a^* \leq (b/b)^* = b/b$ by respectively the first and second equations of (Ta). Hence $a^*;b \leq b$, verifying (S3). ∎

Hence the class of relation algebras with star can be shown to be a finitely axiomatized variety using (S1), (S2), and any one of (Ind), (Ta), or (Ng).

This proposition can be straightforwardly extended to residuated Boolean algebras. A key observation is that $a\backslash a$ is transitive in any residuated order, as noted in the section on residuation.

I do not know whether (S1)-(S2) and (Cl) implies (S3) in an arbitrary Boolean monoid, though I conjecture it does not. I further conjecture that the obvious "left-handed" version of (S3) is not equipollent with (S3), though with (S1)-(S2) it implies (Cl) for the same reasons as (S3) does, and has a corresponding left-handed version of (Ind).

One noteworthy asymmetry between converse and star is that, whereas converse *produces* the equations $a;0 = 0$ and $a;(b+c) = a;b + a;c$, star *consumes* them, requiring $a;0$ (normality) to define reflexive transitive closure via (Ind) and $a;(b+c) = a;b + a;c$ (additivity) in order that (Ind) define no more than reflexive transitive closure. Thus in this equational microeconomy, conservatively regulated by the direction of motion (back one for converse and forward many for star), supply meets demand. In the absence of converse however we must meet star's demand artificially by adding normality and additivity to the basis.

4 Dynamic Algebras

4.1 Boolean Modules

Whereas a Boolean monoid lumps the logical and relative aspects of the calculus of relations into a single sort, a Boolean module [Bri81] maintains the distinction. The resulting system is a little more complex, but has several advantages: free models that are representable, and an equational theory that is decidable and finitely axiomatizable yet complete for the representable modules.

The passage from Boolean monoid to relation algebra involves taking on the operations converse and star. We will set up Boolean modules and dynamic algebras along roughly the same lines, obtaining dynamic algebras from Boolean modules by adding some operations defined *relative to the Boolean module*. This approach differs from that of Kozen [Koz79c, Koz79a] and Brink [Bri81], who assume those operations are given as an algebra of scalars prior to the installation of that algebra in the module.

A Boolean module $(\mathcal{B}, K, \diamond)$ consists of a Boolean algebra $\mathcal{B} = (B, \vee, \neg)$ (following the customary notation used in the dynamic logic literature), a set K of names of operators, and an operation

$\diamond : K \times B \to B$ such that $\diamond(a, p)$, also written $\langle a \rangle p$, is the result of applying to p the Boolean operator named by a. Following Jonsson and Tarski [JT48, JT51], a Boolean operator is a function $f : B \to B$ on a Boolean algebra such that $f(0) = 0$ and $f(p \vee q) = f(p) \vee f(q)$. We write $\langle a \rangle$ for the operator named by a, and write $\Diamond : K \to (B \to B)$ for the function defined by $\Diamond(a)(p) = \diamond(a, p)$.

4.2 Kleenean Algebras

The *type* of a Kleenean algebra $\mathcal{K} = (K, +, 0, ;, \breve{}, ^*)$ consists of five operations with respective arities 2,0,2,1,1 respectively. We abbreviate 0^* to λ, the Kleenean algebra notation for the unit of the monoid notated 1' in **RA**.

The motivating example is a Kleenean algebra of relations on a set X. This is an algebra $\mathcal{K} = \langle K, +, 0, ;, ^*, \breve{} \rangle$ where $K = 2^{X^2}$ and the operations have their standard interpretations for binary relations. Thus a Kleenean algebra of relations on X is obtained as a reduct of **Rel** X by dropping complement, then expanding by adding $*$ defined as reflexive transitive closure of binary relations. A *representable Kleenean algebra* is an algebra isomorphic to a subalgebra of a Kleenean algebra of relations. The delicate distinction that arose for Boolean monoids between the algebra **Rel** X of subsets of the complete relation X^2 and the algebra **Eqv** E of subsets of an arbitrary equivalence relation E does not come up here since we no longer have either intersection or a top element. Any representable relation algebra with star becomes a representable Kleenean algebra when complement is dropped.

Redko [Red64] has shown that the equational theory of the representable Kleenean algebras without converse is not finitely axiomatizable. Moreover Conway [Con71] has enumerated several *finite* models of this theory which do not satisfy axiom (S3) (given in the section on star for Boolean monoids) expressing that a^* is the least reflexive transitive element dominating a. By *Conway's Leap* I shall mean the 4-element Kleenean algebra in which $K = \{0, 1, 2, 3\}$ with both + and ; interpreted as numeric max except for $a; 0 = 0 = 0; a$. Take $a^* = a + even(a)$, i.e. $0^* = 1^* = 1$, $2^* = 3^* = 3$. This satisfies the equational Kleenean theory for representable Kleenean algebras, yet $2^0 \vee 2^1 \vee 2^2 \vee \ldots 2^i = 2$ for $i \leq 3$ (and hence beyond) while $2^* = 3$, defeating S3. Hence if Kleenean algebras satisfy both the equational theory of representable Kleenean algebras and (S3) they do not form a variety.

The question then arises as to the appropriate definition of a Kleenean algebra. The previous paragraph notwithstanding, we shall define Kleenean algebras equationally, albeit in an indirect way that circumvents the above difficulties.

4.3 Dynamic Algebras

A dynamic algebra is a Boolean module $(\mathcal{B}, \mathcal{K}, \diamond)$ in which the set K has been expanded to a Kleenean algebra, satisfying the following five equations, one for each Kleenean operation.[5] In this

[5]Strictly speaking converse is not a Kleenean operation. However it augments Kleene's four operations for regular expressions [Kle56, HU79] without any fuss.

approach the Kleenean algebra is defined as an integral part of a dynamic algebra, as opposed to being defined externally to a dynamic algebra, as is customary with algebras of scalars.

These equations are essentially the Segerberg axioms [Seg77] for Fischer and Ladner's propositional dynamic logic [FL79], translated into equational form. The differences from Segerberg's system are the equations for converse, which are due to Parikh [Par78], and the weakening to monotonicity of the equations that would have expressed normality and finite additivity, namely $\langle a\rangle 0 = 0$ and $\langle a\rangle(p \vee q) = \langle a\rangle p \vee \langle a\rangle q$, which can be recovered from the equation for converse.

We use the abbreviations $p \leq q$ for $p \vee q = q$, $p \wedge q$ for $\neg(\neg p \vee \neg q)$, and $[a]p$ for $\neg(\langle a\rangle \neg p)$.

$$\langle 0\rangle p \;=\; 0 \tag{D1}$$

$$\langle a+b\rangle p \;=\; \langle a\rangle p \vee \langle b\rangle p \tag{D2}$$

$$\langle a;b\rangle p \;=\; \langle a\rangle\langle b\rangle p \tag{D3}$$

$$\langle a\rangle[a\breve{}]p \;\leq\; p \;\leq\; [a\breve{}]\langle a\rangle p \tag{D4}$$

$$p \vee \langle a\rangle\langle a^*\rangle p \;\leq\; \langle a^*\rangle p \;\leq\; p \vee \langle a^*\rangle(\langle a\rangle p \wedge \neg p) \tag{D5}$$

A dynamic algebra *with test* is one with an operation $? : B \to K$, notation $?(p) = p?$, satisfying

$$\langle p?\rangle q = p \wedge q. \tag{D?}$$

The first three equations define $\langle 0\rangle$ to be the constantly zero operator, $\langle a+b\rangle$ to be the pointwise disjunction of $\langle a\rangle$ and $\langle b\rangle$, and $\langle a;b\rangle$ to be the composition of $\langle a\rangle$ and $\langle b\rangle$. If we translate $\langle a\rangle p$ as $a;b$ then equation (D4) can be seen to be exactly the equation rK,Kr of relation algebra, which as can be seen from the lattice diagram for those axioms entails the theory R of right-handed relation algebras. Similarly equation (D5) so translated can be seen to be the relation algebra equation for star.

The arguments showing that the relation algebra equations for converse and star implicitly define those operations carry over to this situation without difficulty; see [Pra79a, Pra80a] for the case of star. Since $[a\breve{}]$ is the right adjoint of $\langle a\rangle$ [Mac71, Thm IV.5-1], $\langle a\breve{}\rangle$ is the *dual* right adjoint of $\langle a\rangle$. $\langle a^*\rangle$ is the reflexive transitive closure of $\langle a\rangle$, satisfying $\langle a\rangle \leq \langle a^*\rangle$, $\langle a^{**}\rangle = \langle a^*\rangle$, $\langle a\rangle \leq \langle b\rangle$ implies $\langle a^*\rangle \leq \langle b^*\rangle$, and $\langle a\rangle$ is reflexive and transitive if and only if $\langle a^*\rangle = \langle a\rangle$. (An operator f is reflexive when $p \leq f(p)$ and transitive when $f(f(p)) \leq f(p)$.)

A *preKleenean* algebra is the Kleenean algebra of a dynamic algebra. A *separable dynamic algebra* [Koz79a] is one for which $\diamond$ is injective, satisfying the Π^0_2 sentence $\forall p[\langle a\rangle p = \langle b\rangle p] \to a = b$. An *intensional* dynamic algebra is an algebra isomorphic to a subalgebra of a separable dynamic algebra. A *Kleenean algebra* is the Kleenean algebra of an intensional dynamic algebra. We denote the corresponding classes **SDA**, **IDA**, and **KA** respectively.

Proposition 10 **IDA** *is a quasivariety.*

Proof: Since **SDA** is defined by equations and a Π^0_2 sentence it is an elementary class closed under direct products. Hence its closure **IDA** under isomorphisms and subalgebras is a quasivariety. ∎

4.4 Examples

In the following example the Boolean and Kleenean elements are respectively unary and binary relations on a set X. This is the motivating example of a dynamic algebra, and corresponds to Example 1 of a Boolean monoid, as well as to our notion of a Kleenean algebra of relations.

Example 1. Let **Kri** $X = (\mathcal{B}, \mathcal{K}, \diamond)$ consist of the Boolean algebra $\mathcal{B} = (2^X, \vee, 0, \wedge, 1, \neg)$ of unary relations on X, and the Kleenean algebra $\mathcal{K} = (2^{X^2}, +, 0, ;, ^*, ^\smile)$ of binary relations on X, with symbols interpreted as for **Rel** X, and such that star is reflexive transitive closure, $\langle a \rangle p = \{x \mid \exists y[xay \wedge p(y)]\}$. This example may be extended to a dynamic algebra with test by adjoining the operation $p? = \{(x, x) | p(x)\}$.

A *Kripke structure on* X is any subalgebra of **Kri** X. The Boolean elements of a Kripke structure are unary relations and the Kleenean binary. It may be verified that Kripke structures satisfy all the equations, those for star and converse being the most challenging. A *representable dynamic algebra* (RDA) is a dynamic algebra isomorphic to a Kripke structure. These form the class **RDA**.

Proposition 11 **RDA** *is a quasivariety* ([Ném82]).

Proof: It suffices to show that the direct product of a family (**Kri** X_i) of Kripke structures is an RDA. Its Boolean component is the power set of $\sum_i X_i$, the disjoint union of the X_i's. Its Kleenean component is isomorphic to the power set of the equivalence relation on $\sum_i X_i$ definable as $\sum_i X_i^2$, relating just those pairs of elements from the same X_i. ∎

Proposition 12 **RDA** $\subseteq$ **IDA**.

Proof: Every **Kri** X is separable. **RDA** is the ISP closure of the **Kri** X while **IDA** is the ISP closure of **SDA**. ∎

It follows from a result of Kozen [Koz81] that the converse does not hold.

In the next example the Boolean and Kleenean elements are languages as sets of strings over a common alphabet X. For languages we must omit converse and test, but see the section on "model robustness" for an approximation to **Lan** X containing converse.

Example 2. Let **Lan** $X = (\mathcal{B}, \mathcal{K}, \diamond)$ consist of the Boolean algebra $\mathcal{B} = (2^{X^\omega}, +, \emptyset, \sim)$ of infinitary languages (sets of infinite-to-the-right strings on X), and the Kleenean algebra $\mathcal{K} = (2^{X^*}, 0, +, ;, ^*)$ of all sets of finite strings, with operations (omitting converse) having their usual meaning for languages [Kle56, HU79]. Take $\langle a \rangle p$ to be the concatenation of languages a and p. We now have a dynamic algebra *without converse*.

All axioms save (D5b) (right hand inequality of (D5)) are easily verified. For (D5b), given any string $s \in \langle a^* \rangle p$ find the least n such that $s = a_1 \ldots a_n t$ for strings $a_i \in a$ and $t \in p$. If $n = 0$ then

$s = t$ and $s \in p$. Otherwise $a_i \ldots a_n t \notin p$ for $1 \leq i \leq n$ or we could find a smaller n. In particular $a_n t \notin p$ whence $a_n t \in \langle a \rangle p - p$, so $s \in \langle a^* \rangle (\langle a \rangle p - p)$.

From any nonempty infinitary language L not containing the symbol 0 we may construct the language $p = 0L$ as a universal separator. If s is in a but not b then $s0L$ is a nonempty subset of $\langle a \rangle p$ but is disjoint from $\langle b \rangle p$. This makes **Lan** X separable.

4.5 Representability and Completeness

Thus far dynamic algebras seem very much like relation algebras, using essentially the same equations to essentially the same effect. However, whereas the relation algebra axioms incompletely axiomatize the representable relation algebras, the dynamic algebra axioms completely axiomatize the representable dynamic algebras. We may put this more graphically as follows.

A representable Boolean algebra is an algebra isomorphic to a field of sets. That is, the class **RBA** of such algebras is the quasivariety ISP(2) generated by the two-element Boolean algebra. A Boolean algebra is a complemented distributive lattice, conditions expressible with finitely many equations and thus making the variety **BA** of Boolean algebras finitely axiomatized. Evidently $\mathbf{RBA} \subseteq \mathbf{BA}$. It is one of nature's little pranks that **RBA** is a variety, but it is a bigger prank that $\mathbf{RBA} = \mathbf{BA}$ [Sto36].

The class **RRA** of representable relation algebras is the quasivariety generated by algebras **Rel** X. The variety **RA** of relation algebras is finitely based, with $\mathbf{RRA} \subseteq \mathbf{RA}$. We again have the little prank, that **RRA** is a variety [Tar55]. The difference is that we no longer have the big prank: $\mathbf{RA} \neq \mathbf{RRA}$ [Lyn50], and although there are infinitely many finitely axiomatizable varieties between these two, **RRA** itself is not finitely axiomatizable [Mon64].

With dynamic algebras the situation is in between these two. The class **RDA** of representable dynamic algebras is the quasivariety generated by algebras **Kri** X. The variety **DA** of dynamic algebras is finitely based, with $\mathbf{RDA} \subseteq \mathbf{IDA} \subseteq \mathbf{DA}$. Now the free IDA's are residually finite [Pra79a] and moreover are representable [Ném82], whence the equational theory of **IDA** completely axiomatizes **RDA**, i.e. **RDA** and **IDA** have the same equational theory. Also **RDA**, **IDA**, and **DA** have the same Boolean equational theory. This is a weaker connection than that of **RBA**=**BA**, but a stronger one than $\mathbf{RRA} \subseteq \mathbf{RA}$, where the equational theory of **RA** is strictly less than that of **RRA**.

To show that every free IDA is representable, let us consider the effect of the passage from dynamic algebras to intensional dynamic algebras on the equational theory. This effect is quite striking: the Boolean theory does not change while the Kleenean theory is transformed at one stroke from the vacuous theory to the theory of Kleenean algebras.

Since the only axioms of **DA** are Boolean, Θ^K_{DA} is trivial, consisting just of all equations $a = a$. Hence the Kleenean variety generated by the preKleenean algebras is just the anarchic variety consisting of all word algebras of type $\mathbf{DA}^K$ and their quotients. As we shall see momentarily however the preKleenean algebras are considerably more organized than their vacuous theory might suggest.

Proposition 13 *A Boolean module has at most one expansion to a separable dynamic algebra.*

Proof: In an SDA, the elements of K serve as distinct names for functions on B. Axioms (D1)-(D3) are easily seen to define the corresponding three operations uniquely, as respectively the constantly zero operation, pointwise disjunction, and composition. Axioms (D4) and (D5) also uniquely define converse and star respectively, for the same reasons as do the corresponding axioms for these operations in **RA**. For completeness we give the proof in full here.

The left half of (D4) can be written equivalently as $p \leq [a]\langle a\breve{}\rangle p$ (p is universally quantified here). The right half says that for any q such that $p \leq [a]q$ (and we have just seen that $\langle a\breve{}\rangle p$ is such a q), we must have $\langle a\breve{}\rangle p \leq q$, i.e. $\langle a\breve{}\rangle p$ is the least q for which $p \leq [a]q$. To see this, let q satisfy $p \leq [a]q$. Then by monotonicity $\langle a\breve{}\rangle p \leq \langle a\breve{}\rangle [a]q$. But the latter is bounded by q, i.e. $\langle a\breve{}\rangle p \leq q$. There can only be one least such q, whence $\langle a\breve{}\rangle$ is uniquely determined.

A similar argument, given in [Pra79a, Pra80a], obtains for star. The left half of (D5) says that $\langle a^*\rangle p$ is among those q's satisfying $p \vee \langle a\rangle q \leq q$. But any such q satisfies $p \leq q$, so by monotonicity $\langle a^*\rangle p \leq \langle a^*\rangle q$. But the latter is bounded by $q \vee \langle a^*\rangle(\langle a\rangle q \wedge \neg q)$, and $\langle a\rangle q \wedge \neg q$ vanishes (since $\langle a\rangle q \leq q$, whence so does $< a^* > (\langle a\rangle q \wedge \neg q)$, yielding $\langle a^*\rangle p \leq q$. Thus $\langle a^*\rangle p$ is the least such q. But there can only be one least such q, whence $\langle a^*\rangle$ is uniquely determined. ∎

We now provide a sense in which the preceding result extends to all **DA**'s. In the following, by a *homomorphism* we mean an operation-preserving function between algebras of the same type, or in the case of an algebra with n sorts, then n "parallel" such functions. For the two sorts of dynamic algebras a homomorphism $f : \mathcal{D} \to \mathcal{D}'$ must be a pair (f_B, f_K) consisting of Boolean and Kleenean homomorphisms $f_B : \mathcal{B} \to \mathcal{B}'$ and $f_K : \mathcal{K} \to \mathcal{K}'$ satisfying $f_K(a+b) = f_K(a) +' f_K(b)$, $f_B(p \vee q) = f_B(p) \vee' f_B(q)$, $f_B(\langle a\rangle p) = \langle f_K(a)\rangle' f_B(p)$, $f_K(p?) = f_B(p)?'$, and similarly for the other operations. By a *quotient* of an algebra $\mathcal{D}$ we mean as usual the equivalence class of all homomorphic images of an algebra isomorphic to a particular such image, or equivalently, the representative $\mathcal{D}/\cong$ of that class whose elements are the congruence classes of some congruence $\cong$ on $\mathcal{D}$ (namely the kernel of the above "particular image").

Proposition 14 $\Diamond$ *is a preKleenean homomorphism.*

Proof: If we denote pointwise disjunction in $B \to B$ by $+'$, then the equation defining $+$ merely asserts $\Diamond + = +'\Diamond$, and similarly for the other four Kleenean operations, bearing in mind the preceding result that these operations are uniquely determined for $B \to B$. But this is then just the assertion that $\Diamond : K \to (B \to B)$ is a homomorphism of preKleenean algebras K and $B \to B$. ∎

It follows that the kernel of $\Diamond$ is a preKleenean congruence. This was first observed for $*$-continuous dynamic algebras by Kozen [Koz79b, Koz80] (see the history section). It was generalized to the weaker Segerberg notion of star by the author [Pra80a].

Proposition 15 *Every dynamic algebra $\mathcal{D}$ has a unique quotient $\mathcal{D}'$ in* **IDA** *such that $\mathcal{B} = \mathcal{B}'$.*

Proof: By the preceding proposition, the unique factorization of the function $\Diamond$ as the composition of an injection $\Diamond'$ with a surjection q is such that q is a quotient. This yields a dynamic algebra $\mathcal{D}'$ as a quotient of $\mathcal{D}$; no finer quotient will land in **IDA**. For uniqueness, the action of $\Diamond$ must be preserved by any homomorphism (via the natural isomorphism relating $\Diamond : K \to (B \to B)$ to $\diamond : K \times B \to B$, "homomorphism" being defined so as to preserve the latter). Hence if $\mathcal{B}$ does not change, distinct operators must remain distinct, whence no coarser quotient will preserve $\mathcal{B}$. ∎

Proposition 16 $\Theta^B_{IDA} = \Theta^B_{DA}$.

Proof: Since **IDA** $\subseteq$ **DA** it suffices to show $\Theta^B_{IDA} \subseteq \Theta^B_{DA}$. From the preceding theorem we infer that every dynamic algebra $\mathcal{D}$ is isomorphic to a subalgebra of the product of an intensional dynamic algebra $\mathcal{D}'$ with a "Boolean-trivial" dynamic algebra $\mathcal{D}''$, namely the quotient of $\mathcal{D}$ fixing K and collapsing B to a point. Hence all equations in Θ_{IDA} hold of $\mathcal{D}'$, and trivially of $\mathcal{D}''$, and hence of $\mathcal{D}$. ∎

But whereas postulating injectivity of diamond does not increase the Boolean theory, it takes the Kleenean theory from the vacuous theory to the theory of Kleenean algebras!

Proposition 17 *An intensional dynamic algebra $(\mathcal{B}, \mathcal{K}, \Diamond)$ for which $\mathcal{B}$ is complete and atomic is representable. In particular every finite IDA is representable.*

Proof: Taking X as the set of atoms of $\mathcal{B}$, interpret $\mathcal{B}$ as 2^X, each $a \in K$ as the relation $paq \equiv (q \leq \langle a \rangle p)$, and $\Diamond$ as in a Kripke structure. Since a complete and atomic Boolean algebra is isomorphic to the power set of its atoms, and since all joins exist and are preserved by $\langle a \rangle$, this Kripke structure is isomorphic to the given dynamic algebra. ∎

By *free dynamic algebra* we shall understand a free algebra of the variety generated by dynamic algebras. Conway's Leap, as defined in the section on Kleenean algebras, shows that this variety contains algebras that do not match our intuition about dynamic algebras. The free algebras of this variety turn out not to so violate intuition, and indeed are not only dynamic algebras, justifying the name, but even more importantly are representable. Showing this will be our main goal, allowing for the occasional digression.

Proposition 18 *Every free dynamic algebra is a subdirect product of finite dynamic algebras* [Pra80a].

Proof: With the assumption of freedom we are able to translate into dynamic algebra terminology Fischer and Ladner's filtration construction [FL79], whereby from any Kripke structure satisfying a particular formula they construct a finite Kripke structure satisfying it. Nothing in their proof makes essential use of attributes of representable dynamic algebras not already possessed by arbitrary dynamic algebras. The reader is referred to the proof of Theorem 5 [Pra80a] for the short (half a page) details of this translation. The case of a free dynamic algebra on the empty set of Boolean generators is treated by Németi [Ném82]. ∎

Corollary 19 *HSP(***RDA***)=HSP(***IDA***). That is,* $\Theta_{RDA} = \Theta_{IDA}$.

Proof: Every intensional dynamic algebra is a quotient of a free intensional dynamic algebra. Each of these in turn is a subdirect product of finite intensional dynamic algebras, which in turn are representable. HSP preserves equations, whence the theory of Kripke structures is a subset of that of intensional dynamic algebras. ∎

Corollary 20 *The Segerberg axioms are sound and complete relative to Kripke structures* [Pra80a]. *That is,* $\Theta^B_{RDA} \subseteq \Theta^B_{IDA}$.

Corollary 21 *Every free intensional dynamic algebra is representable.* [Ném82]

(With the previous corollary as my goal I overlooked this nice strengthening in [Pra80a].)

Proof: For any quasivariety K the free algebras of HSP(K) belong to K. **RDA** is a quasivariety and HSP(**RDA**)=HSP(**IDA**). ∎

4.6 Computational Complexity

Theorem 22 *There exist* $1 < c < d$ *such that* Θ_{DA} *and its complement* $\bar{\Theta}_{DA}$ *are not in DTIME*(c^n) [FL79] *but are in DTIME*(d^n) [Pra79b].

That is, the time required to deterministically test either satisfiability or validity of dynamic algebra equations is one exponential in the number n of occurrences of variables in the formula, a bound that cannot be improved by more than by a polynomial of degree $\log_c d$. For comparison, the best deterministic procedure known for pure Boolean equations, i.e. propositional calculus, requires time $2^{n/4}$ or 1.1892^n [VG88], down to 1.093^n for equations $t = 0$ when t is in conjunctive normal form. Fischer and Ladner do not supply a specific value for c, but their proof is constructive and if pushed hard might conceivably yield a c as high as 1.01. Thus we are still some distance from knowing whether dynamic logic is any harder to decide in practice, i.e. deterministically, than Boolean logic, though close enough that these two bounds may well pass each other within the coming decade.

4.7 The Language Model

It can be shown [Pra79b] that the equational theory of Example 2 is that of **IDA**, in the absence of converse.

We could alternatively have taken $\mathcal{B} = 2^{X^*}$. This is still an intensional dynamic algebra, with any single string serving as a universal separator. The difference is that the equational theory of the resulting class is strictly larger than that of **IDA**. In particular $\langle a^* \rangle([a]p \vee [a]\neg p)$ is now a theorem for any a and p. This asserts that for any predicate it is possible to run any program sufficiently often that at its next execution it is deterministic with respect to that predicate.

We may extend Example 2 to include converse by embedding $\mathcal{K}$ (an algebra of languages) in the larger algebra $\mathcal{K}^+$ of all (normal additive) operators on $\mathcal{B}$. We then define $\langle a^{\smile} \rangle p = \{s \mid \langle a \rangle s \wedge p \neq 0\}$. This defines $\langle a^{\smile} \rangle$ as an operator but it does not define $a^{\smile}$ as a language.

To verify (D4a) (left hand inequality of (D4)), consider any string $t \in \langle a \rangle [a^{\smile}]p$. Then there exists $s \in [a^{\smile}]p$ such that $t \in \langle a \rangle s$. So $s \notin \langle a^{\smile} \rangle \neg p$, whence $\langle a \rangle s \wedge \neg p = 0$. Hence $t \notin \neg p$, i.e. $t \in p$. For (D4b), suppose $s \in p$. Then $\langle a \rangle s \leq \langle a \rangle p$, that is, $\langle a \rangle s \wedge \neg \langle a \rangle p = 0$, whence $s \notin \langle a^{\smile} \rangle \neg \langle a \rangle p$, i.e. $s \in [a^{\smile}] \langle a \rangle p$.

In order to add the test operation $p?$ to this example we evidently require that the structure satisfy the sentence $\forall p \exists a \forall q [\langle a \rangle q = p \wedge q]$. But Example 2 falsifies this at $p = 0X^\omega$. For if $a \neq 0$ then take $q = 1X^\omega$ making $p \wedge q = 0 \neq \langle a \rangle q$, while if $a = 0$ then take $q = p$.

I do not have a completely satisfactory solution. Here is as much as I have been able to do. Modify $\mathcal{B}$ to satisfy the sentence as follows. The infinite string $(AB \ldots YZ)^\omega$ has 26 distinct suffixes, take $\mathcal{B}$ to be the Boolean algebra consisting of the 2^{26} sets of such suffixes. Take $\mathcal{K}$ as before but with X chosen to include the 26 letters. Define $\langle a \rangle p = ap \cap 1$ (1 being the top of $\mathcal{B}$, i.e. the set of 26 suffixes) and $p? = \pi_{26}(p)$ where $\pi_{26}(p)$ is the set of all length-26 prefixes of strings of p. Axiom $(D?)$ is now easily verified, and the arguments for the other axioms are easily modified to accommodate this change to diamond.

But now we have lost separability. By modifying $\mathcal{K}$ along the same lines, changing $a;b$ from concatenation to something more discriminating, we could restore it. But this completely loses the spirit of more **Lan** X. This raises the somewhat vague question, is there a way to define test for a **Lan**-like dynamic algebra?

The beginning of Example 2 (no converse and test) appears in [Pra80a]. The modification used to define test is essentially the result of "the LAN construction" [Pra79b] used to show that the theory of **Lan** X coincides with the theory of dynamic algebras, the difference being that we did not cater for test there, allowing complement to be taken relative to X^ω rather than to L^ω as here.

5 Reflections

For a change of pace let us reflect on some of the philosophical issues bearing on dynamic algebras and their relationship to relation algebras.

5.1 Analysis

Why do Boolean modules and dynamic algebras have so many properties that relation algebras lack? I like to think of it in terms of holding the two essential ingredients of dynamic and relation algebras at the proper distance. Too far apart and all you have is a Boolean algebra and a monoid. Too close and they interfere destructively.

Brink [Bri81] argues that Boolean modules are relatively well-behaved compared to relation algebras. I make a similar point in the context of regular algebras versus dynamic algebras [Pra79a, Pra79b, Pra80a]. Redko [Red64] has shown that the equational theory of regular algebras has no finite basis. Conway [Con71] has observed that this theory has a three-element model in which $x^0 + x^1 + \ldots + x^n$ is constant with increasing $n > 0$ yet $x*$ is not that constant, a discontinuity we refer to as *Conway's Leap*. Replacing one-sorted regular algebras by two-sorted dynamic algebras disposes of both these aberrations, as we will see later in the section on properties of dynamic algebras.

The common idea here seems to be that intersection and composition in too close proximity only "fight" each other. If instead each is moved to an appropriate sort, a logical sort accommodating the Boolean operations and a relative sort for the Kleenean operations, the separation seems to encourage cooperation instead of competition.

5.2 Star, Converse, and Test

Star has turned out to be converse's long-lost fraternal twin. Star should have been included in relation algebras from the outset. Converse without star is a piston without a crankshaft, or $\cos(x)$ without e^{ix}.

Test restores strong connectivity of information flow around the algebra. This flow obtains vacuously in one-sorted relation algebras.

Star and test are standard features of any imperative programming language. Star provides iteration, while test enables the rational performance of choice and iteration, expressed deterministically with **if** p **then** a **else** b and **while** p **do** a respectively, and more generally with guarded commands [Dij76].

The logic-of-programs significance of dynamic algebra is as follows. The set X is viewed as the states of a computer. Binary relations are viewed as programs: the meaning of (x, y) as an element of a program is that when that program starts in state x it *may* stop in y. A *deterministic* program is one that is a partial function. The program $a;b$ performs a then b. The program $a+b$ nondeterministically chooses to perform one of a or b. The program a^* nondeterministically chooses an $i \geq 0$ and performs a^i, that is, $a;a;\ldots;a$ i times. The test program $p?$ changes nothing but stops only if p holds, otherwise it is said to *block*. The programming construct "**if** p **then** a **else** b" can then be expressed as $(p?;a)+(\neg p?;b)$, while "**while** p **do** a" can be written $(p?;a)^*;\neg p$. The program $a^{\smile}$ "runs a backwards;" a may be deterministic without $a^{\smile}$ being deterministic, e.g. the deterministic program that replaces x by its square when run backwards nondeterministically replaces x by one of its square roots.

5.3 Merits of Decidability

I would like to pass judgment on the value of decidability in mathematical theories. One may with considerable precedent take the position that undecidability, if not lack of a finite basis, is necessary in any theory rich enough to serve as a foundation of mathematics [TG87]. After all is not all of mathematics founded on Zermelo-Fränkel set theory, evidently an undecidable theory?

I would like here to question the inevitability of undecidability.

First, if we really did need a universal theory to serve as a foundation for mathematics I would grant that such a theory should be undecidable. I question however the premise that a universal theory is needed in the first place. The link between mathematics and foundations seems more potential than actual. That is, it is a tenet of faith that "conventional" mathematical proofs can be expanded out to a purely set theoretic argument, yet this is almost never done. Moreover category theory has in recent years posed a challenge to set theory as an alternative and strikingly different foundation, indicating the nonuniqueness of such expansions. The possibility then arises that no such foundation is needed. Instead we may consider any given argument as being conducted in one or more relatively small and localized theories.

Second, the purpose of theory is to organize thought, not to drown it, to be constructive without being oracular. There is something of a movement in programming to make programs more like proof systems, and computations more like proofs. Coming in the other direction there is similar enthusiasm in logic for making proof systems more constructive, and proofs more like computations. There is however the distinct possibility that the two movements will rush right past each other and find that they have merely switched places!

Instead of founding mathematics on a single theory such as ZF or RA, why not view mathematics as a large collection of domain-specific theories? The whole of mathematics founded on a single small theory may have the real estate advantages of an inverted pyramid but it also has its structural disadvantages.

I propose that the proper notions of constructivity in a logic are its computational complexity and its human surveyability. These elements should be present in proportions suited to the application, mainly the former for a mechanical theorem prover, mainly the latter for computer aided instruction, and in more even proportions for a mathematician's mechanical apprentice.

This then speaks for computational tractability as an important criterion for judging the merits of any theory. If there is any distinction at all to be made between computation and logic it may well be the respective thresholds of polynomial time and exponential time as criteria for tractability!

5.4 Induction as Termination at Convergence

Kozen's original definition of dynamic algebra included $*$-continuity as a condition. The terminology subsequently standardized on by Kozen and myself is "dynamic algebra" for the equational class with the term "$*$-continuous" added to denote the condition that $\langle a^*\rangle p = p \vee \langle a\rangle p \vee \langle a\rangle\langle a\rangle p \vee \ldots = \bigvee_{i<\omega} a^i; p$.

Kozen [Koz81], p.175. argues for the practical value of the additional $*$-continuity condition as follows. "Looping is inherently infinitary and nonequational; ... Thus the equational approach must eventually be given up if we are ever to bridge the gap between algebraic and operational semantics." To my recent query as to whether he still held this view he replied, "Strictly speaking, no. Practically speaking, yes." I would like to take this opportunity to offer my position on the relative appropriateness of the two definitions of star.

The conditions differ only on the question of when iteration terminates. Under the equational definition, iteration terminates at convergence, namely at q satisfying $\langle a\rangle q \leq q$, whilst under the stronger $*$-continuity condition it terminates at ω.

I prefer convergence rather than ω as the place to stop because ω is not first-order definable (in the same sense that one may say that finiteness is not first-order definable), it is uneconomic for short iterations, it is needlessly restrictive for long iterations, and it is a potential Achilles heel for nonclassical dynamic logics.

While the class of dynamic algebras based on termination at convergence forms a quasivariety, that based on termination at ω is not even first-order-definable. Just as one cuts hair at different lengths for better appearance, and opens electrical circuits slowly for less electrical noise, so should one terminate iteration at convergence to achieve the tameness of a quasivariety rather than always exactly at ω, which goes beyond first-order logic.

In the classical formulation of dynamic logic adhered to in this paper, the economic argument reduces to an esthetic quibble, there being no charge for "gedanken-iterations." In nonclassical frameworks however $*$-continuity may prove unsound. I have no feeling for whether $*$-continuity will prove compatible with intuitionistic dynamic logic, but its nonconservatism seems quite opposed to, and hence likely to be unsound for, the explicit conservatism of (intuitionistic) linear [Gir87] dynamic logic.

I see no point in banning iteration beyond ω, in theory or in practice. The place in mathematics of iteration beyond ω has long since been secured. Computer science has clung more recently than mathematics to the superstition that all its practically accessible objects are finite. However any questions as to the worldly meaning of iteration beyond ω in computation have surely been dispelled by now by such applications as Manna and Dershowitz's multiset orderings, involving iteration up to ϵ_0, and Schwichtenberg's use of infinite ordinals to give an elegant short description of certain rapidly growing functions and hence very large numbers. Today's high level programming languages cannot afford to maintain the fiction that there is no iteration after ω. If the compiler's target language proscribes iteration beyond ω, it should be the compiler's duty to shield the programmer from this low-level restriction.

5.5 Intensionality

Extensionality occasionally gets in the way, and **SDA** would appear to constitute a simple but well-motivated example of this phenomenon. The Kleenean elements of an SDA are extensional in that separability connotes extensionality. Yet in the passage from **SDA** to **IDA** extensionality is lost.

If Kleenean elements were Gödel numbers then we would obtain intensionality as a basic property of acceptable Gödel numberings, that there is no bijective Gödel numbering of partial recursive functions. But any such connection with Gödel numbering or effectiveness can only be made via the computer science origins of dynamic algebra, not via its inherent structure, nothing in which hints of such a connection. We have only an abstract algebra of programs combined with imperative control structures, with no reason to suppose that the programs are not distinct partial recursive functions, or at least relations.

Instead intensionality arises here despite our efforts to achieve extensionality, when we pass from **SDA** to the quasivariety **IDA** by taking subalgebras.

Almost the same transition is made by Kozen [Koz81] when he passes from separable to inherently separable dynamic algebras, where the essential advantages of extensionality are preserved without preserving the extensionality itself. Kozen defines an inherently separable dynamic algebra as one sharing its Kleenean component with a separable dynamic algebra. The notion of intensional dynamic algebra introduced in this paper makes it in effect an inherently separable dynamic algebra for which the sharing is mediated via an inclusion between the two Boolean algebras, this being a roundabout way of describing a subalgebra of a separable dynamic algebra. Clearly intensional implies inherently separable for dynamic algebras, but I do not know whether the converse holds; if it does the "almost" at the start of this paragraph may be removed.

5.6 Origins of Dynamic Algebra

My adoption of universal algebra in 1979 represented for me the transfer to logic of a principle I previously understood only as a programmer. Unsound logic, meaning a discrepancy between a theory T and a class of (real or fictitious) worlds W, is in programming terms a bug. The connection between programming and logic can be made by substituting abstract program for theory, concrete program for proof system, computation for proof, and instruction step for proof step, leaving the notion of model unchanged.

Whereas a beginning programmer fixes a bug by fixing the program, experience teaches the principle that fixing the world is also an option. Unix for example was perceived by some in the early 1980's as a bug in the world of operating systems. This bug has been fixed, or at least attenuated, by adapting both Unix and the world to each other.

So it goes with logic. Given a proof system denoting a theory T and a class W of typical worlds, to prove T complete for W, an amateur logician such as myself would think only to compare T to other T's while holding W fixed. The first completeness proof [Par78] of Segerberg's axiomatization [Seg77] of PDL however proceeded by changing both, reflecting Parikh's extensive logical background.

Failing to understand Parikh's proof, and also wanting to understand the relationship of the proof theory to the problem of deciding PDL theoremhood, I undertook to find a structure that would work for me both as an understandable completeness proof and a decision method. This resulted in [Pra78], extensively revised as [Pra80b]. The former axiomatized dynamic logic in the language of Gentzen sequents, the latter extended this approach to a theory whose atomic formulas were $u \models p$ meaning "state u satisfies proposition p" and $u\langle a\rangle v$ meaning "from state u program a can halt in state v." In these proofs I held Kripke structures themselves to be the only models and all the rest as various proof systems of varying distances from Kripke structures, with completeness proved for the remoter ones via those closer to Kripke structures where completeness was more obvious.

It was at about this time in 1978 that Dexter Kozen conceived the notion of dynamic algebra. He mentioned the concept to me, not by name that I recall, towards the end of 1978 when I visited IBM Yorktown Heights, and said there might be a representation theorem there. At the time I saw no connection with algorithms and completeness proofs for dynamic logic. I had no idea then of the role played by representation theorems in completeness proofs that work with many W's and one T.

At STOC-79 in April I recounted an intriguing equational derivation to a group of about eight dynamic logic enthusiasts, without however being able to formulate an associated theorem. (It turned out to be the proof that the inductive definition of star entailed the definition as local reflexive transitive closure.) Thinking that the derivation might play a role in an algebraic completeness proof for an equational theory, but never having seen an equational completeness proof before, I asked Janos Makowsky's advice. He recommended Henkin's paper on the logic of equality, a pedagogically drawn-out universal algebra proof that the equational theory of the monoid of natural numbers is completely axiomatized by the theory of commutative monoids. I found it the perfect Rosetta stone for learning how to translate syntactic insights about proofs into semantic ones.

This led to my formulation and submission to FOCS-79 in early May of a semantic proof that half a dozen models of programming logic were all completely axiomatized by the equational theory of PDL [Pra79b]. The proof proceeded by showing that any algebra of those classes could be constructed from the algebras of a neighboring class by homomorphisms, subalgebras, and direct products, thereby establishing a strongly connected graph of inclusions between the equational theories of the classes. One of the classes consisted of the models of Segerberg's axioms, which I then called Hoare algebras. That this semantic proof method is complete is an immediate corollary of Birkhoff's theorem [Bir35] that every class closed under homomorphisms, subalgebras, and direct products forms a variety.

The one step in this proof that I did not supply was the inclusion $\Theta_{FKRI} \subseteq \Theta_{DA}$, FKRI denoting the class of finite Kripke structures. This was not a lacuna in the proof since it is the statement of completeness of the Segerberg axioms, which by then had no shortage of published proofs. Nevertheless I thought it would be nice to make the whole proof purely algebraic by finding the proper HSP formulation of the completeness proof. This however I was unable to do in time for the FOCS-79 deadline.

Shortly after submitting that paper I received a manuscript from Dexter Kozen [Koz79b]. It gave the full details of Kozen's dynamic algebra (a term I subsequently adopted for the proceedings version of my paper). Kozen's notion was the same as mine in most respects. The biggest difference from mine was that Kozen, like Brink, modeled his definition on that of an R-module over a ring R, where the notion of a ring is presumed to be given *a priori*. Thus Kozen took K to be what amounted to a semiring with star satisfying $a^* = \bigvee_{i<\omega} a^i$, with $a; b^* = \bigvee a; b^i$ and $a^*; b = \bigvee a^i; b$. In contrast my definition satisfied *no* Kleenean equations. Since that paper was about completeness of the Boolean theory the missing Kleenean theory presented no problem.

This led two months later to my proof [Pra79a] that every free separable dynamic algebra was a subdirect product of finite separable, hence representable, dynamic algebras. I later rewrote this to reduce the length of the proof proper to only half a proceedings page [Pra80a], at which point I felt I had a good grip on why Segerberg's axioms were complete.

Further reflection on the meaning of Segerberg's induction axiom led me to propose a formulation of the least-fixpoint or μ-calculus for Boolean modules [Pra81]. This would appear to be the first time that the notions of least fixpoint and Boolean module were brought together. This juxtaposition has since enjoyed considerable attention from the computer science community, most notably in its expression as Kozen's $L\mu$ calculus [KP83].

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References

[Bir35] G. Birkhoff. On the structure of abstract algebras. *Proc. Cambridge Phil. Soc*, 31:433–454, 1935.

[Bir67] G. Birkhoff. *Lattice Theory*, volume 25. A.M.S. Colloq. Publications, 1967.

[Bri81] C. Brink. Boolean modules. *J. of Algebra*, 71:291–313, 1981.

[CCMP91] R.T Casley, R.F. Crew, J. Meseguer, and V.R. Pratt. Temporal structures. *Math. Structures in Comp. Sci.*, 1(2):179–213, July 1991.

[Con71] J.H. Conway. *Regular Algebra and Finite Machines.* Chapman and Hall, London, 1971.

[CT51] L.H. Chin and A. Tarski. Distributive and modular laws in the arithmetic of relation algebras. *Univ. Calif. Publ. Math.*, 1:341–384, 1951.

[dBdR72] J.W. de Bakker and W.P. de Roever. A calculus for recursive program schemes. In M. Nivat, editor, *Automata, Languages and Programming*, pages 167–196. North Holland, 1972.

[Ded01] R. Dedekind. *Essays on the Theory of Numbers*. Open Court Publishing Company, 1901. Translation by W.W. Beman of *Stetigkeit und irrationale Zahlen* (1872) and *Was sind und was sollen die Zahlen?* (1888), reprinted 1963 by Dover Press.

[Dij76] E.W. Dijkstra. *A Discipline of Programming*. Prentice-Hall, Englewood Cliffs, N.J., 1976.

[Dil39] R.P. Dilworth. Noncommutative residuated lattices. *Trans. AMS*, 46:426–444, 1939.

[DM60] A. De Morgan. On the syllogism, no. IV, and on the logic of relations. *Trans. Cambridge Phil. Soc.*, 10:331–358, 1860.

[Ekl77] P.C. Eklof. Ultraproducts for algebraists. In *Handbook of Mathematical Logic*, pages 105–137. North Holland, 1977.

[FL79] M.J Fischer and R.E. Ladner. Propositional dynamic logic of regular programs. *JCSS*, 18(2), 1979.

[Fuc63] L. Fuchs. *Partially Ordered Algebraic Systems*. Pergamon Press, 1963.

[Gir87] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[HU79] J.E. Hopcroft and J.D. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley, 1979.

[Jón82] B. Jónsson. Varieties of relation algebras. *Algebra Universalis*, 15:273–298, 1982.

[JT48] B. Jónsson and A. Tarski. Representation problems for relation algebras. *Bull. Amer. Math. Soc.*, 54:80,1192, 1948.

[JT51] B. Jónsson and A. Tarski. Boolean algebras with operators. Part I. *Amer. J. Math.*, 73:891–939, 1951.

[JT52] B. Jónsson and A. Tarski. Boolean algebras with operators. Part II. *Amer. J. Math.*, 74:127–162, 1952.

[Kle56] S.C. Kleene. Representation of events in nerve nets and finite automata. In *Automata Studies*, pages 3–42. Princeton University Press, Princeton, NJ, 1956.

[Koz79a] D. Kozen. On the duality of dynamic algebras and Kripke models. In E. Engeler, editor, *Proc. Workshop on Logic of Programs 1979, LNCS 125*, pages 1–11. Springer-Verlag, 1979.

[Koz79b] D. Kozen. A representation theorem for models of $*$-free PDL. Manuscript, May 1979.

[Koz79c] D. Kozen. A representation theorem for models of $*$-free PDL. Technical Report RC7864, IBM, September 1979.

[Koz80] D. Kozen. A representation theorem for models of *-free PDL. In *Proc. 7th Colloq. on Automata, Languages, and Programming*, pages 351–362, July 1980.

[Koz81] D. Kozen. On induction vs. *-continuity. In D. Kozen, editor, *Proc. Workshop on Logics of Programs 1981, LNCS 131*, pages 167–176. Spring-Verlag, 1981.

[KP83] D. Kozen and R. Parikh. A decision procedure for the propositional μ-calculus. In E. Clarke and Kozen D., editors, *Proc. Workshop on Logics of Programs 1983, LNCS 164*, pages 313–325. Springer-Verlag, 1983.

[KT89] D. Kozen and J. Tiuryn. Logics of programs. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*. North Holland, Amsterdam, 1989.

[Lyn50] R.C. Lyndon. The representation of relational algebras. *Ann. of Math., Ser 2*, 51:707–729, 1950.

[Mac71] S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, 1971.

[Mak64] M. Makkai. On PC_Δ classes in the theory of models. *Publications of Math. Inst. Hung. Acad. Sci.*, 9:159–194, 1964.

[Mon64] J.D. Monk. On representable relation algebras. *Michigan Math. J.*, 11:207–210, 1964.

[Ném82] I. Németi. Every free algebra in the variety generated by the representable dynamic algebras is separable and representable. *Theoretical Computer Science*, 17:343–347, 1982.

[Ng84] K.C. Ng. *Relation Algebras with Transitive Closure*. PhD thesis, University of California, Berkeley, 1984. 157+iv pp.

[NT77] K.C. Ng and A. Tarski. Relation algebras with transitive closure, Abstract 742-02-09. *Notices Amer. Math. Soc.*, 24:A29–A30, 1977.

[Par78] R. Parikh. A completeness result for a propositional dynamic logic. In *LNCS 64*, pages 403–415. Springer-Verlag, 1978.

[Pei33] C.S. Peirce. Description of a notation for the logic of relatives, resulting from an amplification of the conceptions of Boole's calculus of logic. In *Collected Papers of Charles Sanders Peirce. III. Exact Logic*. Harvard University Press, 1933.

[Pra76] V.R. Pratt. Semantical considerations on Floyd-Hoare logic. In *Proc. 17th Ann. IEEE Symp. on Foundations of Comp. Sci.*, pages 109–121, October 1976.

[Pra78] V.R. Pratt. A practical decision method for propositional dynamic logic. In *Proc. 10th Ann. ACM Symp. on Theory of Computing*, pages 326–337, San Diego, May 1978.

[Pra79a] V.R. Pratt. Dynamic algebras: Examples, constructions, applications. Technical Report MIT/LCS/TM-138, M.I.T. Laboratory for Computer Science, July 1979.

[Pra79b] V.R. Pratt. Models of program logics. In *20th Symposium on foundations of Computer Science*, San Juan, October 1979.

[Pra80a] V.R. Pratt. Dynamic algebras and the nature of induction. In *12th ACM Symposium on Theory of Computation*, Los Angeles, April 1980.

[Pra80b] V.R. Pratt. A near optimal method for reasoning about action. *Journal of Computer and System Sciences*, 2:231–254, April 1980. Also MIT/LCS/TM-113, M.I.T., Sept. 1978.

[Pra81] V.R. Pratt. A decidable mu-calculus. In *Proc. 22nd IEEE Conference on Foundations of Computer Science*, pages 421–427, October 1981.

[Pra86] V.R. Pratt. Modeling concurrency with partial orders. *Int. J. of Parallel Programming*, 15(1):33–71, February 1986.

[Red64] V.N. Redko. On defining relations for the algebra of regular events (Russian). *Ukrain. Mat. Z.*, 16:120–126, 1964.

[Sch95] E. Schröder. *Vorlesungen über die Algebra der Logik (Exakte Logik). Dritter Band: Algebra und Logik der Relative*. B.G. Teubner, Leipzig, 1895.

[Seg77] K. Segerberg. A completeness theorem in the modal logic of programs. *Notices of the AMS*, 24(6):A–552, October 1977.

[Sto36] M. Stone. The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40:37–111, 1936.

[Tar41] A. Tarski. On the calculus of relations. *J. Symbolic Logic*, 6:73–89, 1941.

[Tar55] A. Tarski. Contributions to the theory of models. III. *Indag. Math*, 17:56–64, 1955.

[TG87] A. Tarski and S. Givant. *A Formalization of Set Theory Without Variables*. American Math. Soc., 1987.

[VG88] A. Van Gelder. A satisfiability tester for non-clausal propositional calculus. *Information and Computation*, 79(1), October 1988.

[WD39] M. Ward and R.P. Dilworth. Residuated lattices. *Trans. AMS*, 45:335–354, 1939.