Homework 4: IERG 6300

Due date: March 1, 2022.

Some useful definitions

Definition 1. Mutual independence of events: We say events $A_1, ..., A_n$ to be mutually independent if for all $1 \leq i_1 < \cdots i_k \leq n$ we have $\prod_{l=1}^{k} P(A_{i_l}) = P(\cap_l A_{i_l})$.

Exercises

1. Construct an example of three events such that $P(A \cap B \cap C) = P(A)P(B)P(C)$, but the events are not mutually independent.

2. Let $\{X_i\}$ be an *uniformly-integrable* zero-mean sequence of independent random variables. Define $S_n := X_1 + \cdots + X_n$. Show that

$$\phi_{\frac{S_n}{n}}(t) \to 1 \quad \forall t,$$

   and, using Levy's continuity theorem about weak convergence, deduce that $\frac{S_n}{n}$ converges in measure to 0, the constant random variable.

3. Let X be a random variable that takes countable infinite many values with positive probability, i.e. $P(X = i) = p_i > 0 \quad \forall i \in \mathbb{N}$. Let $X_1, ..., X_n, \ldots$ be independent random variables distributed identically to X. Let D_n be the number of *distinct* elements seen in the first n observations $X_1, .., X_n$. Show that

   (a) $D_n \to \infty \quad a.s.$

   (b) $\frac{1}{n}E(D_n) \to 0$, and hence deduce that $\frac{D_n}{n} \to 0$ in measure.

4. Let $X_1, .., X_n, ..$ be a sequence of mutually independent and identically distributed $U[0, 1]$ random variables (i.e. uniformly distributed on $[0, 1]$). Let $B_1 = 1$ and for $i \geq 2$ let $B_i = 1_{X_i > \max\{X_1, .., X_{i-1}\}}$. Further

let $R_n = \sum_{i=1}^n B_i$, it denotes the number of times the previous highest was beaten.

(a) Argue that B_i's are mutually independent Bernoulli random variables with $\mathrm{P}(B_i = 1) = \frac{1}{i}$.

(b) Show that $\frac{b_n}{\log n} \to 1$ where $b_n = \mathrm{var}(R_n)$.

(c) Show that Lindeberg's CLT applies to $X_{n,k} := \frac{1}{\sqrt{\log n}}(B_k - \frac{1}{k}), n \geq 2, 1 \leq k \leq n$.

(d) Argue that $\frac{R_n - \log n}{\sqrt{\log n}} \to G$, where G is the standard Gaussian.