CONNECTED LIE GROUPS AND PROPERTY RD

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Abstract
For a locally compact group, the property of rapid decay (property RD) gives a control on the convolutor norm of any compactly supported function in terms of its L^2-norm and the diameter of its support. We characterize the Lie groups that have property RD.

0. Introduction

The property of rapid decay (property RD) emerged from the work of U. Haagerup in [15] and was first studied systematically by P. Jolissaint in [21], mostly in the context of finitely generated groups. Property RD gives a control on the convolutor norm of any compactly supported function in terms of its L^2-norm and the diameter of its support. Before Haagerup's work, C. Herz stated and proved in [17, Théorème 1] that connected semisimple real Lie groups with finite center have property RD. (Of course, he did not use this terminology.) The terminology *rapid decay* comes from the fact that a group has property RD if and only if any *rapidly decaying* function is an L^2-convolutor (see Definition 2.3 and Lemma 2.4). Property RD is useful in the theory of C*-algebras. Connes and Moscovici used it in [7] to prove the Novikov conjecture for word hyperbolic groups, and V. Lafforgue used it in [25] to prove the Baum-Connes conjecture for some groups having property (T). Property RD is also relevant to the study of random walks on nonamenable groups. This is used in Section 7 to relate property RD to Varopoulos's work [37] and developed further in [6].

The main result of this article is a precise algebraic description of those connected (real) Lie groups that have property RD.

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THEOREM 0.1 (Main theorem)
Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and universal cover $\widetilde{G}$. The following are equivalent:

(a) *G has property RD;*

(b) *$\mathfrak{g} = \mathfrak{s} \times \mathfrak{q}$, where $\mathfrak{s}$ is semisimple and $\mathfrak{q}$ is an algebra of type R;*

(c) *$\widetilde{G} = \widetilde{S} \times \widetilde{Q}$, where the connected Lie groups $\widetilde{S}$ and $\widetilde{Q}$ are, respectively, semisimple and of polynomial volume growth.*

This result is extended in Theorem 7.4 to compactly generated almost connected groups. The equivalence between Theorem 0.1(b) and (c) is well known (see, e.g., [14], [19], [36]). That Theorem 0.1(a) implies (b) follows from Varopoulos's work in [37]; this is explained in Section 7. That Theorem 0.1(c) implies (a) occupies a large portion of this article. A short description of the article is as follows. Notation is set in Section 1. Sections 2 and 3 discuss property RD in the context of locally compact groups (see also [21] and [20]). In Section 4 we consider a locally compact unimodular group $G = PK$, where P is amenable and K is compact. Theorem 4.4 gives a necessary and sufficient condition for property RD in terms of the growth of an elementary spherical function of G. Theorem 4.4, together with a fundamental estimate due to Harish-Chandra, implies property RD for semisimple Lie groups with finite center and for semisimple k-groups over a local field k. Section 5 establishes the stability of property RD under some central extensions, a result proved by Jolissaint [21] in the case of finitely generated groups. Section 6 shows that, referring to Theorem 0.1, (c) implies (a).

1. Basic notation

Throughout this article all groups are locally compact, Hausdorff, and separable. Let $\mathscr{C}_c(G)$ denote the algebra of all continuous functions on the group G with values in $\mathbb{C}$ and with compact support. Let ν be a left Haar measure on G (unique up to a multiplicative constant), and define the modular function m by $\nu(Bg) = m(g)\nu(B)$ (for any Borel set B and $g \in G$). Let $L^2(G)$ be the Hilbert space $L^2(G, \nu)$ equipped with the inner product $\langle f, g\rangle = \int_G f(x)\overline{g(x)}\,dx$.

1.1. Convolutions

For details on the following, see [18, Chapter V] and [11, Paragraphe XIII]. For $f, g \in L^1(G)$, the convolution $f * g \in L^1(G)$ is defined as

$$f * g(x) = \int_G f(xy)g(y^{-1})\,dy = \int_G f(y)g(y^{-1}x)\,dy.$$

Let

$$f^*(x) = \frac{1}{m}(x)\overline{f(x^{-1})}$$

be the canonical isometric involution of $L^1(G)$. Let $B(L^2(G))$ be the involutive Banach algebra of bounded operators on $L^2(G)$. Recall that if $f \in L^1(G)$ and $g \in L^2(G)$, the inequality $\|f * g\|_2 \leq \|f\|_1\|g\|_2$ shows that left convolution by f defines a bounded operator $\lambda_G(f)$ on $L^2(G)$ and that the representation

$$\lambda_G : L^1(G) \to B\big(L^2(G)\big)$$

is a $*$-homomorphism; that is, $\lambda_G(f^*) = \lambda_G(f)^*$. (We write λ in place of λ_G whenever no confusions can arise.) The formula

$$\iota(f)(x) = m(x)^{-1/2} f(x^{-1})$$

defines an automorphism of $L^2(G)$ which is an isometric involution. If f is a measurable function on G such that

$$\int_G m(y)^{-1/2}|f(y)|\, dy < \infty,$$

write $\rho_G(f)$ or $\rho(f)$ for the right convolution by f. It is easy to check that

$$\rho(f) = \iota \circ \lambda\big(\iota(f)\big) \circ \iota. \tag{1.1}$$

This is an element of $B(L^2(G))$ because

$$\|\rho(f)\|_{2\to 2} = \big\|\lambda\big(\iota(f)\big)\big\|_{2\to 2} \leq \|\iota(f)\|_1 = \int_G m(y)^{-1/2}|f(y)|\, dy < \infty. \tag{1.2}$$

The right convolution should not be confused with the extension to $L^1(G)$ of the unitary right regular representation on $L^2(G)$,

$$\big(\rho(t)f\big)(x) = m(x)^{1/2} f(xt), \quad t \in G,\ f \in L^2(G),$$

of G on $L^2(G, \nu)$.

1.2. Length functions

A *length function* on a locally compact group G is a Borel map $L : G \to \mathbb{R}^+$ satisfying $L(1) = 0$, $L(gh) \leq L(g) + L(h)$, and $L(g) = L(g^{-1})$, $g, h \in G$. We set $B_L(r) = \{g \in G : L(g) \leq r\}$. A length function L on G is locally bounded if, for any compact set U, $M_U = \sup\{L(u) : u \in U\} < \infty$. If G contains a compact set K

satisfying $G = \bigcup_{n\in\mathbb{N}} K^n$, we say that G is compactly generated. If G is compactly generated and K is a compact symmetric generating set, the word length on G associated to K is defined as

$$L_K(g) = \inf\{n : g \in K^n\}, \qquad L_K(1) = 0.$$

In the first paragraph of Sec. 1.2, please identify what "11" in [40] represents.

Such length functions are locally bounded. Indeed, note that there exists an integer N such that K^N has positive Haar measure (otherwise, G would have measure 0) and thus $K^N K^{-N} = K^{2N}$ is a neighborhood of the identity (see, e.g., [40, Chapter III, 11]). It follows that the interior of K^{2N+1} generates G. This implies that L_K is locally bounded.

Fix a locally bounded length function L, and fix a compact symmetric generating set K. For $g = s_1 \cdots s_n$ with $s_i \in K$ and n minimal, we have

$$L(g) = L(s_1 \cdots s_n) \le \sum_{i=1}^{n} L(s_i) \le M_K L_K(g).$$

Hence, word-length functions are all comparable to each other (we sometimes talk about "the" word length without specifying K), and they are, in a sense, the largest locally bounded length functions on G. The formula

$$d(g,h) = L_K(g^{-1}h)$$

defines a metric (with integer values) on G, and the action of G on itself by left translations is free and isometric. Further examples of length functions (with real values) can be obtained by letting G act continuously by isometries on a metric space (X,d) and by setting $L(g) = d(x_0, g(x_0))$ for some base point $x_0 \in X$. Such length functions can be very different from word lengths and are not always proper.

If G is a connected Lie group, any left-invariant Riemannian metric induces a locally bounded length function on G by letting $L(g)$ be the geodesic distance between g and the identity element. If L is such a Riemannian length function and K is as in the previous paragraph, then there are constants $c, C \in (0,\infty)$ such that $cL_K(g) \le L(g) \le CL_K(g)$ for all for g outside a large enough compact neighborhood of the identity. In words, at large scale, Riemannian and word-length functions are always comparable on G (see, e.g., [39]).

A compactly generated group has polynomial volume growth if, for any compact symmetric generating set K, there exist $C, D > 0$ such that $\nu(B_{L_K}(r)) \le Cr^D$ for all $r \ge 1$.

2. Property RD

In what follows, by support of a measurable function, we mean its essential support.

Definition 2.1
Let L be a locally bounded length function on G. Let E be a subset of $L^2(G)$. We say that the pair (G, L) has *property* RD_E if there exist two constants $C, D \geq 0$ such that for any function $f \in E$ with compact support in $B_L(R)$, $R \geq 1$, we have

$$\|\lambda(f)\|_{2\to 2} \leq CR^D \|f\|_2. \tag{2.3}$$

For simplicity, when $E = L^2(G)$, we write RD for $\mathrm{RD}_{L^2(G)}$.

Definition 2.2
Let G be compactly generated. We say that G has *property RD* if (G, L_K) has property RD for some (equivalently, any) compact symmetric generating set K.

Similar definitions with $\lambda(f)$ replaced with $\rho(f)$ lead to the same concepts. This can be proved directly using (1.1) or deduced from (1.1) and [20, Theorem 2.2], where Ji and Schweitzer prove that property RD implies unimodularity.

Definition 2.3
Let L be a length function on a locally compact group G. For $k \geq 0$, define

$$H_L^k(G) = \Big\{ f \in L^2(G) : \int_G \big(1 + L(x)\big)^{2k} |f(x)|^2 \, dx < \infty \Big\},$$

and define $H_L^\infty(G) = \bigcap_{k\geq 0} H_L^k(G)$. The space $H_L^\infty(G)$ is called the *space of rapidly decaying functions*.

The space $H_L^\infty(G) \subseteq L^2(G)$ is a Fréchet space for the projective limit topology induced by the sequence of norms $\|f\|_{2,L,k} = \|(1+L)^k f\|_2$. Recall that the *reduced C^*-algebra* $C_r^*(G)$ of a locally compact group G is the operator norm closure of compactly supported continuous functions on G, viewed as acting on $L^2(G)$ via the left regular representation (i.e., as $\lambda(f)$, where $f \in \mathscr{C}_c(G)$). In the following lemma, we collect equivalent definitions of property RD. In particular, it implies that Definition 2.1 of property RD coincides with the one given by Jolissaint in [21] and used by Ji and Schweitzer in [20].

LEMMA 2.4
Let G be a locally compact group, and let L be a locally bounded length function on G. The following are equivalent:
(1) *(G, L) has property RD;*
(2) *(G, L) has property RD_E for $E = \mathscr{C}_c(G)$;*
(3) *(G, L) has property RD_E for $E = \{f \in \mathscr{C}_c(G) : f = mf^*\}$;*

(4) *there are* $k > 0$ *and* $C > 1$ *such that, for any* $f \in \mathscr{C}_c(G)$,

$$\|\lambda(f)\|_{2\to 2} \leq C\|(1+L)^k f\|_2;$$

(5) $H_L^\infty(G) \subseteq C_r^*(G)$.

The proofs are elementary or easily adapted from the literature.

3. Elementary stability results

Property RD is not stable under arbitrary extensions. (Abelian groups have property RD, but not all solvable groups have it; see Proposition 4.1.) The next result says that property RD is stable under direct products (for some central extensions, see Proposition 5.5).

LEMMA 3.1
Let G_1, G_2 *be compactly generated groups equipped with length functions* L_1, L_2. *Set* $G = G_1 \times G_2$, *and set* $L = L_1 + L_2$. *Then* (G, L) *has property RD if and only if* (G_1, L_1) *and* (G_2, L_2) *do.*

Proof
For $f \in L^2(G)$ compactly supported, define

$$f_1(x) = \Big(\int_{G_2} |f(x,y)|^2\, dy\Big)^{1/2} \in L^2(G_1).$$

Then $\|f\|_{L^2(G)} = \big(\int_{G_1} |f_1(x)|^2\, dx\big)^{1/2} = \|f_1\|_{L^2(G_1)}$. Now assume that G_i has property RD with constants C_i, D_i, $i = 1, 2$. Let $f \in L^2(G)$ be supported in the ball of radius $R > 1$ for the length $L = L_1 + L_2$. Fixing x_1, write

$$\begin{aligned}
&\int_{G_2} \Big| \int_{G_1\times G_2} f(y_1, y_2) g(y_1^{-1}x_1, y_2^{-1}x_2)\, dy_1\, dy_2 \Big|^2 dx_2 \\
&\leq \bigg(\int_{G_1} \Big(\int_{G_2} \Big| \int_{G_2} f(y_1, y_2) g(y_1^{-1}x_1, y_2^{-1}x_2)\, dy_2 \Big|^2 dx_2 \Big)^{1/2} dy_1 \bigg)^2 \\
&\leq C_2^2 R^{2D_2} \Big| \int_{G_1} f_1(y_1) g_1(y_1^{-1}x_1)\, dy_1 \Big|^2,
\end{aligned}$$

where the first inequality is due to Minkowsky (see [32, Theorem 3.29]) and the last inequality follows from property RD on (G_2, L_2). Since f_1 is supported in $B_{L_1}(R)$, integrating with respect to x_1 and using property RD on (G_1, L_1) yields

$$\|\lambda(f)(g)\|_{L^2(G)} \leq C_2 R^{D_2} \|\lambda(f_1)(g_1)\|_{L^2(G_1)} \leq C R^D \|f\|_{L^2(G)} \|g\|_{L^2(G)},$$

where $C = C_1 C_2$ and $D = D_1 + D_2$.

Conversely, assume that G has property RD. Let $f_1 \in \mathscr{C}_c(G_1)$ be supported in $B_{L_1}(R)$ with $R \geq 1$. Fix a compact neighborhood U of $1 \in G_2$, and let $M_U = \sup_U L_2$. Define $f \in L^2(G)$ by

$$\forall (y_1, y_2) \in G, \quad f(y_1, y_2) = f_1(y_1)\mathbf{1}_U(y_2),$$

where $\mathbf{1}_U$ denotes the characteristic function of U. As $\|\lambda(f)\|_{2\to 2} \geq \|\lambda(\mathbf{1}_U)\|_{2\to 2}\|\lambda(f_1)\|_{2\to 2}$ with $C_U = \|\lambda(\mathbf{1}_U)\|_{2\to 2} < \infty$, we obtain

$$\begin{aligned}\|\lambda(f_1)\|_{2\to 2} &\leq C_U^{-1}\|\lambda(f)\|_{2\to 2} \leq C_U^{-1}C(M_U + R)^D\|f\|_{L^2(G)}\\ &\leq C'R^D\|f_1\|_{L^2(G_1)},\end{aligned}$$

where C, D are the constants in property RD on G. □

Property RD is not closed under passing to general subgroups, but the next lemma shows that it passes to open subgroups.

LEMMA 3.2
Let (G, L) have property RD, and take $H < G$ to be an open subgroup. Then (H, L') has property RD, where L' is the length function L restricted to H.

Proof
Since H is open, the Haar measure on H is the restriction of the one on G. Let $f \in L^2(H)$ be supported on $B_{L'}(R)$ for some $R \geq 1$. Extend f to $\tilde{f} \in L^2(G)$ by setting $\tilde{f} = 0$ on $G \setminus H$, so that $\|f\|_{L^2(H)} = \|\tilde{f}\|_{L^2(G)}$ and $\tilde{f}$ is supported on $B_L(R)$. Then

$$\|\lambda(f)\|_{2\to 2} \leq \|\lambda(\tilde{f})\|_{2\to 2} \leq CR^D\|\tilde{f}\|_{L^2(G)} = CR^D\|f\|_{L^2(H)}.$$

This shows that (H, L') has property RD. □

Property RD is not always inherited by cocompact closed subgroups. (In fact, we see that a noncompact semisimple group with Iwasawa decomposition *NAK* has property RD, whereas *NA* does not). The following result is useful to treat almost connected groups.

LEMMA 3.3
Let G be a compactly generated group, and let H be a closed finite-index subgroup. Then G has property RD if and only if H does.

Proof
Since H has finite index in G, it is an open subgroup. Hence, if G has property RD, then so does H by Lemma 3.2.

Conversely, first note that if $f, g \in L^2(G)$ have disjoint compact supports contained in the ball of radius R with center the identity and satisfy (2.3) with constants C and D, then

$$\|\lambda(f+g)\|_{2\to 2} \le \sqrt{2}CR^D(\|f\|_2^2 + \|g\|_2^2)^{1/2} = \sqrt{2}CR^D\|f+g\|_2^2.$$

For $t \in G,\ f \in L^2(G)$, write $(\lambda(t)f)(x) = f(t^{-1}x)$ for the left regular representation. As we may assume that all groups are unimodular in the lemma, the right regular unitary representation on $L^2(G,\nu)$ is just $(\rho(t)f)(x) = f(xt)$. Hence,

$$\|\lambda(f)\|_{2\to 2} = \|\lambda(f)\circ\lambda(t^{-1})\|_{2\to 2} = \big\|\lambda\big(\rho(t)f\big)\big\|_{2\to 2}.$$

Also, $\operatorname{supp}(\rho(t)f) = \operatorname{supp}(f)t^{-1}$.

Hence, if we choose a set $T \subset G$ of right H-coset representatives (which is finite by hypothesis) and consider the orthogonal decomposition $L^2(G) = \bigoplus_{t\in T} L^2(Ht)$, we see that it is enough to prove the RD inequality for $\lambda(f)$ when $f \in L^2(H)$. But in this case, $\lambda(f)$ preserves each orthogonal subspace $L^2(Ht)$, and if $g \in L^2(Ht)$, then

$$\|\lambda_G(f)g\|_{L^2(G)} = \big\|\lambda_H(f|_H)\big(\rho(t)g\big)|_H\big\|_{L^2(H)}.$$

As $\|g\|_{L^2(G)} = \|(\rho(t)g)|_H\|_{L^2(H)}$, $\|f\|_{L^2(G)} = \|f|_H\|_{L^2(H)}$, and as the restriction to the finite-index subgroup H of a length function on G is bounded below by a fixed multiple of a length function on H, the proof is finished. □

The following lemma is used later to extend our results on connected Lie groups to almost connected compactly generated groups.

LEMMA 3.4
Let $1 \to K \to G \to Q \to 1$ be a short exact sequence of compactly generated groups, and assume that K is compact. Then G has property RD if and only if Q has property RD.

Before proving this lemma, we collect some facts that are used in the proof and again in Section 4. We start with a classical observation used, for instance, in the study of the Kunze-Stein phenomenon (see [8]). The proof is included for the convenience of the reader.

LEMMA 3.5
Let G be a locally compact group, and let K be a compact subgroup. Let $f \in \mathscr{C}_c(G)$. We set

$$f_K(x) = \Big(\int_K |f(xk)|^2 \, dk \Big)^{1/2}, \qquad {}_K f_K(x) = \Big(\int_K \int_K |f(kxk')|^2 \, dk \, dk' \Big)^{1/2},$$

where dk denotes the normalized Haar measure on K. Then $f_K, \ {}_K f_K \in L^2(G)$, $\|f_K\|_2 = \|{}_K f_K\|_2 = \|f\|_2$, and

$$\|\lambda(f)\|_{2\to 2} \le \|\lambda(f_K)\|_{2\to 2}, \qquad \|\lambda(f)\|_{2\to 2} \le \|\lambda({}_K f_K)\|_{2\to 2},$$
$$\|\rho(f)\|_{2\to 2} \le \|\rho(f_K)\|_{2\to 2}, \qquad \|\rho(f)\|_{2\to 2} \le \|\rho({}_K f_K)\|_{2\to 2}.$$

Proof
The equality of the norms of f_K, ${}_K f_K$, and f follows from the fact that $m(k) = 1$ for $k \in K$. (Indeed, $m(k^n) = m(k)^n$ is bounded and bounded away from zero for all $n \in \mathbb{N}$ since K is compact.) Concerning convolutor norms, the equality in (1.2), the fact that λ is a $*$-homomorphism, and the identity $(\iota(f)^*)_K = \iota(f_K)^*$ easily show that it suffices to treat left convolutors. For any function $\xi \in \mathscr{C}_c(G)$ and any $x \in G$, $k \in K$, we have

$$f * \xi(x) = \int_G f(xy)\xi(y^{-1}) \, dy = \int_G f(xyk)\xi(k^{-1}y^{-1}) \, dy.$$

Hence,

$$\begin{aligned} |f * \xi(x)| &= \Big| \int_K \int_G f(xyk)\xi(k^{-1}y^{-1}) \, dy \, dk \Big| \\ &\le \int_G \Big(\int_K |f(xyk)|^2 \, dk \Big)^{1/2} \Big(\int_K |\xi(ky^{-1})|^2 \, dk \Big)^{1/2} dy \\ &= \int_G f_K(xy)\xi_K(y^{-1}) \, dy = f_K * \xi_K(x). \end{aligned}$$

It follows that $\|\lambda(f)\|_{2\to 2} \le \|\lambda(f_K)\|_{2\to 2}$. Set

$${}_K f(x) = \Big(\int_K |f(kx)|^2 \, dk \Big)^{1/2},$$

and check that $(f^*)_K = ({}_K f)^*$. Hence, we obtain

$$\begin{aligned} \|\lambda(f)\|_{2\to 2} = \|\lambda(f^*)\|_{2\to 2} &\le \big\|\lambda\big((f^*)_K\big)\big\|_{2\to 2} = \big\|\lambda\big((f^*)_K^*\big)\big\|_{2\to 2} \\ &= \|\lambda({}_K f)\|_{2\to 2} \le \big\|\lambda\big(({}_K f)_K\big)\big\|_{2\to 2} = \|\lambda({}_K f_K)\|_{2\to 2}, \end{aligned}$$

as desired. □

We recall some elementary facts concerning operators on homogeneous spaces (for details, see, e.g., [33]). Let G be a locally compact group that acts continuously and transitively on a space X with compact stabilizers. Fix $o \in X$, and let K denote the stabilizer of o so that $X = G/K$. For $x \in X$, let $\bar{x}$ be an element of G so that $\bar{x}o = x$. Let $p(x, y)$ be a locally integrable nonnegative kernel that is G-invariant (i.e., $p(gx, gy) = p(x, y)$ for any $g \in G$). Let dx be the G-invariant measure on X so that $dg = dx\, dk$, where dk is the normalized Haar measure on K. Set

$$\phi(g) = p(go, o) = p(o, g^{-1}o).$$

Note that ϕ satisfies $\phi(gk) = \phi(kg) = \phi(g)$ for all $g \in G$, $k \in K$. One checks that the right convolution operator $\rho(\phi)$ realizes on G the operator $Pf(x) = \int_X p(x, y) f(y)\, dy$ defined on $\mathscr{C}_c(X)$. In particular, $\|P\|_{2\to 2} = \|\rho(\phi)\|_{2\to 2}$, and

$$\int_G |\phi(g)|^2\, dg = \int_X |p(x, o)|^2\, dx.$$

Given G, X, and p as above, if Q is another locally compact group that acts continuously and transitively on X with compact stabilizers and such that p is Q-invariant as well, we get right convolution operators $\rho_G(\phi^G)$ and $\rho_Q(\phi^Q)$ on G and Q, respectively, with

$$\|P\|_{2\to 2} = \|\rho_G(\phi^G)\|_{2\to 2} = \|\rho_Q(\phi^Q)\|_{2\to 2} \qquad \text{and} \qquad \|\phi^G\|_2 = \|\phi^Q\|_2. \tag{3.4}$$

Proof of Lemma 3.4
First, assume that G has property RD. Let A be a compact symmetric neighborhood of the identity generating Q. Let $f \in L^2(Q)$ be nonnegative, supported on $B(R)$ with $R \geq 1$. Let $\pi : G \to Q$ be the projection with kernel K in the short exact sequence of Lemma 3.4. Then $f \circ \pi$ has its support in $B(R+1)$ for the compact generating set $\pi^{-1}(A)$. Applying equality (3.4) yields

$$\|\rho(f)\|_{2\to 2} = \|\rho(f\circ\pi)\|_{2\to 2} \leq C(R+1)^D \|f\circ\pi\|_{L^2(G)} = C(R+1)^D \|f\|_{L^2(Q)}.$$

Conversely, assume that Q has property RD, and fix a compact generating set as above. Let $f \in L^2(G)$ be supported on $B(R) \subseteq G$ with $R \geq 1$. Define $f_K \in L^2(Q)$ as in Lemma 3.5, so that f_K is supported on $B(R) \subseteq Q$. Then

$$\|\rho(f)\|_{2\to 2} \leq \|\rho(f_K)\|_{2\to 2} \leq CR^D \|f_K\|_{L^2(Q)} = CR^D \|f\|_{L^2(G)}.$$

This shows that G has property RD. □

Remark 3.6
According to Jolissaint [21, Proposition A.3], if a compactly generated group G has a discrete cocompact subgroup with property RD, then G has property RD as well. For instance, Jolissaint proved that discrete groups acting properly and cocompactly on Riemannian manifolds with pinched negative sectional curvature have property RD. He deduced (in [21, Corollary A.4]) that $\mathrm{SL}_2(\mathbb{R})$, as well as any connected noncompact Lie group of real rank one and finite center, has property RD.

Remark 3.7
It is not known whether property RD passes to cocompact lattices, and this appears to be an interesting question. So far, only a few cocompact lattices in semisimple Lie groups are known to have property RD (see [31], [24], and [5]), and the methods used to establish property RD for those groups are quite different from what we do here for connected groups. A. Valette [35] conjectures that cocompact lattices in real and p-adic semisimple Lie groups have property RD.

4. Unimodular groups of type *PK*

In this section, we present a necessary and sufficient condition for property RD on unimodular groups of the form $G = PK$, where K is a compact subgroup and P is a closed and amenable subgroup. This condition involves the growth of the elementary spherical function ϕ_0 (see Theorem 4.4). If L is a locally bounded length function on the group G, rad denotes the space of all radial functions in $\mathscr{C}_c(G)$, that is, functions f such that $L(x) = L(y)$ implies $f(x) = f(y)$.

4.1. Amenability
Recall that a locally compact group G is amenable if and only if, for any nonnegative $f \in L^1(G)$,

$$\|\lambda(f)\|_{2\to 2} = \|f\|_1. \tag{4.5}$$

In particular, if G is amenable and $f \in L^1(G)$ is nonnegative such that $\int_G m(y)^{-1/2} f(y)\,dy < \infty$, we have

$$\|\rho(f)\|_{2\to 2} = \int_G m(y)^{-1/2} f(y)\,dy < \infty. \tag{4.6}$$

In fact, G is amenable if and only if (4.5) holds for one nonnegative $f \in L^1(G)$ with support S such that the closure of the subgroup generated by SS^{-1} is G (see, e.g., [26] and [2, Theorem 4]).

PROPOSITION 4.1 (Jolissaint [21, Corollary 3.1.8])

Let G be a locally compact, amenable group, and let L be a locally bounded length function on G. The following are equivalent:

(1) *there are constants $c, d > 0$ such that $\nu(B_L(n)) \leq cn^d$;*

(2) *(G, L) has property RD;*

(3) *(G, L) has property RD_{rad}.*

In particular, the only amenable groups with property RD are those with polynomial volume growth.

Proof

As G is amenable, by (4.5) we have

$$\|\lambda(f)\|_{2\to 2} = \int_G f(x)\,dx \leq \sqrt{\nu\big(\mathrm{supp}(f)\big)}\,\|f\|_2. \tag{4.7}$$

Thus (1) implies (2). Obviously, (2) implies (3). To see that (3) implies (1), apply property RD_{rad} to the radial function $f = \mathbf{1}_{B_L(R)}$, and use the equality in (4.7). □

4.2. Nonamenability and nonunimodularity

Throughout Section 4.2, let G be a unimodular locally compact group having two closed subgroups P and K such that $G = PK$. We assume that P is amenable and that K is compact.

PROPOSITION 4.2

Let f be a nonnegative integrable function on G such that $f(ks) = f(sk) = f(s)$ for all $k \in K$ and $s \in G$. Let $f|_P$ be the restriction of f to P. Then

$$\|\lambda_G(f)\|_{2\to 2} = \int_P m(y)^{-1/2} f|_P(y)\,dy, \tag{4.8}$$

where m is the modular function of P. Moreover, G is nonamenable if and only if P is nonunimodular.

Statements of this sort are folklore in the theory of semisimple Lie groups. The following short and completely elementary proof of (4.8) is given for the convenience of the reader.

Proof

We first prove (4.8). Let $X = G/K$ be equipped with the G-invariant measure dx such that $ds = dx\,dk$ (where ds is a Haar measure on $G\,dk$ and is the normalized Haar measure on K). Note that X can also be realized (as a measure space) as $X = P/P\cap K$.

Because of the bi-invariance of f under K, we can set

$$p(xK, yK) = f(y^{-1}x).$$

This kernel on X is invariant under the action of G (hence, also under the action of P). Thus, by (3.4), we have

$$\|\rho_G(f)\|_{2\to 2} = \|\rho_P(f|_P)\|_{2\to 2}. \tag{4.9}$$

Note that (4.9) only requires the compactness of K and the bi-invariance of f but neither the unimodularity of G nor the amenability of P. As G is unimodular, we have $\|\rho_G(f)\|_{2\to 2} = \|\lambda_G(f)\|_{2\to 2}$, and as P is amenable, $\|\rho_P(f|_P)\|_{2\to 2} = \int_P m(y)^{-1/2} f|_P(y)\,dy$ (see (4.6)). Equality (4.8) follows. By construction, we have

$$\int_P f|_P(y)\,dy = \int_G f(s)\,ds = \|f\|_1 \tag{4.10}$$

and, by the unimodularity of G,

$$\|f\|_1 = \int_G f(s^{-1})\,ds = \int_P f|_P(y^{-1})\,dy = \int_P m(y)^{-1} f|_P(y)\,dy. \tag{4.11}$$

Now, if P is unimodular, we get $\|\lambda_G(f)\|_{2\to 2} = \int_P f|_P(y)\,dy = \|f\|_1$, which shows that G is amenable. If P is not unimodular and f is such that m is not constant on the support of $f|_P$, then we get

$$\begin{aligned}\|\lambda(f)\|_{2\to 2} &= \int_P m(y)^{-1/2} f|_P(y)\,dy \\ &< \Big(\int_P f\Big|_P(y)\,dy \int_P m(y)^{-1} f\Big|_P(y)\,dy\Big)^{1/2} = \|f\|_1,\end{aligned}$$

where the last equality follows from (4.10) and (4.11). The characterization of amenability given by (4.5) shows that G is nonamenable. □

Formula (4.8) appears difficult to use directly for our purpose, and we need the following variation. The function $\phi(s) = m^{-1/2}(x)$ is well defined because of two independent facts. First, any $s \in G$ can be written as $s = xk$, $x \in P$, $k \in K$. Second, $P \cap K$ is compact. By definition, this function on G is right K-invariant. Set

$$\phi_0(s) = \int_K \phi(ks)\,dk, \quad s \in G. \tag{4.12}$$

PROPOSITION 4.3
For any nonnegative integrable function f on G such that $f(ks) = f(sk) = f(s)$, for all $k \in K$ and $s \in G$, we have

$$\|\lambda_G(f)\|_{2\to 2} = \int_G \phi_0(s) f(s)\, ds. \tag{4.13}$$

Proof
Using the K-bi-invariance of f, we have

$$\int_P m^{-1/2}(x) f|_P(x)\, dx = \int_G \phi(s) f(s)\, ds = \int_G \phi_0(s) f(s)\, ds.$$

Thus the desired equality follows from Proposition 4.2. □

THEOREM 4.4
Let G be a unimodular locally compact group having two closed subgroups P, K such that $G = PK$, P is amenable, and K is compact. Let L be a locally bounded length function on G. Then (G, L) has property RD if and only if there are constants $c, d \in (0, \infty)$ such that, for all $r \geq 1$,

$$\int_{B_L(r)} \phi_0^2(s)\, ds \leq c r^d.$$

Proof
Assume that $\int_{B_L(r)} \phi_0^2(s)\, ds \leq cr^d$ for all $r \geq 1$. Let $f \in L^2(G)$ be nonnegative, supported in $B_L(r)$, and K-bi-invariant. By Proposition 4.3, we have

$$\|\lambda(f)\|_{2\to 2} = \int_G \phi_0(s) f(s)\, ds \leq \|\phi_0 \mathbf{1}_{B_L(r)}\|_2 \|f\|_2 \leq c^{1/2} r^{d/2} \|f\|_2.$$

By Lemma 3.5, the same inequality holds without the hypothesis that f is K-bi-invariant, which proves that G has property RD. In order to prove the converse, let

$$\widetilde{L}(x) = \int_K \int_K L(kxk')\, dk\, dk',$$

and let $M = \sup_{k\in K} L(k)$. Notice that for all $r > 2M$,

$$B_L(r - 2M) \subseteq \widetilde{L}^{-1}([0, r]) \subseteq B_L(r + 2M).$$

The function $f = \phi_0 \mathbf{1}_{\widetilde{L}^{-1}([0,r])}$ is K-bi-invariant. Hence,

$$\|\lambda(f)\|_{2\to 2} = \int_G \phi_0(s) f(s)\, ds.$$

We conclude by applying the RD inequality to f. □

4.3. Semisimple groups with finite center

The first part of the following theorem is due to Herz [17, Théorème 1]. Herz's proof is different from ours in that it uses the dual viewpoint of matrix coefficients and a reduction modulo a parabolic subgroup P.

THEOREM 4.5

Connected semisimple real Lie groups with finite center have property RD. If k is a local field, the group of k-points of a connected linear algebraic semisimple group defined over k has property RD.

Proof

We treat the real case and then give the necessary references for the algebraic case. Let G be equipped with its canonical K-bi-invariant Riemannian metric, and let $G = NAK$ be an Iwasawa decomposition (see [16]). Theorem 4.4 applies with $P = NA$, which is amenable and nonunimodular (if nontrivial). Moreover, the function ϕ_0 defined at (4.12) is the elementary spherical function or Harish-Chandra function that is almost L^2 in the sense that for all $r \geq 1$, it satisfies

$$\int_{B(r)} |\phi_0(x)|^2 \, dx \leq Cr^\gamma, \tag{4.14}$$

where $\gamma = 2b + \ell = 2 \times \sharp\{\text{indivisible positive roots}\} + \dim(A)$ (see, e.g., [1], [9], or [23]).

For an algebraic semisimple group over a local field, we also have a decomposition $G = PK$ with the desired property (see [34, Section 0.6] and [27, Theorem 2.2.1(2)]). A version of (4.14) is given by [34, Lemma 4.2.5]. □

Remark. Let G be a noncompact semisimple Lie group, and let $G = NAK$ be an Iwasawa decomposition. The group $P = NA$ is amenable and not unimodular and thus can have neither property RD nor property $\mathrm{RD}_{\mathrm{rad}}$ with respect to any locally bounded length function (see Proposition 4.1). However, if L is the length function associated to the canonical Riemannian metric on G, the K-invariant functions on NA are precisely the L-radial functions, and any L-radial function $f \in L^2(NA)$ supported in $B_L(r)$ with $r \geq 1$ satisfies $\|\rho(f)\|_{2\to 2} \leq Cr^\gamma \|f\|_2$. Estimates derived in [29] show that Damek-Ricci NA groups also have this property, although they are not associated with a semisimple group.

5. Central extensions and property RD

The aim of this section is to establish the stability of property RD under central extensions having polynomially distorted center. This is a generalization to locally compact groups of [21, Proposition 2.1.9] for the case of central extensions. We start with the following general result.

PROPOSITION 5.1
Let $p : E \to G$ be a surjective homomorphism of compactly generated groups. There exists a Borel section $\sigma : G \to E$ of p which is locally bounded (i.e., if K is compact, then $\sigma(K)$ is relatively compact) and which is Lipschitz with respect to word lengths and such that $\sigma(1) = 1$.

For the proof, we need the following.

LEMMA 5.2
Let G be a compactly generated group, and let K be a compact symmetric neighborhood of 1 generating G. Then there is a countable pointed partition (G_n, g_n) *that is a partition*

$$G = \coprod_{n\in\mathbb{N}} G_n,$$

where the G_n's are relatively compact Borel subsets of G and $g_n \in G_n$ such that $g_n^{-1}G_n \subseteq K$.

Proof
Let $\{g_n\} \subseteq G$ be a maximal subset of elements with the property that $d(g_n, g_m) = L_K(g_n^{-1}g_m) > 1$. Notice that since the ball of radius 1 is a neighborhood of 1, the set of g_n's is discrete in G. Since a ball of finite radius is compact, there are only finitely many g_n's in each ball of finite radius, so there are countably many altogether. Since $\{g_n\}$ is maximal, the union of balls of radius 1 centered at the g_n's cover G. (If not, then there would be $g \in G$ not in $\{g_n\}$ and at distance greater than 1 to any g_n, which contradicts maximality.) We write $B(g_n, r) = g_n B(r)$ for the ball of radius r centered at g_n. We define the G_n's as

$$G_0 = K = B(1), \qquad G_1 = B(g_1, 1) \setminus G_0, \ldots, \qquad G_n = B(g_n, 1) \setminus \Big(\bigcup_{k<n} G_k\Big), \ldots.$$

It is a partition of G by construction, and $g_n \in G_n$ because for any $n \neq m$, we have that $d(g_n, g_m) > 1$, so that $g_n \notin B(g_m, 1)$. Finally, $g_n^{-1}G_n \subseteq g_n^{-1}B(g_n, 1) = g_n^{-1}g_n K = K$, and the proof is complete. □

Proof of Proposition 5.1
Let K be a symmetric compact generating neighborhood of the identity in G, and let (G_n, g_n) be a pointed partition of G as in Lemma 5.2. Let S be a symmetric compact generating set for E. For each $n \in \mathbb{N}$, let $e_n \in E$ be a preimage of g_n of minimal length in the alphabet S. Let σ_K be a Borel section of p on K whose image is relatively

compact (see [22, Lemma 2]). We may assume that $\sigma_K(1) = 1$. Define

$$\sigma_n : G_n \to E, x \mapsto e_n \sigma_K(g_n^{-1} x)$$

and $\sigma : E \to E$ by $\sigma = \coprod_{n \in \mathbb{N}} \sigma_n$, so that σ is a Borel map. We check that it is a section for p. For $g \in G_n$, we have

$$p\sigma(g) = p\sigma_n(g) = p\big(e_n \sigma_K(g_n^{-1} g)\big) = p(e_n) p\sigma_K(g_n^{-1} g) = g_n g_n^{-1} g = g.$$

Now, let us prove that the section σ that we just obtained is Lipschitz. Let $C = \sup\{L_S(g) | g \in \sigma_K(K)\}$. Since $\sigma_K(K)$ is relatively compact in E, we have that $C < \infty$. For g_n of length m, if we write $g_n = k_1 \cdots k_m$ with all $k_i \in K$, we have that $L_E(\sigma_K(k_1) \cdots \sigma_K(k_m)) \leq Cm$ and $p(\sigma_K(k_1) \cdots \sigma_K(k_m)) = g_n$. Since e_n is a shortest preimage of g_n, we deduce

$$L_S(e_n) \leq L_S\big(\sigma_K(k_1) \cdots \sigma_K(k_m)\big) \leq Cm = CL_K(g_n).$$

Finally, take $g \in G$ and $n \in \mathbb{N}$ such that $g \in G_n$. We have

$$\begin{aligned} L_S\big(\sigma(g)\big) &= L_S\big(e_n \sigma_K(g_n^{-1} g)\big) \leq L_S(e_n) + C \\ &\leq CL_K(g_n) + C \leq C\big(L_K(g) + 2\big) \end{aligned}$$

since $L_K(g_n^{-1} g) = L_K(g^{-1} g_n) \leq 1$ because $g_n^{-1} g \in K$ if $g \in G_n$. □

Definition 5.3
Let $D : \mathbb{N} \to \mathbb{N}$ be a nondecreasing function. Let A and E be two compactly generated groups. Assume that $A < E$ (i.e., A is a subgroup of E). We say that A has *distortion at most* D if there are two compact symmetric generating sets U and S for A and E, respectively, such that for all $a \in A$,

$$L_U(a) \leq D\big(L_S(a)\big) L_S(a).$$

We say that the subgroup A has *polynomial distortion* in E if D can be chosen to be a polynomial and *undistorted* if D can be chosen constant.

Notice that our definition of distortion is equivalent to the one given by Gromov in [13, Chapter 3], as he defines (under the hypothesis of Definition 5.3) the distortion function as

$$\mathrm{DISTO}(r) := \frac{\mathrm{diam}_A(A \cap B_E(r))}{r},$$

and one easily checks that A has distortion at most DISTO because $2D(n) \geq \mathrm{DISTO}(n/2)$. We need the following simple lemma.

LEMMA 5.4

Let $p : G \to Q$ be a surjective homomorphism of compactly generated groups. Let H be a subgroup of G which contains $\ker(p)$. *Then the distortion of $p(H)$ in Q is bounded by the distortion of H in G.*

Proof

Let S and T be compact symmetric generating sets for G and H, respectively. Then $p(S)$ and $p(T)$ are compact symmetric generating sets for Q and $p(H)$, respectively. Let D be the distortion of H in G relative to the word lengths L_S and L_T. Take $q \in p(H)$; we want to estimate $L_{p(T)}(q)$ in terms of $L_{p(S)}(q)$. Since $\ker(p) < H$, we can choose $h \in H$ of minimal S-length, so that $p(h) = q$ and $L_S(h) = L_{p(S)}(q)$. Hence, we obtain

$$L_{p(T)}(q) \leq L_T(h) \leq D\big(L_S(h)\big)L_S(h) = D\big(L_{p(S)}(q)\big)L_{p(S)}(q). \qquad \square$$

PROPOSITION 5.5

Let $1 \to A \to E \to G \to 1$ be an exact sequence of compactly generated groups with A closed and central. If G has property RD, and if A has polynomial distortion in E, then E has property RD as well.

Proof

As G has property RD, it is unimodular, and it follows from [3, Chapitre VII, Paragraphe 2, Numéro 7, Corollaire] that E is also unimodular. First, notice that a compactly generated abelian group is of polynomial growth for any word length and thus has property RD. Let T, S, and U be respective compact symmetric generating neighborhoods of the identity in E, A, and G. Let C_G, D_G and C_A, D_A be the constants needed for the RD inequality (as in Definition 2.1) for G and A, respectively. Let σ be a section of the canonical projection $p : E \to G$ with the same properties as in Proposition 5.1. Each element in E can be written in a unique way as $a\sigma(x)$ with $a \in A$ and $x \in G$. Recall that the formula

$$c(x, y) = \sigma(x)\sigma(y)\sigma(xy)^{-1}, \quad \forall x,\ y \in G,$$

defines an inhomogeneous 2-cocycle on G with values in A. For $f, g \in \mathscr{C}_c(E)$, we define $f_y(a) = f(a\sigma(y))$ and

$$g'_{(y,x)}(a) = g_{y^{-1}x}\big(a - c(y, y^{-1}x)\big).$$

For all x, y, the elements f_y and $g'_{(y,x)}$ belong to $L^2(A)$, and since c is measurable, we have

$$f * g\big(a\sigma(x)\big) = \int_G \Big(\int_A f_y(b) g'_{(y,x)}(a - b)\, db\Big)\, dy = \int_G f_y * g'_{(y,x)}(a)\, dy.$$

If we square and integrate over E, we obtain

$$\begin{aligned}\|f * g\|^2_{L^2(E)} &= \int_G \Big(\int_A \Big|\int_G f_y * g'_{(y,x)}(a)\,dy\Big|^2 da\Big)^{(1/2)2}\,dx\\ &\le \int_G \Big(\int_G \|f_y * g'_{(y,x)}\|_{L^2(A)}\,dy\Big)^2\,dx.\end{aligned}$$

Now assume that the support of f is contained in the ball of radius r, and for $y \in G$, let us look at the support of f_y. Take a in the support of f_y. Then $L_T(a\sigma(y)) \le r$, so that $L_T(a) \le L_T(a\sigma(y)) + L_T(\sigma(y)) \le C''r$, where $C'' > 1$ is a constant that depends only on the Lipschitz constants of σ and p. Hence, the hypothesis on the distortion of A implies the existence of constants $k > 1, C > 1$, depending only on the word lengths, such that $L_S(a) \le C(1+r)^k$. Applying property RD for A to $\|f_y * g'_{(y,x)}\|_{L^2(A)}$, we obtain

$$\|f * g\|^2_{L^2(E)} \le \int_G \Big(\int_G C_A\big(C(1+r)^k\big)^{D_A}\|f_y\|_{L^2(A)}\|g'_{(y,x)}\|_{L^2(A)}\,dy\Big)^2\,dx.$$

Finally, define $\tilde{f}, \tilde{g} \in L^2(G)$ by $\tilde{f}(y) = \|f_y\|_{L^2(A)}$ and $\tilde{g}(y) = \|g_y\|_{L^2(A)}$, so that, clearly, $\|\tilde{f}\|_{L^2(G)} = \|f\|_{L^2(E)}$ and $\|\tilde{g}\|_{L^2(G)} = \|g\|_{L^2(E)}$. Notice that $\tilde{f}$ is supported on the ball of radius $C'r$, where C' is the Lipschitz constant of p. Concerning g, we have

$$\|g'_{(y,x)}\|_{L^2(A)} = \|g_{y^{-1}x}\|_{L^2(A)} = \tilde{g}(y^{-1}x).$$

Going back to the computation of $\|f * g\|^2_{L^2(E)}$, we now get

$$\begin{aligned}\|f * g\|^2_{L^2(E)} &\le C_A^2\big(C(1+r)^k\big)^{2D_A}\|\tilde{f} * \tilde{g}\|^2_{L^2(G)}\\ &\le C_A^2\big(C(1+r)^k\big)^{2D_A}C_G^2 C'r^{2D_G}\|f\|^2_{L^2(E)}\|g\|^2_{L^2(E)}.\end{aligned}$$

We conclude that E has property RD by Lemma 2.4(2). □

6. Lie groups with property RD

In this section we prove that (c) implies (a) in our main theorem, Theorem 0.1.

THEOREM 6.1
Let G be a connected Lie group whose universal cover $\widetilde{G}$ decomposes as $\widetilde{S} \times \widetilde{Q}$, where $\widetilde{S}$ is semisimple and $\widetilde{Q}$ has polynomial growth. Then G has property RD.

COROLLARY 6.2
Semisimple Lie groups have property RD.

We start with the following lemma.

LEMMA 6.3

Let Z be the center of a simply connected semisimple Lie group $\widetilde{G}$. Then Z is undistorted in $\widetilde{G}$.

Proof

Let $G = \widetilde{G}/Z$, and let $p : \widetilde{G} \to G$ be the canonical projection. As $\widetilde{G}$ is semisimple, Z is discrete, and this implies that G has trivial center. Let $G = NAK$ be an Iwasawa decomposition. Since G has trivial center, K is compact. Let S be the simply connected group $S = NA$ by $\widetilde{K} = p^{-1}(K)$ and by $\widetilde{S}$, the connected component of 1 in $p^{-1}(S)$. Consider the map

$$\varphi : G \to S \times K,$$
$$g \mapsto (s, k).$$

On G, we fix a left-invariant Riemannian metric. We consider $S \times K$ as the direct product of the Lie groups S and K and choose a left-invariant Riemannian metric on this product. According to [30, Lemma 3.1], and since K is compact, the map φ is bi-Lipschitz. Notice, for further reference, that $\widetilde{G} = \widetilde{S}\widetilde{K}$ and $Z \subseteq \widetilde{K}$ (see [16, Theorem 5.1 and its proof]). The map

$$\widetilde{\varphi} : \widetilde{G} \to \widetilde{S} \times \widetilde{K},$$
$$\tilde{s}\tilde{k} \mapsto (\tilde{s}, \tilde{k})$$

is well defined since $\widetilde{G} = \widetilde{S}\widetilde{K}$. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{G} & \xrightarrow{\ \widetilde{\varphi}\ } & \widetilde{S} \times \widetilde{K} \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p_1} \\ G & \xrightarrow{\ \varphi\ } & S \times K \end{array}$$

where p_1 is the product of the Z-regular cover $\widetilde{K} \to K$ with the trivial cover $\widetilde{S} \to S$. On $\widetilde{G}$, we choose the left-invariant Riemannian metric that turns p into a local isometry. On $\widetilde{S} \times \widetilde{K}$, we choose the left-invariant metric (for the product structure) which turns p_1 into a local isometry. As $\widetilde{\varphi}$ covers φ, it is also bi-Lipschitz. Since $Z \subseteq \widetilde{K}$ is cocompact, it is undistorted, and since the inclusion $\widetilde{K} \subseteq \widetilde{S} \times \widetilde{K}$ is totally geodesic, it is undistorted as well, and we conclude because $\widetilde{\varphi}$ is bi-Lipschitz. □

Proof of Theorem 6.1
We show that G is an extension of a group with property RD by a central subgroup with polynomial distortion. A group G as in Theorem 6.1 is of the form $G = \widetilde{G}/\Gamma$, where Γ is a discrete subgroup of $Z(\widetilde{G})$, the center of $\widetilde{G}$. Now, $Z(\widetilde{S})$, the center of $\widetilde{S}$, is discrete in $\widetilde{S}$ (see [12]), and hence the semisimple group $\widetilde{S}/Z(\widetilde{S})$ has trivial center. The following diagram is commutative:

$$\begin{array}{ccc} \widetilde{G} & & \\ {\scriptstyle p_\Gamma}\downarrow & \searrow^{p_Z} & \\ G = \widetilde{G}/\Gamma & \xrightarrow[p]{} & \widetilde{G}/Z(\widetilde{G}) \end{array}$$

where the bottom arrow $p : G \to \widetilde{G}/Z(\widetilde{G})$ is the quotient of G by $Z(\widetilde{G})/\Gamma$. Since $Z(\widetilde{G})/\Gamma$ is central in G, we have a central extension to which we want to apply Proposition 5.5. To start with,

$$\widetilde{G}/Z(\widetilde{G}) = \widetilde{S}/Z(\widetilde{S}) \times \widetilde{Q}/Z(\widetilde{Q})$$

has property RD because it is a product of two groups with property RD (see Lemma 3.1 combined with Theorem 4.5 and Proposition 4.1). In [38], Varopoulos proved that any closed subgroup of a connected Lie group with polynomial volume growth is at most polynomially distorted. Combined with Lemma 6.3 and the fact that if A is a subgroup of X and B is a subgroup of Y, then

$$\mathrm{DISTO}(A \times B, X \times Y) = \max\bigl(\mathrm{DISTO}(A, X), \mathrm{DISTO}(B, Y)\bigr),$$

it implies that the center $Z(\widetilde{G})$ is at most polynomially distorted in $\widetilde{G}$. Hence, according to Lemma 5.4, the subgroup $A = Z(\widetilde{G})/\Gamma$ is at most polynomially distorted in G. We conclude using Proposition 5.5. □

7. The structure of connected Lie groups with property RD

In this section we finish the proof of our main theorem, Theorem 0.1. To do so, we start by explaining the terms used in Theorem 0.1(b). Recall that a Lie algebra is of type R if all the weights of the adjoint representation are purely imaginary. A Lie group is of type R if its associated Lie algebra is of type R. According to Guivarc'h and Jenkins, a Lie algebra is of type R if and only if the associated Lie group has polynomial volume growth (see [14] and also [19]). Thus by the fundamental theorem of Lie (see [36, Theorem 2.8.2]), the statements (b) and (c) in Theorem 0.1 are equivalent. We now turn to the problem of whether (a) implies (b) in the proof of Theorem 0.1. This part relies on Varopoulos's work in [37]. Varopoulos introduces a dichotomy among finite-dimensional real Lie algebras. Namely, he divides them into B-algebras and NB-algebras. We now quote the two results of [37] which are crucial for our purpose.

THEOREM 7.1 (Varopoulos [37, Proposition, page 824])
Let $\mathfrak{g}$ be a unimodular algebra. Then either $\mathfrak{g}$ is a B-algebra, or $\mathfrak{g}$ is the direct product $\mathfrak{s} \times \mathfrak{q}$, where $\mathfrak{q}$ is an algebra of type R and $\mathfrak{s}$ is semisimple.

A connected Lie group is called a B-group if its Lie algebra is a B-algebra. Those groups have the following property.

THEOREM 7.2 (Varopoulos [37, Theorem B])
Let G be a B-group, and let ϕ be a continuous compactly supported probability density on G. Assume that $\phi^* = \phi$. Then there exists $c > 0$ such that the convolution powers $\phi^{(n)}$, where $n \in \mathbb{N}$, satisfy

$$\phi^{(n)}(1) = O\big(\|\lambda(\phi^{(n)})\|_{2\to 2} \exp(-cn^{1/3})\big). \tag{7.15}$$

This theorem has an easy corollary.

COROLLARY 7.3
B-groups cannot have property RD.

Proof
Let G be a B-group. We choose ϕ as in Theorem 7.2. We have $\lambda(\phi)^* = \lambda(\phi^*) = \lambda(\phi)$ and $\|\lambda(\phi^{(2n)})\|_{2\to 2} = \|\lambda(\phi^{(n)})\|_{2\to 2}^2$. By (7.15), it follows that

$$\phi^{(2n)}(1) \le A\|\lambda(\phi^{(2n)})\|_{2\to 2} \exp(-cn^{1/3}) = A\|\lambda(\phi^{(n)})\|_{2\to 2}^2 \exp(-cn^{1/3})$$

for some constant $A \ge 1$. Set $f = \phi^{(n)}$, so that $\phi^{(2n)}(1) = \|f\|_2^2$. Now assume that G has property RD; then

$$\begin{aligned}\|\lambda(f)\|_{2\to 2}^2 &\le C^2 n^{2D}\|f\|_2^2 = C^2 n^{2D}\phi^{(2n)}(1)\\ &\le AC^2 n^{2D}\|\lambda(\phi^{(n)})\|_{2\to 2}^2 \exp(-cn^{1/3})\\ &= AC^2 n^{2D}\|\lambda(f)\|_{2\to 2}^2 \exp(-cn^{1/3}),\end{aligned}$$

In the paragraph following the proof of Cor. 7.3, please let us know if the C heading is OK.

where C and D are the constants coming from the definition of property RD and the second inequality follows from the assumption that G is a B-group. We conclude that $1 \le AC^2 n^{2D} \exp(-cn^{1/3})$, which is a contradiction for n big enough. It follows that G does not have property RD. □

End of the proof of Theorem 0.1 (Main theorem)
We can now finish the proof of our main result. It remains to show that Theorem 0.1(a) implies (b). Let G be a connected Lie group. If G has property RD, then according to [20], it must be unimodular. By Corollary 7.3, a unimodular group having property RD cannot be a B-group. By Theorem 7.1, it follows that G must have the structure described in (b). The proof of Theorem 0.1 is now complete. □

Recall that a group G is almost connected if the connected component of the identity in G is cocompact. Recall also that any almost connected group G admits a compact normal subgroup K such that G/K is a Lie group (see [28, Chapter IV, Paragraph 4.6]). Notice that G/K has finitely many connected components. We can now give a complete classification for almost connected compactly generated groups (given by Theorem 0.1 combined with Lemmas 3.3 and 3.4).

THEOREM 7.4
Let G be an almost connected compactly generated group. Let K be a normal compact subgroup such that $L = G/K$ is a Lie group. Let L_0 be the connected component of the identity in L. The following are equivalent:
(a) *G has property RD;*
(b) *the Lie algebra $\mathfrak{l}$ of L decomposes as a direct product $\mathfrak{l} = \mathfrak{s} \times \mathfrak{q}$, where $\mathfrak{s}$ is semisimple and $\mathfrak{q}$ is an algebra of type R;*
(c) *the universal cover $\widetilde{L}_0$ of L_0 decomposes as a direct product $\widetilde{S} \times \widetilde{Q}$, where $\widetilde{S}$ is semisimple and $\widetilde{Q}$ has polynomial volume growth.*

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References

[1] J.-P. ANKER, *La forme exacte de l'estimation fondamentale de Harish-Chandra,* C. R. Acad. Sci. Paris Ser. I Math. **305** (1987), 371–374. MR 0910372

[2] C. BERG and J. P. R. CHRISTENSEN, *Sur la norme des opérateurs de convolution*, Invent. Math. **23** (1974), 173–178. MR 0338685

[3] N. BOURBAKI, *Éléments de mathématique, fasc. 29, livre 6: Intégration, chapitre 7: Mesure de Haar; chapitre 8: Convolution er représentations*, Actualités Sci. Indust. **1306**, Hermann, Paris, 1963. MR 0179291

[4] ———, *Éléments de mathématique: Espaces vectoriels topologiques*, chapitres 1 – 5, new ed., Masson, Paris, 1981. MR 0633754

[5] I. CHATTERJI, *Property (RD) for cocompact lattices in a finite product of rank one Lie groups with some rank two Lie groups*, Geom. Dedicata **96** (2003), 161 – 177. MR 1956838

[6] I. CHATTERJI, CH. PITTET, and L. SALOFF-COSTE, *Heat decay and property RD*, in preparation.

[7] A. CONNES and H. MOSCOVICI, *Cyclic cohomology, the Novikov conjecture and hyperbolic groups*, Topology **29** (1990), 345 – 388. MR 1066176

[8] M. COWLING, "Herz's 'principe de majoration' and the Kunze-Stein phenomenon" in *Harmonic Analysis and Number Theory (Montreal, 1996)*, CMS Conf. Proc. **21**, Amer. Math. Soc., Providence, 1997, 73 – 88. MR 1472779

[9] M. COWLING, S. GIULINI, A. HULANICKI, and G. MAUCERI, *Spectral multipliers for a distinguished Laplacian on certain groups of exponential growth*, Studia Math. **111** (1994), 103 – 121. MR 1301761

[10] M. COWLING, U. HAAGERUP, and R. HOWE, *Almost L^2 matrix coefficients*, J. Reine Angew. Math. **387** (1988), 97 – 110. MR 0946351

[11] J. DIXMIER, *Les C^*-algèbres et leurs représentations*, reprint of the 2nd ed., Grands Class. Gauthier-Villars, Éditions Jacques Gabay, 1996. MR 1452364

[12] V. V. GORBATSEVICH, A. L. ONISHCHICK, and E. B. VINBERG, *Foundations of Lie Theory and Lie Transformation Groups*, reprint, Springer, Berlin, 1997. MR 1631937

[13] M. GROMOV, "Asymptotic invariants of infinite groups" in *Geometric Group Theory, Vol. 2 (Sussex 1991)*, London Math. Soc. Lecture Note Ser. **182**, Cambridge Univ. Press, Cambridge, 1993, 1 – 295. MR 1253544

[14] Y. GUIVARC'H, *Croissance polynomiale et périodes des fonctions harmoniques,* Bull. Soc. Math. France **101** (1973), 333 – 379. MR 0369608

[15] U. HAAGERUP, *An example of a nonnuclear C*-algebra, which has the metric approximation property*, Invent. Math. **50** (1978/79), 279 – 293. MR 0520930

[16] S. HELGASON, *Differential geometry, Lie groups, and symmetric spaces*, Pure Appl. Math. **80**, Academic Press, New York, 1978. MR 0514561

[17] C. HERZ, *Sur le phénomène de Kunze–Stein*, C. R. Acad. Sci. Paris Sér. A-B **271** (1970), A491 – A493. MR 0281022

[18] E. HEWITT and K. A. ROSS, *Abstract Harmonic Analysis, Vol I: Structure of Topological Groups, Integration Theory, Group Representations*, 2nd ed., Grundlehren Math. Wiss. **115**, Springer, Berlin, 1979. MR 0551496

[19] JENKINS, J. W., *Growth of connected locally compact groups*, J. Functional Analysis **12** (1973), 113 – 127. MR 0349895

[20] R. JI and L. B. SCHWEITZER, *Spectral invariance of smooth crossed products, and rapid decay locally compact groups*, *K*-Theory **10** (1996), 283 – 305. MR 1394381

[21] P. JOLISSAINT, *Rapidly decreasing functions in reduced C*-algebras of groups*, Trans. Amer. Math. Soc. **317** (1990), 167 – 196. MR 0943303

[22] E. T. KEHLET, *Cross sections for quotient maps of locally compact groups*, Math. Scand. **55** (1984), 152 – 160. MR 0769031

[23] A. W. KNAPP, *Representation Theory of Semisimple Groups: An Overview Based on Examples*, reprint of the 1986 original, Princeton Landmarks Math. **36**, Princeton Univ. Press, Princeton, 1986. MR 1880691

[24] V. LAFFORGUE, *A proof of property (RD) for discrete cocompact lattices of* $\mathrm{SL}_3(\mathbf{R})$ *and* SL (3, **C**), J. Lie Theory **10** (2000), 255 – 277. MR 1774859

[25] ———, *K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes,* Invent. Math. **149** (2002), 1 – 95. MR 1914617

[26] H. LEPTIN, *On locally compact groups with invariant means,* Proc. Amer. Math. Soc. **19** (1968), 489 – 494. MR 0239001

[27] G. A. MARGULIS, *Discrete Subgroups of Semisimple Lie Groups*, Ergeb. Math. Grenzgeb. (3) **17**, Springer, Berlin, 1991. MR 1090825

[28] D. MONTGOMERY and L. ZIPPIN, *Topological Transformation Groups*, Interscience, New York, 1955. MR 0073104

[29] S. MUSTAPHA, "Multiplicateurs spectraux sur certains groupes non-unimodulaires" in *Harmonic Analysis and Number Theory (Montreal, 1996)*, CMS Conf. Proc. **21**, Amer. Math. Soc., Providence, 1997, 11 – 30. MR 1472776

[30] CH. PITTET, *The isoperimetric profile of homogeneous Riemannian manifolds*, J. Differential 'Geom. **54** (2000), 255 – 302. MR 1818180

[31] J. RAMAGGE, G. ROBERTSON, and T. STEGER, *A Haagerup inequality for* $\tilde{A}_1 \times \tilde{A}_1$ *and* $\tilde{A}_2$ *buildings*, Geom. Funct. Anal. **8** (1998), 702 – 731. MR 1633983

[32] W. RUDIN, *Functional Analysis*, 2nd ed., Internat. Ser. Pure Appl. Math. McGraw-Hill, New York, 1991. MR 1157815

[33] L. SALOFF-COSTE and W. WOESS, *Transition operators on co-compact* G*-spaces*, to appear in Rev. Mat. Iberoamericana, preprint, 2005.

[34] A. J. SILBERGER, *Introduction to Harmonic Analysis on Reductive* p*-adic groups*, Math. Notes **23**, Princeton Univ. Press, Princeton, 1979. MR 0544991

[35] A. VALETTE, *Introduction to the Baum-Connes Conjecture,* Lectures Math. ETH Zürich, Birkhäuser, Basel, 2002. MR 1907596

[36] V. S. VARADARAJAN, *Lie groups, Lie algebras, and Their Representations*, reprint of the 1974 ed., Grad. Texts Math. **102**, Springer, New York, 1984. MR 0746308

[37] N. TH. VAROPOULOS, *Analysis on Lie groups*, Rev. Mat. Iberoamericana **12** (1996), 791 – 917. MR 1435484

[38] N. TH. VAROPOULOS, "Distance distortion on Lie groups" in *Random Walks and Discrete Potential Theory (Cortona, Italy, 1997)*, Sympos. Math. **39**, Cambridge Univ. Press, Cambridge, 1999, 320 – 357. MR 1802438

[39] N. TH. VAROPOULOS, L. SALOFF-COSTE, and TH. COULHON, *Analysis and Geometry on Groups*, Cambridge Tracts in Math. **100**, Cambridge Univ. Press, Cambridge, 1993. MR 1218884

[40] A. WEIL, *L'intègration dans les groupes topologiques et ses applications*, Actual. Sci. Ind. **869**, Hermann et Cie., Paris, 1940. MR 0005741

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