Disordered Systems

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Chapter 1

Introduction

Very simple book. Introduction. Written with starting PhD students in mind.

Because in no way I could find a book like this.

not too much material. only what looks sensible to me to teach in a one term intensive course.

very good given together or just after and advanced course in statistical mechanics (Renormalization Group).

Things in the order in which I teach them. I try not to be boring: sometimes I postpone basic things not to spend too much time on basics without small adventures in the "exciting" stuff. Sometimes I postpone technical computations not to have too much of a technical load all together on the poor students.

I am interested in teaching results, but also in giving tools and techniques (for example the transfer matrix approach, the use of scaling arguments, super-symmetry techniques with Parisi-Sourlas).

Lecturing style: I explain in some detail things that the students seem to be asking more about.

Important ideas are repeated in different chapters of the book, explaining them at different levels: *repetita juvant*, as they used to say many years ago.

Student questions and comments have been very useful.

I give many details of the computations. And many explanations about the reasoning.

Disorder. Scaling. Statistical mechanics. Dilution, RFIM, Spin Glasses and Parisi theory: phenomenology and paradigm.

The way I discuss Monte Carlo. Very idiosyncratic: mainly for intro-

ducing dynamics, correlation times, finite size scaling behavior of correlation times.

Slow dynamics.

Quenched disorder and complexity.

Same chapters can (and maybe should) read in a different order: for example FSS and LY theorem maybe before Harris and Griffiths, basics of Monte Carlo before Griffiths dynamical,...

Main idea of the book. Basis of physics of disordered systems: 1. scaling theory. 2. Parisi mean field theory 3. RFIM (we do not know what it is). In the book I discuss 3 before 2 (I do things in order of growing complexity). I do not enter in "modern" disputes (who is scaling theory, who is Parisi): I only introduce the basic cases (temperature dilution for scaling, SK for Parisi) where we are sure about what happens, and I discuss PT of RFIM.

Acknowledgments

all students from the course Physics of Disordered Systems.... during the years

Chapter 2

An Introduction to Disordered Systems

2.1 An Application of Mean Field: Minority Games

2.2 Disorder in the Statistical Mechanics of Biological Systems: the Example of DNA and RNA

Chapter 3

Harris Criterion

Harris Criterion [6] will be one of the building blocks of our construction, since it will be one of the first tools used to distinguish when quenched disorder plays a relevant role.

Let us start by state the criterion, that we will then derive in detail in a more general case, following [7]. Let us consider a d dimensional ferromagnetic spin model. If ν, the critical exponent of the correlation length $\xi \sim_{T\to T_c} (T-T_c)^{-\nu}$ is such that $\nu < \frac{2}{d}$ then the critical behavior of the system diluted with quenched uncorrelated (site or link) disorder is different (i.e. it has different critical exponents) from the one of the pure system. If $\nu > \frac{2}{d}$ the quenched disorder is, on the contrary, *irrelevant*. This is Harris criterion. Notice that this is a perturbative statement, valid for small disorder: for large amount of disorder non-perturbative effects can play an important role and completely change the nature of the system.

If we use the *hyper-scaling relation* (that we will derive only in chapter (5)), telling us that $2-d\nu = \alpha$, where α is the critical exponent of the specific heat, i.e. $C_V \sim_{T\to T_c} (T-T_c)^{-\alpha}$, we find that $\boxed{\text{the disorder is relevant}}$ if $\frac{2-\alpha}{d} < \frac{2}{d}$, i.e. $\boxed{\text{if}}$

$$\boxed{\alpha > 0}\,, \tag{3.1}$$

that is the most standard way to phrase $\boxed{\text{Harris Criterion}}$: if the specific heat of the pure system diverges at criticality, a small amount of dilution quenched disorder has to change the critical exponents of the system. For example this is true for the $3d$ Ising model.

Harris criterion is very well seen as a consistency requirements: one asks

in which situation it is consistent that the insertion of a small amount of quenched disorder does not change the universal physics of the system, i.e. the critical exponents. The answer is, as we have already said, that if $\alpha_{\rm pure} < 0$ the weakly disordered system can keep the same properties than the original, pure system. Thinks could very well change before the point of inconsistency, but on general grounds in physics things happen for good reasons: if we have not overlooked some important feature the criterion will give the right prediction. In all cases that can be solved analytically or understood by different techniques the Harris criterion turns out to give the correct answer. For example a Renormalization Group approach to the n component vector spin models in $d = 4 - \epsilon$ finds that the crossover derived through the Harris criterion is indeed the correct one.

3.1 Correlated Impurities

We will derive Harris Criterion in a more general approach than the one used in the original Harris work. Following Weinrib and Halperin [7] we will include the case of *correlated* quenched impurities, when considering the introduction of a small amount of quenched disorder in a system that when pure undergoes a second order phase transition.

On general grounds we can think that we have in mind a "random temperature" type of disorder: different regions of the material feel a different effective temperature or, in an equivalent way, have a (quasi-) transition at a different value of the local critical temperature T_c(region). This kind of disorder can be realized in nature in many different ways: one can for example have a small density of impurities or a small amount of randomness in the link interaction, causing in this way a variation of local transition temperature.

In figure (3.1) we show different small pieces of the system that have a transition at different values of the temperature. Obviously we know that finite parts of the system do not have a true transition, but if these elementary parts (of which we have many) are large enough the observed behavior will be very similar to the one of a true phase transition. We are not including in this framework models with random fields and models like spin glasses, where the disorder can have more subtle effects.

So, as we said we will not consider only uncorrelated dilutions, or very short range correlations. We will include in our derivation T_c fluctuations due

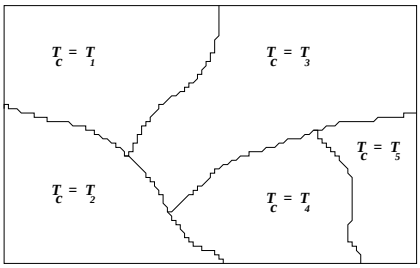

Figure 3.1: A system with different local critical temperatures.

to defects with long range correlations, i.e. with correlations that decay as a function of distance like power laws with not too large exponents (we will see that the definition of a "not too large exponent" will become quantitative in the following). In our models:

$$g\left(\vec{x}-\vec{y}\right) \equiv \langle T_c(\vec{x}) T_c(\vec{y})\rangle_c \equiv \langle T_c(\vec{x}) T_c(\vec{y})\rangle - \langle T_c\rangle^2 \simeq |\vec{x}-\vec{y}|^{-a} \ , \qquad (3.2)$$

where a is a constant exponent and the last relation is valid for large values of the distance $|\vec{x}-\vec{y}|$. From the experimental point of view one can think about models where the fluctuations in the local T_c depend from many independent phenomena, characterized from very different typical scales: such a situation can generate power law correlations on many decades of distances.

So let us try to verify the criterion in a system with "temperature disorder" and, as we have define before, with a connected correlation function $g(\vec{x}) \simeq |\vec{x}|^{-a}$. Let ξ be the correlation length in the pure system. We start by dividing the system in regions of linear size ξ, and volume $V = \xi^d$. We ask if *the typical variation of the critical temperature T_c of these regions becomes small when $T \to T_c$*: only if this is true the scaling laws can consistently stay unchanged after adding the quenched disorder. We define a reduced temperature

$$\theta \equiv \frac{T - T_c}{T_c} \ , \qquad (3.3)$$

and a local reduced temperature

$$\theta(\vec{x}) \equiv \frac{T - T_c(\vec{x})}{T_c} \ . \qquad (3.4)$$

We take as transition temperature of a region of size ξ the average of $T_c(\vec{x})$ over this region: this makes sense because spins are correlated up to distance

of order ξ. So we write

$$\theta_V \equiv \frac{1}{V}\int_V d^d x\ \theta(\vec{x}) \tag{3.5}$$

for the reduced temperature of a given spatial domain. θ_V is a random variable, of which we can compute the fluctuations Δ^2:

$$\begin{aligned}
\Delta^2 &\equiv \left\langle \theta_V^2 \right\rangle_c = \frac{1}{V^2}\int_V d^d x\ \int_V d^d y\ \left\langle \theta(\vec{x})\theta(\vec{y})\right\rangle_c \\
&= \frac{1}{V^2}\int_V d^d x\ \int_V d^d y \\
&\quad \left(\left\langle \left(\frac{T-T_c(\vec{x})}{T_c}\right)\left(\frac{T-T_c(\vec{y})}{T_c}\right)\right\rangle - \left\langle \left(\frac{T-T_c(\vec{x})}{T_c}\right)\right\rangle^2 \right) \\
&= \frac{1}{T_c^2 V^2}\int_V d^d x\ \int_V d^d y\ \left(\langle T_c(\vec{x})T_c(\vec{y})\rangle - \langle T_c(\vec{x})\rangle^2 \right) \\
&= \frac{1}{T_c^2 V^2}\int_V d^d x\ \int_V d^d y\ g(\vec{x}-\vec{y})\ .
\end{aligned} \tag{3.6}$$

Now with a change of variables we find that the $2d$ integrals above give us

$$V\int_V d^d z\ g(\vec{z}) = \xi^d \left(\int d^{d-1}\Omega\right)\int_0^\xi g(\rho)\ \rho^{d-1}\ d\rho\ , \tag{3.7}$$

where the first integral is the angular integral that gives a constant, and where in the second integral we have assumed that the function g becomes isotropic and that ξ is large (that is true when T approaches T_c). So we have found that, forgetting irrelevant constants,

$$\Delta^2 \sim \xi^{-d}\int_0^\xi g(\rho)\ \rho^{d-1}\ d\rho\ . \tag{3.8}$$

Our hypothesis is that for large ρ we have that $g(\rho)\sim\rho^{-a}$. So when $T\to T_c$ and ξ is large we have to evaluate $\int_0^\xi \rho^{d-a-1}\ d\rho$. We find that

$$\Delta^2 \simeq \begin{cases} \xi^{-d}\cdot \text{constant} & \text{for } a > d\ , \\ \xi^{-d}\cdot \log(\xi) & \text{for } a = d\ , \\ \xi^{-d}\cdot \xi^{d-a} & \text{for } a < d\ . \end{cases} \tag{3.9}$$

Do not forget that for large values of a the model is short ranged, and we expect to recover the original, a independent Harris criterion. As usual we

insist on the fact that the use of the logarithm in the marginal case $a = d$ is close to formal. Consistency criteria give important indications above and below marginality: in the marginal case different things could happen, and basically a simple consistency criterion does not tell.

Now we notice that it is consistent that the critical behavior does not change from the one of the pure system if

$$\frac{\Delta^2}{\theta^2} \longrightarrow_{\theta \to 0} 0 \tag{3.10}$$

i.e. if the local fluctuations of T_c are becoming small on the scale given by the distance from T_c.

Since $\xi \sim \theta^{-\nu}$ (3.9) becomes

$$\frac{\Delta^2}{\theta^2} \simeq \begin{cases} \theta^{d\nu-2} & \text{for } a > d \ , \\ \theta^{d\nu-2} \cdot \log(\theta^{-\nu}) & \text{for } a = d \ , \\ \theta^{a\nu-2} & \text{for } a < d \ . \end{cases} \tag{3.11}$$

We can check now when the requirement (3.10) is satisfied. If $a > d$ (i.e. in the case of a fast decay of correlation functions) the critical point turns out to be stable if $d\nu_{\text{pure}} > 0$ i.e. if $\alpha_{\text{pure}} < 0$. This is the original Harris criterion. For $a > d$ the tail of $g(\vec{x})$ does not have any effect, since the decay is too fast. In this way we are also giving a quantitative meaning to a "not too large exponent", that is, in this case, an exponent smaller than the spatial dimensionality of the system, d.

If, on the contrary, $a < d$ we have found that the critical point can be stable if $a\nu_{\text{pure}} > 0$. This is a different requirement than the original one. Notice that this requirement is stronger than the original one, since for $a < d$ one has that $a\nu_{\text{pure}} - 2 < d\nu_{\text{pure}} - 2$.

Chapter 4

Griffiths Singularities

We will discuss now a few implications of a crucial result originally due to Griffiths [8]. We will see that when one considers randomly diluted ferromagnets, with dilution p, already beyond the critical temperature of the diluted system, i.e. for T larger than

$$T_c(p) < T_c(p=1) \equiv T_c^{(\text{pure})}$$

"something" happens to the free energy. This something is not strong enough, as we will see, to force a phase transition.

In a normal, non disordered spin model, let T_c be the critical point (second order phase transition). In this case T_c is the end-point of a line of first order phase transitions (we have already discussed this issue in same detail). The line of first order phase transitions is characterized by the presence of a non-zero order parameter, that goes to zero at the critical point:

$$\delta m \equiv m^+ - m^- = 2\tilde{m} \longrightarrow_{T \to T_c^-} 0 \ ,$$

where $m^{\pm}$ are respectively the magnetizations of the plus and of the minus state, $m^+ \equiv \tilde{m} = -m^-$. This fact implies, as we have discussed, that the correlation length is diverging, $\xi \to \infty$. In this "usual" situation for $T > T_c$ the order parameter has a vanishing expectation value, i.e. $m(T) = 0$, and the free energy density $f(T, H)$ is an analytic function. What we mean is that at a given fixed $T > T_c$ there is "no trace" of the phase transitions: when changing T one can detect that a phase transition is coming since ξ is diverging (there are *precursor* signs of the phase transition, as opposed to the

case of a first order phase transition where one has no precursor phenomena), but there are no (even weak) residual singularities after T has crossed T_c.

We will show here that is a large class of disordered systems things are different: the value of the so-called *Griffiths temperature* T_G, defined as the temperature where the non-analyticity ends when increasing T, can be shown to be strictly larger than T_c, i.e. $T_G > T_c$ (and typically it coincides with the critical temperature of the pure model).

Let us start by considering the usual ferromagnetic Ising model (for example a simple cubic d dimensional lattice, for example $d = 2$ or larger, with a first neighbor interaction). When considering the diluted model some sites will be empty, with probability $(1 - p)$. Only the occupied sites interact, with the same interaction of the pure Ising model. The quenched occupation probability for the site i, p_i does not depend on H, on T and on the occupation probabilities of the other sites, p_j. We have already discussed in chapter (2) that the critical temperature of the diluted model, $T_c(p)$, decreases with decreasing p, and that for p smaller than the percolation threshold probability, $p < p_c$, the model cannot have a ferromagnetic phase transition anymore (since there is not an infinite cluster anymore: it is clear that there cannot be a spontaneous magnetization at $T(p) > 0$ if p is smaller of the p_c where a connected infinite cluster appears). One can show that in this situation the *Griffiths Singularities* appear:

> for each value of the dilution $p < 1$ the magnetization M is a non-analytic function of the magnetic field H at $H = 0$ for each $T < T_G \equiv T_c(p = 1)$.

4.1 The Existence of the Singularities: Statics

We will start by discussing Griffiths Theorem, and the existence of singularities in the free energy density: in this way we will be discussing the static, equilibrium situation, as opposed to the dynamical regime that we will be discussing in section (4.2).

We will try to sketch a rigorous proof of Griffiths Theorem (basically the one given in chapter 2 of [5]) by stressing the ones that appear to be the main physical issues. We will also use directly the original Griffiths derivation of the theorem [8] and the version of the theorem proved in [9].

Our discussion will be divided in four main points:

1. we will express the spontaneous magnetization of a spin system as a sum over the zeroes of a polynomial in the activity ρ;

2. we will discuss the fact that, because of the Lee-Yang theorem (that we will prove in chapter (6)), these zeroes have to lie in the unitary circle in the plane of complex activity;

3. we will exhibit a special class of connected clusters of spins, $\mathcal{C}$;

4. we will show that the presence alone of configurations of the class $\mathcal{C}$ is enough to make an analytic continuation through the unitary circle, $|\rho| = 1$.

We insist again: we will try to stress the most important physical issues that allow to give a rigorous proof, without trying to be rigorous ourselves.

So, we will show that in a diluted Ising model there are Griffiths singularities. We will call Λ a periodic box in $\mathbb{Z}^d$. We consider the usual Ising Hamiltonian:

$$H[\sigma] \equiv -\sum_{ij} \sigma_i J_{ij} \sigma_j - h \sum_i \sigma_i \ . \tag{4.1}$$

The couplings J_{ij} are quenched random variables, that dilute the lattice. There are two main possible choice for the couplings:

A a *site dilution*, where the quenched random variables are site variables τ_i and

$$J_{ij} = \begin{cases} J\tau_i\tau_j & \text{if } |i-j| = 1 \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

where the a priori probability of the τ_i is $P(\tau_i = 1) = p$ and $P(\tau_i = 0) = (1-p)$.

B a *link dilution*, where the quenched random variables are defined on links and

$$J_{ij} = \begin{cases} 0 & \text{if } |i-j| > 1 \ , \\ 0 & \text{if } |i-j| = 1 \text{ with probability } (1-p) \ , \\ J & \text{if } |i-j| = 1 \text{ with probability } p \ . \end{cases}$$

The main result one can show rigorously is that for $T < T_c(p = 1)$ but T not too small the imaginary axis in the complex h plane (i.e. the unitary circle in the complex activity plane) contains at least a singularity at $h = 0$, but the spontaneous magnetization goes to zero when $h \to 0$ along the real axis. It is also known that in spite of this singularity m can be C^∞ in h at $h = 0$ [10] (the magnetization does not have any visible change at $T = T_c(p = 1)$, and it can still be infinitely differentiable). On one side because of that the standard high temperature expansion ceases to be absolutely convergent in the Griffiths region: in these conditions it is interesting to look for improved expansion methods that can converge even in presence of Griffiths singularities [5]. On the other side the main physical effects of such dilution disorder and of Griffiths singularities is on the dynamical behavior of the system. As we will see in the next section the dynamics of the system becomes very slow, and decays are not exponential anymore.

We will discuss the case of site dilution (the same kind of treatment works for link dilution). As from item (1) in our to do list we will start by computing the magnetization of the site diluted system. Site i is *occupied* if $\tau_i = 1$ and *empty* if $\tau_i = 0$.

A *connected cluster* C is a group of $|C|$ occupied lattice sites with the property that all the sites in C are connected to another occupied site of C with a first neighbor link, and they are not connected from a first neighbor link to any of the occupied sites that do not belong to C. We show in figure (4.1) a configuration of clusters C on a (two dimensional) volume Λ.

Let C be a connected clusters of occupied sites in Λ. Let $|C|$ be the number of sites in C, and $P_{C,\Lambda}$ the probability of C (the clusters C are constructed by independent site percolation, see the discussion in chapter (2)). We call m_Λ the average magnetization per site of Λ (averaged also over the disorder), and m_c the average magnetization per site of the connected cluster C. Now

$$m_\Lambda = |\Lambda|^{-1} \sum_C P_{C,\Lambda}\, m_C \ , \tag{4.2}$$

were the sum runs over all the connected clusters that can appear in the region Λ.

Let the *activity* ρ be defined as

$$\rho \equiv e^{-2\beta h} \ , \tag{4.3}$$

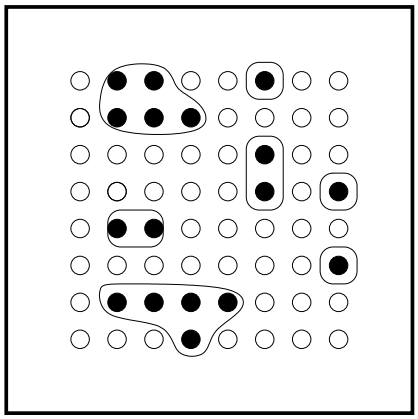

Figure 4.1: A volume Λ with a configuration of connected clusters of occupied sites. Occupied sites are represented with filled dots, empty sites with empty dots. Connected clusters are enclosed by continuous contours.

and let f_C be the free energy density restricted to the connected cluster C, i.e.

$$f_C \equiv -\frac{1}{\beta|C|}\log\left(Z_C\right) \ ,$$

where Z_C is the partition function computed by summing only on the spins that belong to the connected cluster C. Now

$$m_C = -\frac{\partial f_C}{\partial h} = -\frac{\partial \rho}{\partial h}\,\frac{\partial f_C}{\partial \rho} = 2\beta\rho\,\frac{\partial f_C}{\partial \rho} \ .$$

Now we notice that $\left(e^{-\beta h|C|}\ Z_C\right)$ *is a polynomial in* ρ *of degree* $|C|$. To show that we write (as we frequently do in these notes)

$$e^{-\beta h\sigma_i} = e^{\beta h}\ e^{-\beta h(\sigma_i+1)} \ ,$$

and we notice that taken away the first term in the right hand side of the equation (it gets canceled by the $e^{-\beta h|C|}$), we are left with a term that can take the two values 1 and ρ: so when we sum over the $\sigma_i = \pm 1$ variables we obtain a polynomial containing all the powers of $\rho \equiv e^{-2\beta h}$ up to the term $\rho^{|C|}$.

Let us write this polynomial of degree $|C|$ as a product over its zeroes:

$$Z_C = \frac{K}{\rho^{\frac{|C|}{2}}}\prod_{\alpha=1}^{|C|}\left(\rho - R_\alpha\left(C\right)\right) \ ,$$

where K is a constant (not relevant here), the power of ρ at the denominator leaves us with the polynomial of degree $|C|$, and where $R_\alpha(C)$ is the α^{th} zero of Z_C. Notice that here we are summing over all zeroes, also coincident ones, all with multiplicity one.

According to item two of our list we notice that Lee-Yang Theorem, that we will prove in chapter (6), tells us that

$$|R_\alpha(C)| = 1 \quad \forall \alpha \ , \tag{4.4}$$

i.e. the zeroes of Z_C are on the unitary circle of the activity complex plane.

We have that

$$\begin{aligned} m_C &= -\frac{\partial f_C}{\partial h} = 2\beta\rho \, \frac{\partial f_C}{\partial \rho} = 2\beta\rho \, \left(-\frac{1}{\beta|C|}\right) \frac{\partial}{\partial \rho} \log\left(Z_C\right) \\ &= \frac{2\rho}{|C|} \left(\frac{|C|}{2\rho} - \sum_{\alpha=1}^{|C|} \frac{1}{\rho - R_\alpha(C)} \right) \ , \end{aligned}$$

i.e.

$$m_C = 1 - \frac{2\rho}{|C|} \sum_{\alpha=1}^{|C|} \frac{1}{\rho - R_\alpha(C)} \ ,$$

and using (4.2) the magnetization per site on Λ is

$$m_\Lambda = \frac{1}{|\Lambda|} \sum_C P_{C,\Lambda} \left(|C| - 2\rho \sum_{\alpha=1}^{|C|} \frac{1}{\rho - R_\alpha(C)} \right) \ .$$

To construct m_Λ we are considering all connected clusters C allowed in Λ, and weighting their magnetization with the probability for C to occur. In this way we have obtained the average magnetization of the diluted lattice Λ, that has been obtained by performing both a thermal average and a quenched disorder average (i.e. an average over the site dilution disorder).

We note now that

$$\frac{1}{|\Lambda|} \sum_C |C| \ P_{C,\Lambda} = p \ , \tag{4.5}$$

i.e. it is equal to the a priori probability for a site to be occupied (this is clear since it is the expected total number of occupied sites divided times the

total number of sites in $|\Lambda|$), and

$$m_\Lambda(\rho) = p - 2\rho \sum_{A=1}^{N_\Lambda} \frac{\eta_A(\Lambda)}{\rho - R_A} , \tag{4.6}$$

where we have done a number of steps. In first we have summed over the first term getting p. In the second term we have introduced some major changes: now the sum runs over all the zeroes that can occur on connected clusters C defined on Λ, all of them counted only once with their correct multiplicity. N_Λ is the finite number of different zeroes that can occur on the connected clusters on Λ, and η_A is the residue of the pole A and is defined as

$$\eta_A(\Lambda) \equiv \frac{1}{|\Lambda|} \left(\sum_{C \text{ for which } R_A \text{ is a zero}} m_A(C)\ P_{C,\Lambda} \right) > 0, \tag{4.7}$$

where the sum runs over all clusters C where the partition function Z_C has a zero in R_A, and $m_A(C)$ is the number of times that R_A appears as a zero of the connected cluster C (i.e. the multiplicity of the zero R_A on cluster C). It turns out to be useful to notice that

$$\sum_{A=1}^{N_\Lambda} \eta_A(\Lambda) = \frac{1}{|\Lambda|} \sum_C \left(\sum_{A=1}^{N_\Lambda} m_A(C) \right) P_{C,\Lambda} ,$$

where the term in round brackets gives $|C|$, implying that, because of relation (4.5)

$$\sum_{A=1}^{N_\Lambda} \eta_A(\Lambda) = p . \tag{4.8}$$

Now from equations (4.6) and (4.8) and from Lee-Yang theorem we can conclude that for $|\rho| \neq 1$ (i.e. if there are no zeroes in the denominator) $m_\Lambda(\rho)$ is bounded uniformly in Λ. This is true since in this case all η_A are positive and their sum is finite and equal to p, and because of (4.4) the factors $1/(\rho - R_A)$ are finite, implying that m_Λ can grow too much.

The fact that the free energy density $f_\Lambda(\rho)$ exists in the thermodynamical limit, i.e. for $\Lambda \to \mathbb{Z}^d$, the fact that $m_\Lambda(\rho) = 2\beta\rho \frac{\partial f_\Lambda}{\partial \rho}$ and standard theorems tell us that

$$m_\Lambda(\rho) \to m(\rho) \text{ for } \Lambda \to \mathbb{Z}^d \text{ if } |\rho| \neq 1 .$$

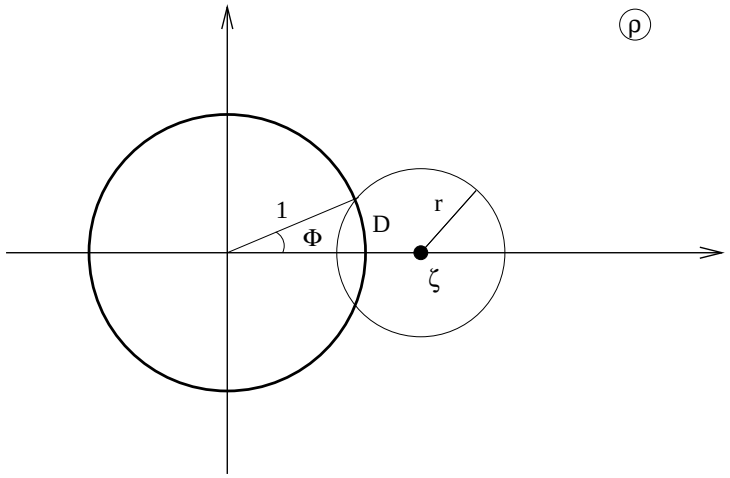

Figure 4.2: The complex ρ plane with the unitary circle, and the construction where one (wrongly) assumes that it exists a point ζ with $|\zeta| > 1$ such that the Taylor series centered around ζ has a convergence radius r that enters the unitary circle.

Now we are left with the task to show that for $T < T_c(p = 1)$ the magnetization $m(\rho)$ cannot be analytically continued along the real axis from the interior of the unit circle of the complex ρ plane to the region with $|\rho| > 1$. We will prove that by showing that the assumption that an analytic continuation can be done is false. If we assume that an analytic continuation is possible we are implying that there exist a point $\zeta > 1$ on the real axis in the ρ complex plane and a radius r, with $\zeta - r < 1$ (i.e. such that the circle around ζ of radius r has a non-zero intersection with the unitary circle around the origin), such that the Taylor series for $m(\rho)$ around the point $\rho = \zeta$ has a convergence radius r (see figure (4.2)). In this case the convergence radius of the series will contain in its interior an arc D of the unitary circle, i.e.:

$$D = \left\{ \rho = e^{i\phi} \mid |\phi| < \Phi \right\} \ ,$$

for some $\Phi > 0$.

Consider now the case where $T < T_c(p = 1)$, and assume $\Phi > 0$. Our item three can be first phrased in a very synthetic way by considering a sequence of regular clusters $\mathcal{C}$, where all spins are occupied in a d dimensional cube and all spins are empty out of the cube. The probability of finding such a

cluster, large at leisure, in the sum over C that forms m_Λ is finite. For such regular clusters, since $T < T_c(1)$, we have that $\min(|R-1|) \to 0$, so that the point $\rho = 1$ is an accumulation point of the zeros.

It is interesting to make the point in better detail. To do so we consider a large volume Λ, large enough that there are singularities R_A of $m_\Lambda(\rho)$ on the arc D: this is surely true for Λ large enough, since at $T < T_c(1)$ singularities from very large compact clusters are approaching the point $\rho = 1$. Let η_A the residue of the pole of R_A, that can be computed (and we will bound) through (4.7). Let $\mathcal{C}_A$ be the cluster such that R_A is a zero of $Z_{\mathcal{C}_A}$, i.e. such that $m_{\mathcal{C}_A}$ has a singularity at R_A. Let $\mathcal{A}$ be the smallest cube including completely $\mathcal{C}_A$ (in the simple sequence discussed before it would be the filled cube itself). Let us take now Λ as the union of n disjoint translates $\mathcal{A}(x)$ of $\mathcal{A}$ (we duplicate $\mathcal{A}$ by centering it around points x, such that the different copies do not overlap). We can get a lower bound for the residue of the pole at R_A by only including these configurations.

We note that the probability of $\mathcal{C}_A$ is

$$P_{\mathcal{C}_A} = p^{|\mathcal{C}_A|} \, (1-p)^{|\partial \mathcal{C}_A|} \, ,$$

where $\partial \mathcal{C}_A$ are the sites in $\mathcal{A}$ at distance 1 from $\mathcal{C}_A$ and not in $\mathcal{C}_A$. This tells us that sites in the cluster are occupied, and that happens with probability p, and that the connected cluster is bordered by empty sites, and sites are empty with probability $(1-p)$. Now by considering only the n translates we have just discussed we get that

$$\eta_A(\Lambda) \geq \frac{n}{|\Lambda|} P_{\mathcal{C}_A} = \frac{1}{\mathcal{A}} P_{\mathcal{C}_A} \equiv \Pi > 0 \, , \tag{4.9}$$

i.e. η_A is strictly positive.

We are left now with the task of completing the discussion of our fourth item. This is done by proving the Lemma telling us that

$$m_\Lambda(r R_A) \geq 2 \frac{\eta_A(\Lambda)}{r-1} \geq 2 \frac{\Pi}{r-1} \, , \tag{4.10}$$

for $r > 1$, uniformly in Λ. So, since $\Pi > 0$, m diverges as $r \to 1^+$. This means that $m(\rho)$ cannot be holomorphic in any neighborhood of the arc D. Since the model enjoys an explicit symmetry we know that $m_\Lambda(h) = -m_\Lambda(-h)$, i.e. $m_\Lambda(\rho) = -m_\Lambda(\rho^{-1})$: so $m(\rho)$ cannot be continued from the interior to

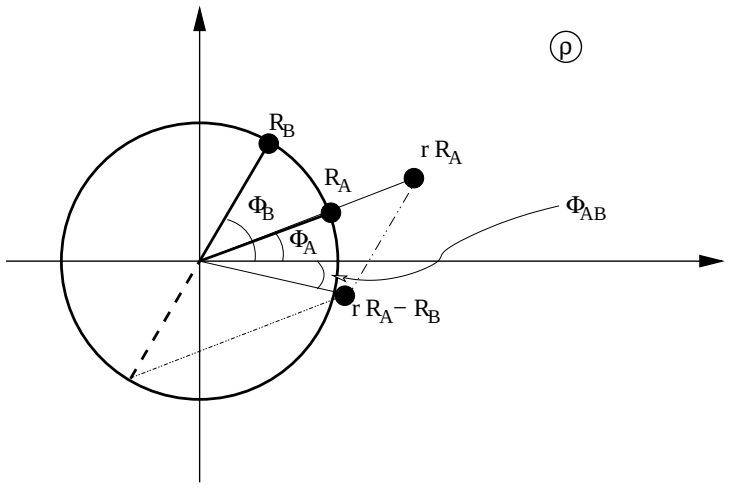

Figure 4.3: The complex ρ plane with the unitary circle, and details useful to prove Lemma (4.10).

the exterior of the unitary ρ circle (even along the real axis). This proves our theorem.

We are left with Lemma (4.10) to prove. We define the angle Φ_A by $R_A \equiv e^{i\Phi_A}$, since R_A is on the unitary disk. Let $r > 1$, $r \in \mathbb{R}$. As shown in figure (4.3) for a generic $R_B \equiv e^{i\Phi_B}$ on the unitary circle we define

$$R_{AB}\ e^{i\Phi_{AB}} \equiv rR_A - R_B\ .$$

The convexity of the unit circle implies that

$$|\Phi_{AB} - \Phi_A| < \frac{\pi}{2}\ . \tag{4.11}$$

To get convinced of this fact we can first thing about the situation where $r = 1$ and R_A and R_B are very close on the unitary circle. In this case $|\Phi_{AB} - \Phi_A|$ is maximum, close to $\frac{\pi}{2}$. Now when R_B goes far from R_a on the unitary circle the difference decreases, to reach zero when the two zeroes are opposite on a diameter. When the two zero approach on the other side $|\Phi_{AB} - \Phi_A|$ starts increasing again to eventually reach $\frac{\pi}{2}$. When $r > 1$ keeping R_A and R_B fixed the resulting Φ_{AB} becomes closer to Φ_A, making (4.11) still valid.

We also notice that $\forall\ u = \rho\ e^{i\phi} \in \mathbb{C},\ \psi \in \mathbb{R}$,

$$\Re\left(u\ e^{i\psi}\right) = \Re\left(\rho\ e^{i(\phi+\psi)}\right) = \rho\ \cos\left(\phi + \psi\right) \leq |u|\ . \tag{4.12}$$

Now we consider the sum that appears in the expression for m_Λ, (4.6). We have that, because of (4.12),

$$\left| \sum_{B=1}^{N_\Lambda} \frac{\eta_B(\Lambda)}{rR_A - R_B} \right| \geq \Re \left(e^{i\Phi_A} \sum_{B=1}^{N_\Lambda} \frac{\eta_B(\Lambda)}{rR_A - R_B} \right) ,$$

that by taking out of the sum the term with $B = A$ we can rewrite as

$$\begin{aligned} &= \frac{\eta_A(\Lambda)}{r-1} + \sum_{B \neq A} \Re \left(\frac{e^{i(\Phi_A - \Phi_{AB})}}{R_{AB}} \, \eta_B(\Lambda) \right) \\ &= \frac{\eta_A(\Lambda)}{r-1} + \sum_{B \neq A} \frac{\eta_B(\Lambda)}{R_{AB}} \cos\left(\Phi_A - \Phi_{AB}\right) . \end{aligned}$$

Now since all the η_B are larger than zero because of (4.7), and (4.11) implies that all the cosines in the sum are larger than zero, by inserting relation (4.9) we have shown that

$$\left| \sum_{B=1}^{N_\Lambda} \frac{\eta_B(\Lambda)}{rR_A - R_B} \right| \geq \frac{\eta_A(\Lambda)}{r-1} \geq \frac{\Pi}{r-1} ,$$

that proves Lemma (4.10) and Griffiths theorem.

4.2 Dynamical Effects of Griffiths Singularities

As we have already suggested Griffiths singularities show their most powerful effect in changing the nature of the dynamics of diluted systems, making it far slower than the usual dynamics of magnetic material. We will follow here the work of references [11, 12, 13] in trying to clarify the nature of these effects. We will even learn here something more than the physics of Griffiths singularities, since we will see that in diluted systems already for values of the temperature that would be in the paramagnetic phase of the pure material the dynamics deeply changes in nature, ceasing to be a pure exponential dynamics.

If one considers ferromagnetic systems away from the critical point, or even typical systems with quenched random disorder at very high temperatures, one finds that connected correlation functions decay exponentially,

i.e.

$$C(t'-t) \equiv \overline{\left\langle \frac{1}{V}\sum_{i=1}^{V}\sigma_i(t)\sigma_i(t') \right\rangle_c} \sim e^{-\frac{|t'-t|}{\tau}} ,$$

where we have considered the disorder averaged correlation function (this is irrelevant for systems without quenched disorder). We know that τ diverges at the critical point

$$\tau \sim_{T\longrightarrow T_c} \xi^z ,$$

where z is the *dynamical critical exponent.*

We will discuss here the fact that, on general grounds, if quenched disorder is present one can get a more complex behavior. Maybe the main result we will discuss will be that, in the Griffiths phase, for an Ising model,

$$C(t) \sim e^{-A(\log(t))^{\frac{d}{d-1}}} , \tag{4.13}$$

where d is, as usual, the spatial dimension of the system. This behavior can be seen as a dynamical signature of the Griffiths singularities. We will discuss this and related results by using scaling arguments, from [11, 12, 13]: recently this behavior has been proved rigorously in [14]. The fact that $\log C \sim -A\left(\log(t)\right)^{\frac{d}{d-1}}$ means that C decays here more slowly than en exponential or a stretched exponential ($\sim \exp(-\frac{t}{\tau}^{\alpha})$).

An instructive observation is that the behavior of equation (4.13) has not yet been observed in numerical simulations (even if eventually, for very large times, it has to emerge since it has been proved rigorously): fits of real experimental data are usually not precise enough to allow a clear cut distinction of a power of a logarithmic behavior from a power law decay. In numerical simulations, even on very large time scales (on nowadays computers) one always observes a stretched exponential: the $(\log(t))^{\frac{d}{d-1}}$ will only eventually emerge at larger times. This is instructive about the care that is frequently needed to rightly interpret data from numerical simulations.

Let us state more precisely the body of results we will discuss here. We will consider *ferromagnetic diluted systems*, and we will look both at the region $T \longrightarrow T_c^+(p=1)$, i.e. at the paramagnetic phase when T approaches the critical temperature of the pure system (that is larger than the critical temperature of the diluted system) and at the Griffiths phase , i.e. for $T_c < T < T_c(p=1)$. The idea behind these findings is the same that is behind the static treatment: there exist large, rare regions, that are very ordered

and sometimes are very weakly connected. These regions relax very slowly to equilibrium: typically a single region decays exponentially, and the non-exponential behavior is given by the need to bring the different regions at thermal equilibrium.

So, we summarize, mainly following Alan Bray [12, 13]. First, defining z_p the dynamical critical exponent of the pure system,

$$\boxed{\text{for } T \longrightarrow T_c^+(p=1) \equiv T_G^+ \text{ one finds that } C(t) \sim e^{-t^{\frac{d}{z_p+d}}},}$$

i.e. a slower decay than in the pure system. For large times, and $\frac{|T-T_G|}{T_G} \neq 0$ the scaling function has an atypical form, i.e.

$$\boxed{C(t) \sim \exp\left\{-t^{\frac{d}{z_p+d}} f\left(\frac{t}{\xi^{z_p+d}}\right)\right\}.}$$

The argument of function f is t/ξ^{z_p+d}, with an atypical dependence over the dimension d (usually the adimensional argument would be $t\xi^{-z}$).

Secondly

in the Griffiths phase, i.e. for $T_c < T < T_c(p=1)$, the relaxation toward equilibrium is even slower.

If we consider a dynamics *without conservation laws* (for example the usual Monte Carlo dynamics where we propose to update spins, one at the time, but not the Kawasaki flipping dynamics where magnetization is conserved) we find that in this case, for $t \gg \xi^{z_p+d}$,

$$\log C(t) \sim -A\,(\log(t))^{\frac{d}{d-1}} \text{ for Ising spins ,} \tag{4.14}$$

and

$$\log C(t) \sim -(Bt)^{\frac{1}{2}} \text{ for Heisenberg spins .} \tag{4.15}$$

Again, to be specific, we will consider an Hamiltonian $H = -\sum_{\langle ij\rangle} J_{ij}\ \vec{S}_i \cdot \vec{S}_j$, where the sum runs over first neighbors of a simple cubic d-dimensional lattice whose sites are labeled from the index i, the $\vec{S}_i$ are n-dimensional vector such that $|\vec{S}_i|^2 = 1$. For $n \geq 2$ these are the so-called Heisenberg of $O(n)$ (from the global symmetry they enjoy) models. For $n = 1$ we have that

$S_i = \pm 1$, i.e. we are dealing with the Ising model. The $\{J_{ij}\}$ are independent quenched independent random variables, that can carry a site or a link dilution (see the discussion of the static case) with occupation probability p. As we have already started to discuss slow anomalous relaxations are implied from the fact that large regions of the system with a fraction of connected sites larger than the average (p) appear in a large lattice with finite probability. These large clusters are typically weakly connected to the rest of the system. The *quenched* nature of the disorder implies that these *traps* cannot get modified with time, and slow down the dynamics on all time scales.

The *main assumption* behind our derivation will be that *the large time behavior of the system is determined by some precise class of clusters*, of a given density, size, shape, that we will try to determine. We will also assume that *compact clusters* control the long time dynamics (when this assumption breaks one can get a very different kind of physics).

We will use a saddle point approach to determine the typical size of the cluster that are dominating the dynamics at time t: these will typically be clusters that will locally have a density p', larger than the average density p. In other words: for $T < T_G \equiv T_C^{\rm pure}$, i.e. in the Griffiths phase, the dynamics at asymptotically long times is governed by cluster with density p' such that $T_c(p') > T$ (remember that the cluster with all occupied sites has $T_c = T_c(p = 1) = T_G$). These clusters are regions of local order that does not change in time (we have quenched noise). We will also see that also in the paramagnetic phase close to T_G, i.e. for $T \longrightarrow T_G^+$, the dominating clusters are the ones with density one.

We can start writing that, on general grounds, the time dependent correlation function $C(t)$ can be written as a sum over clusters of connected spins of all sizes and densities:

$$C(t) = \sum_{L,p'} P(L,p') \; e^{-\frac{t}{\tau(L,p')}} \; , \tag{4.16}$$

where $P(L,p')$ is the probability that a site belongs to a cluster of size L and density p', and τ is the relaxation time of such a cluster. We are looking at the decay by separating it in different decays over connected clusters of all possible densities and sizes: each one of these decays is exponential, and the non-exponential character of the dynamics is given by the sum over a large number of different decay channels. The fact that the lattice is non-homogeneous generates the non-exponential decay.

4.2.1 Dynamics in the Paramagnetic Phase

We start by discussing the paramagnetic phase, when T approaches T_G from above. In this case the Ising model and the Heisenberg model have the same behavior.

In this situation the slowest mode is given by clusters with $p' = 1$. T is getting close to T_G from above, where the pure system is critical: because of that clusters with $p' = 1$ are being the first to become critical.

The probability of finding in the lattice a cluster of size L^d with all sites occupied is

$$P(L,1) \sim p^{L^d} = e^{-cL^d} \ , \quad \text{with} \quad c \equiv \log \frac{1}{p} \ , \tag{4.17}$$

since each site is occupied with probability p independently from the others. The relaxation time of such a cluster (with $p' = 1$, i.e. with all occupied sites) scales as

$$\tau(L,1) \sim \xi_p^{z_p} \ f\left(\frac{L}{\xi_p}\right) \ , \tag{4.18}$$

where ξ_p is the correlation length of the pure system, z_p is the dynamical critical exponent of the pure system, and f is a scaling function. As usual f is such that

$$\tau(L,1) \sim \begin{cases} \xi_p^{z_p} & \text{for } L \gg \xi_p \quad \text{i.e. for large clusters ,} \\ L^{z_p} & \text{for } L \ll \xi_p \quad \text{i.e. for a large correlation length .} \end{cases}$$

L, the cluster size, has the role of the lattice size in usual finite size scaling, and the behavior changes when the correlation length of the fully occupied connected cluster reaches its size L. So, we have claimed that fully occupied clusters, i.e. clusters with $p' = 1$: we can select the contribution given by these clusters in equation (4.16), and use a saddle point to determine the typical size of such clusters. Let us insert the probability (4.17) and the correlation time (4.18) in (4.16), and find

$$C(t) \simeq e^{-cL^d - t \ \xi_p^{-z_p} f\left(\frac{L}{\xi_p}\right)^{-1}} \ . \tag{4.19}$$

The saddle point is determined by the condition

$$\frac{\partial}{\partial L}\left(c \ L^d + t \ \xi_p^{-z_p} f_1\left(\frac{L}{\xi_p}\right)\right) = c \ d \ L^{d-1} + t \ \xi_p^{-z_p} \ f_2\left(\frac{L}{\xi_p}\right) \xi_p^{-1} = 0 \ ,$$

and so

$$L \simeq t^{\frac{1}{d-1}} \; \xi_p^{-\frac{z_p+1}{d-1}} \; g\left(\frac{L}{\xi_p}\right) \; ,$$

where g is a scaling function. By inserting this result in (4.19) we find that

$$\log C \simeq -c \; t^{\frac{d}{d-1}} \; \xi_p^{-\frac{d(z_p+1)}{d-1}} \; g\left(\frac{L}{\xi_p}\right) - t \; \xi_p^{-z_p} \; h\left(t^{\frac{1}{d-1}} \; \xi_p^{-\frac{z_p+1}{d-1}} \; \tilde{h}\left(\frac{L}{\xi_p}\right) \xi_p^{-1}\right) \; , \tag{4.20}$$

where g, h and $\tilde{h}$ are again scaling functions. We will determine the scaling relation of time and correlation length by looking at the second contribution to the right hand side of (4.20) (and later we will verify that the first contribution behaves in a consistent way: it better does!). The argument of the function h is adimensional: since $\tilde{h}$ is also adimensional this implies that

$$t^{\frac{1}{d-1}} \; \xi_p^{-\frac{z_p+1}{d-1}-1} = \left(t \; \xi_p^{-(z_p+d)}\right)^{\frac{1}{d-1}}$$

is adimensional, i.e. that

$$t \sim \xi_p^{z_p+d} \; . \tag{4.21}$$

This result is a truly remarkable atypical scaling. We are indeed used to the fact that the time t scales as a power of the correlation length, but here we have found an additional precise, explicit dependence over the dimensionality of the space, d. Now under (4.21) the scaling function h rightly has an adimensional argument, while the dimensional argument that multiplies it scales as

$$t \; \xi_p^{-z_p} \sim t \; t^{-\frac{z_p}{z_p+d}} = t^{\frac{d}{z_p+d}} \; . \tag{4.22}$$

As one would have expected it is trivial to verify that the first contribution to the sum in the right hand side of (4.20) has the same scaling found in (4.22): indeed we have that

$$t^{\frac{d}{d-1}} \; \xi_p^{-\frac{d(z_p+1)}{d-1}} \simeq t^{\frac{d}{d-1}} \; t^{-\frac{d(z_p+1)}{(d-1)(z_p+d)}} \simeq t^{\frac{d}{z_p+d}} \; ,$$

QED. We have established that, in the paramagnetic region, when T approaches T_G from below,

$$\log C(t) \simeq -t^{\frac{d}{z_p+d}} g\left(t \; \xi_p^{-(z_p+d)}\right) \; , \tag{4.23}$$

where g is a scaling function. Now we can discuss the two most relevant limits. For very large times, i.e. $t \gg \xi_p^{z_p+d}$, i.e. times where we are exploring

distances that are far larger than the correlation length of the problem) the relevant lengths are of order ξ_p, since the typical size of a correlated clusters has reached ξ_p and has stopped to grow. In these conditions from (4.19) we find that:

$$\log C(t) \simeq -c\ \xi_p^d - t\ \xi_p^{-z_p},\ \text{for } t \gg \xi_p^{z_p+d}\ . \tag{4.24}$$

The opposite limit is the one where the time we have waited has not allowed to explore lengths of the order of the size determined by $\xi_p^{z_p+d}$. In this regime the scaling function g of equation (4.23) behaves as a constant, i.e. $g(0) \sim$ constant, and

$$\log C(t) \simeq -t^{\frac{d}{z_p+d}},\ \text{for } t \ll \xi_p^{z_p+d}\ . \tag{4.25}$$

Exactly at $T = T_G$ equation (4.25) is valid at all times, since $\xi_p(T_G) = \infty$.

The argument presented in this subsection is valid both for Ising and Heisenberg like systems: only the value of z_p will depend on the details of the model, but the new features due to the presence of dilution are unchanged.

A few words about the sense of this procedure are in order. We are working here to connect time scales (the amount of time we have left to the system to equilibrate in the phase space) and distances (which typical size of domains we system is exploring at a given time). When we start with, for example, a high T disordered sample, spins are uncorrelated, and the correlation length at time zero is of the order of one lattice spacing. Correlation grows in time, allowing to study regimes of larger distance: eventually this growth will be saturated when we reach the asymptotic correlation length of the system, if it is finite.

4.2.2 Dynamics in the Griffiths Phase

We will discuss now what happens in the Griffiths phase, i.e. for $T_c(p) < T < T_G = T_c(p = 1)$. Here the result will depend crucially, for reasons we will discuss, from the details of the model (Ising or Heisenberg). We will assume in the following that the dynamics does not possess special conservation laws: our results will be valid, for example, for the usual Glauber dynamics, and for example for a Metropolis like evolution, while they will not apply, for example to the spin flip Kawasaki dynamics (that conserves the total magnetization of the system).

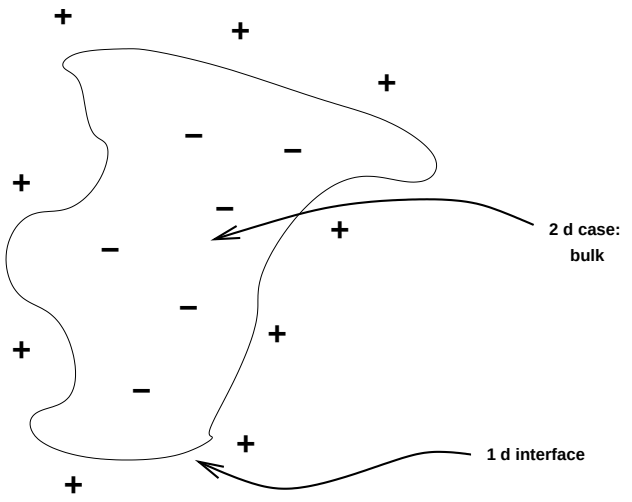

Figure 4.4: A cluster that has to be reversed ($2d$ case): the relevant interaction is on the $1d$ interface.

Dynamics in the Griffiths Phase: the Ising Case

Here the dominating clusters are again clusters with $p' = 1$, i.e. local connected cluster where sites are fully occupied. Let us discuss why.

We start by giving an estimate for $\tau(L, p')$, the relaxation time of a cluster of size L and average density p'. We analyze the $p' = 1$ case: for doing that we have to determine the time needed to flip the full cluster. This time needed for a coherent reversal of all cluster spin is the mode that determine the process rate: since spins can only take the values ± 1 this is the only way we have to flip the cluster. In order to flip the cluster we will have to pay the price of breaking an interface of size L^{d-1} (see figure (4.4) for the two dimensional case). So we find that the relaxation time for flipping a compact cluster

$$\tau(L, 1) \sim \tau_p \; e^{\sigma L^{d-1}} \; , \tag{4.26}$$

where τ_p is a microscopic time that characterizes a spin flip in the ordered phase of the pure system, σ is the surface tension and σL^{d-1} the free energy carried by the interface. The T^{-1} factor *(Arrhenius law)* has been included in σ. What is crucial in equation (4.26) is that τ depends exponentially from L (we will see that this will be the feature that will select as dominating clusters the one with $p' = 1$).

The entropic term (that favors clusters with smaller local density) is $\log P(L,p')$, and it depends on L only like a power: we have already noticed in (4.17) that $\log\left(P(L,1)\right) \sim -cL^d$. Again, the entropic term that favors clusters with smaller local density only increases as a power in L, while the energetic term, that makes cluster with $p'=1$ important grows as en exponential in L: asymptotically, for large L, fully occupied clusters will be the most important ones in slowing down the dynamics. So we can consider clusters with $p'=1$, and inserting (4.17) and (4.26) in (4.16) we find that

$$C(t) \sim \sum_L e^{-c\ L^d - \frac{t}{\tau_p} e^{-\sigma L^{d-1}}} \quad . \tag{4.27}$$

For large t we have to determine the value of L that maximizes the contributions to the sum. For doing that we go to the scaling variable $y \equiv \sigma^{\frac{1}{d-1}} L$, and we solve

$$c\ d\ \sigma^{-\frac{d}{d-1}}\ y^{d-1} - \frac{t}{\tau_p}\ (d-1)\ y^{d-2} e^{-y^{d-1}} = 0\ ,$$

that gives

$$L \sim \left[\frac{1}{\sigma} \log\left(\frac{t}{\tau_p}\ \sigma^{\frac{d}{d-1}}\right)\right]^{\frac{1}{d-1}} \quad . \tag{4.28}$$

Now (4.27) and (4.28) give

$$\log C(t) \sim -\left[\frac{1}{\sigma} \log\left(\frac{t}{\tau_p}\ \sigma^{\frac{d}{d-1}}\right)\right]^{\frac{d}{d-1}} \quad . \tag{4.29}$$

It is interesting to discuss the limit in which $T \longrightarrow T_G^-$, i.e. when the system is reaching the critical temperature of the pure system from below. In this limit

$$\sigma \sim \xi_p^{-(d-1)} \text{ and } \tau_p \sim \xi_p^{z_p}\ ,$$

that in (4.28) gives

$$L \sim \xi_p\ \log\left(\frac{t}{\xi_p^{z_p+d}}\right)^{\frac{1}{d-1}} \quad ,$$

and in (4.29) gives

$$\log C(t) \sim -\xi_p^d\ \log\left(\frac{t}{\xi_p^{z_p+d}}\right)^{\frac{d}{d-1}} \quad \text{for} \quad t \gg \xi_p^{z_p+d}\ . \tag{4.30}$$

It is interesting that in (4.30) there is no sign of the criticality that the system undergoes when $T \longrightarrow T_c(p)$: the divergence of the correlation length in the Griffiths phase is controlled by the criticality at T_G, i.e. by the presence of large connected clusters of density 1.

Dynamics in the Griffiths Phase: the Heisenberg Case

We will discuss now what happens for Heisenberg models, i.e. when spin are vectors defined on a n-dimensional sphere, with $n \geq 2$. The big difference is that in this case the elementary spins can be changed with continuity, by small increments: in this case we do not need a complete flip, $1 \rightarrow -1$ or vice-versa, to move a spin. Since it is possible to go by using small changes from one value of a spin to a different value $\tau(L, p')$ turns out to be far smaller than in the Ising case.

The main mechanism behind the relaxation of Heisenberg spins is very different then for Ising spin: here the thermal noise drives a diffusion of the order parameter on the n-dimensional sphere. Let us call $\vec{M}(t)$ the magnetization of a given connected cluster of spins at time t, and $\delta\vec{M}$ the variation of the magnetization in a single time step (where all spins of the cluster change once). Since the spins are moving thanks to an incoherent thermal noise we have that in one time step

$$\delta\vec{M} \sim L^{\frac{d}{2}} \ . \tag{4.31}$$

After t steps

$$\delta\vec{M}(t) \sim L^{\frac{d}{2}} \ t^{\frac{1}{2}} \ ,$$

where the power of L is from (4.31), and the power of t is because of the random walk in time (different time steps are uncorrelated).

Which is the time that a cluster of size L needs to flip completely? It is the time τ needed for the change of M to become of order volume, i.e. such that $\delta\vec{M}(\tau) \sim L^d$, that is $L^{\frac{d}{2}} \ \tau^{\frac{1}{2}} \sim L^d$, i.e.

$$\tau \sim L^d \ .$$

We can extend our estimate to the behavior of cluster with density $p' \leq 1$. In this case we have to remember that the time scale is given by $\tilde{\tau}(p')$, the relaxation time of a bulk system of density p', and the length scale is given by the correlation length of a system at density p', $\tilde{\xi}(p')$. So the scaling law

becomes now

$$\tau(L,p') \sim \tilde{\tau}(p') \left(\frac{L}{\tilde{\xi}(p')}\right)^d \quad . \tag{4.32}$$

Now that we have the behavior of $\tau(L,p')$ we need the probability that a connected cluster composed of $N = L^d$ sites contains Np' occupied sites. It is

$$P(L,p') = \left(\begin{array}{c} N \\ N_{p'} \end{array} \right) p^{Np'} (1-p)^{N(1-p')} \quad . \tag{4.33}$$

Developing the Newton binomial we find that

$$\begin{aligned} \log P(L,p') \simeq -L^d \left(p' \log \frac{p'}{p} + (1-p') \log \frac{1-p'}{1-p} \right) \\ + \text{sub-leading corrections } \equiv -L^d f(p') \ . \end{aligned} \tag{4.34}$$

It is easy from (4.32) and (4.34) to use a saddle point to compute the asymptotic behavior of $C(t)$ (we leave that as an exercise to the reader) and find that

$$\log C(t) \sim (Bt)^{\frac{1}{2}} \quad . \tag{4.35}$$

Chapter 5

Scaling Laws

In the following we will discuss the phenomenological theory of scaling. We will discuss about scale invariance, scaling and hyperscaling, about the role of fluctuations. We will discuss critical exponents and derive relations among them.

We have already used the methods that we describe here when discussing, for example, the dynamical effects of Griffiths singularities, and that chapter should be reread and rethought after a careful reading of the following sections.

The foundations of the subject can be found in references [15] and [16]. The reprint collection contained in [17] is a very useful tool. We will follow here closely the way in which [2] deals with the subject: also the discussion of [1] is very relevant to the points we are making here.

5.1 Scale Invariance

Second order phase transitions and critical phenomena are the main foundation on which our work is built. We are basing the most of our analysis of disordered systems by studying how criticality is modified by the presence of disorder, and by generalizing the concept of phase transitions (Parisi theory being the crucial contribution in this direction). Universality is here a crucial issue. A large number of completely different physical phenomena, where a correlation length diverges, $\xi \to \infty$ for some precise value of the external parameters that appear in the Hamiltonian, have the same kind of behavior. On one side the Curie point if transitions from a paramagnetic

to a ferromagnetic regime is a typical example; on the other side the case of *critical opalescence* in a gas-liquid transition makes clear the strength of cooperative effects, since regions of the scale of microns cooperate to produce a visible effect on a very macroscopic scale. One can easily produce a large list of very well known and important critical phenomena, ranging from magnets, to crystalline structures and conductivity in random media, to super-conductors and super-fluidity, up to polymers, proteins, liquid crystals and percolative phenomena, to end with gauge systems describing the fundamental physics of elementary particles.

One observation about this list opens an important road. All these phenomena are organized in few *universality classes*, characterized by the same *critical exponents*: non-universal quantities, like the value of the critical temperature (and in general of the critical parameters) or of the amplitude that multiply the diverging quantities, do depend from the details of the interaction, but the exponents of the power divergence only depend from a few relevant features like the dimensionality of the system, the symmetries of the Hamiltonian and the space where the order parameter is defined. This crucial fact can be naturally connected to the fact that the physics of the system is dominated, at criticality, by a diverging correlation length. In these conditions all the small distance structure of the system (like the exact degree and pattern of connectivity, for example) becomes irrelevant and is forgotten: only what happens at very large distances is important.

We will try in the following to see how from a few main theoretical hypothesis (mainly the dominance of a single, diverging correlation length at the critical point and the existence of the thermodynamical limit) one can derive a number of physical implications. The relation of this scaling theory to the theory of renormalization group is very reminiscent of the relation of thermodynamics (a phenomenological theory, where one never says how things work, but starts from a few principles and use a consistent mathematical apparatus to derive consequences) to statistical mechanics (where one use the law of forces together with the law of large numbers).

In the mean field theory, that we have discussed in detail, one finds the existence of a critical point (and this is a crucial and non-trivial feature), and a well defined critical behavior. There is an *upper critical dimension*, d_c^u, that is equal to 4 for the universality class of the Ising model (ϕ^4 theory), and for $d > d_c^u$ critical exponents do not change, and things remain basically the same, since fluctuations are not effective (this is connected to the fact that generically two random travelers never meet in $d \geq 4$, where one is very

lonely).

We start by considering the critical temperature T_c, and the effective temperature

$$\theta \equiv \frac{T - T_c}{T_c} \ . \tag{5.1}$$

For $\theta > 0$ we are sitting in the disordered, symmetric phase, while when $\theta < 0$ we move to the ordered phase, where the original symmetry of the Hamiltonian is broken. In field theory θ is connected to m^2, a bare squared mass. In a continuous transition a correlation length ξ diverges when $\theta \to 0$. Let a be the elementary lattice spacing of our system.

In this framework let us start by studying the regime where $\xi \gg a$, i.e. where the correlation length, that diverges at T_c, is far larger than the microscopic scale of the theory. When θ is small we have a large value of ξ (the typical size of a cluster of aligned spins). When $\theta = 0$ the scale given by ξ disappears, since $\xi \to \infty$: the theory becomes *scale free*. We will also assume that there are no large anisotropies, and that the theory is becoming isotropic when approaching the critical point.

Let us consider for example a two point correlation function $G(\vec{x})$. For example for a scalar field $\phi(\vec{x})$ we have

$$G\left(\vec{x}\right) \equiv \langle \phi(\vec{0})\, \phi(\vec{x}) \rangle \ ,$$

where lengths are expressed in terms of a. We assume, by ignoring anisotropies, that

$$\frac{G(|\vec{x}|)}{G(|\vec{y}|)} = \tilde{R}\left(\frac{|\vec{x}|}{|\vec{y}|}, \frac{|\vec{y}|}{a}\right) \ , \tag{5.2}$$

i.e. that nothing changes if we double both the length of $\vec{x}$, the length of $\vec{y}$ and the microscopic scale. Now we use the crucial assumption that the critical limit exists, i.e. that for fixed $|\vec{x}| \gg a$ and $|\vec{y}| \gg a$ and $a \to 0$

$$\frac{G(|\vec{x}|)}{G(|\vec{y}|)} = R\left(\frac{|\vec{x}|}{|\vec{y}|}\right) \ , \tag{5.3}$$

i.e. that in the critical limit the dependence over a disappears (since the relevant scales are determined by the diverging ξ). Now because of (5.3) the obvious relation

$$\frac{G(|\vec{x}|)}{G(|\vec{y}|)} = \frac{G(|\vec{x}|)}{G(|\vec{z}|)} \cdot \frac{G(|\vec{z}|)}{G(|\vec{y}|)}$$

implies that in the critical limit

$$R\left(\frac{|\vec{x}|}{|\vec{y}|}\right) = R\left(\frac{|\vec{x}|}{|\vec{z}|}\right) \cdot R\left(\frac{|\vec{z}|}{|\vec{y}|}\right)$$

that in turn implies that

$$R(r) \simeq r^{\gamma} \ ,$$

i.e. that at criticality correlations behave as power laws. In other terms we have found that, under our hypothesis, at $T = T_c$, i.e. at $\theta = 0$ we have that

$$G(|\vec{x}|) = G(|\vec{y}|) \left(\frac{|\vec{x}|}{|\vec{y}|}\right)^{-(d-2+\eta)} \ , \tag{5.4}$$

that gives the usual definition of the critical exponent η. η has been defined in (5.4) in such a way to be zero in mean field (where the power exponent is $d-2$ from naive power counting): the *anomalous dimension* η measures the deviation from the Gaussian approximation. When $\eta \neq 0$ we are dealing with a scale invariant (since correlations behave as power laws), non-trivial (since the exponent is different from $d-2$) field theory.

5.2 Relevant Parameters, Scaling and Hyperscaling

To understand better what is happening let us seat at small $\theta > 0$, in the disordered phase close to the critical point $\theta = 0$. Here ξ is large, and $\xi \gg a$, i.e. the system has developed a correlation length that is far large than the natural, microscopic scale. In this region we can discuss three different regimes for the distance $|\vec{x}|$:

1. the region of large correlation length, where $\xi \gg |\vec{x}| \gg a$;
2. a transient region, where $\xi \simeq |\vec{x}| \gg a$;
3. the region of large distances, where $|\vec{x}| \gg \xi \gg a$.

In the three cases we are looking at distances very large on the scale of the lattice spacing a. When θ becomes smaller and we approach the critical point ξ increases, enlarging the first region and shrinking the third one, but

till $\theta > 0$ we find the three different regions that we have enumerated: only exactly at $\theta = 0$ we are always in the first region. Let us discuss what happens in the three regions:

1. in the first region we can consider ξ as infinite. Everything works as if we were sitting at $\theta = 0$;

2. this is the transition region, where we start to observe violations from the power law behavior. The behavior in this region does not have clear signatures: things are murky;

3. here correlation function decay exponentially. This region is not feeling the effects of criticality anymore. On these length scales the system behaves as a normal paramagnet. There are no traces of the scale invariant, power law behavior.

This discussion shows that θ is a *relevant parameter* that decides about the infrared (i.e. at large distances) behavior of the system. When we look at larger and larger distances the system becomes more and more sensitive to the fact that $\theta \neq 0$, i.e. that $T \neq T_c$, even of a small amount: when we look at distances large enough even a small deviation from the exact point of criticality, $\theta = 0$, dramatically change the behavior of the system making correlation functions to decay exponentially. We can summarize by saying that

$$G(|\vec{x}|) \simeq \frac{1}{|\vec{x}|^{d-2+\eta}} \; g\left(\frac{|\vec{x}|}{\xi(\theta)}\right) \; , \tag{5.5}$$

where the scaling function g is regular in the origin (the first of our three regions), and decreases exponentially at infinity (the third region).

A last brief summary of what we have done could maybe help *(repetita juvant)*. We seat at T very close to T_c, just slightly above it. We look at correlation functions, and find that for distances that are not too large they scale in a way very similar to the critical, $T = T_c$ one, i.e. they go to zero as a power law, $|\vec{x}|^{-(d-2+\eta)}$ times a smooth, regular function. When we increase the distance we have a crossover region, and for very large distance we do not feel the critical point anymore, and correlation functions decay exponentially again.

As $\theta \to 0$ we expect, for example from mean field theory, that the correlation length diverges as a power law, i.e. that

$$\xi(\theta) \sim \xi_+ \; \theta^{-\nu} \; , \quad \text{for } \theta \longrightarrow 0 \; , \tag{5.6}$$

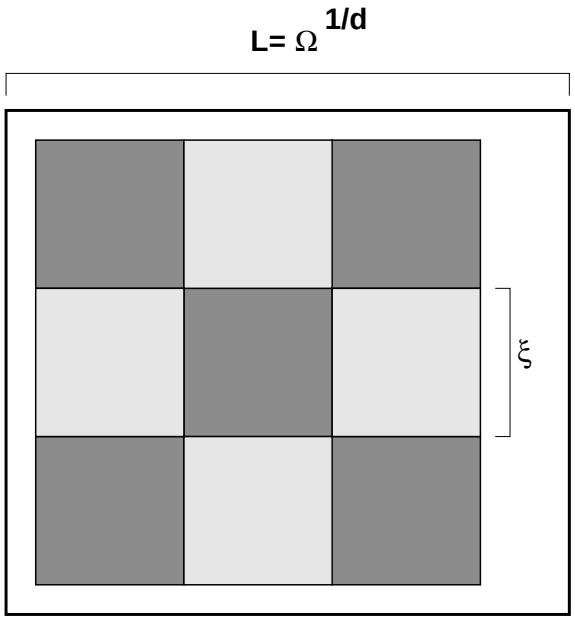

Figure 5.1: The system in a volume Ω is divided in blocks of size ξ.

that relates a length scale to a temperature scale.

Notice that we have already, in this discussion, introduced two (fundamental) critical exponents, η and ν. ξ_+ is what we have called a *critical amplitude*: it is not universal and it depends on the details of the system.

The real meaning of this discussion is found when one assumes that our main hypothesis of a single diverging correlation length determines the behavior of the system for different physical quantities. So we will assume the *scaling hypothesis*:

close to the critical point ξ is the only relevant length scale.

For our next step we will need an additional, independent hypothesis. Let us consider the free energy of our system in the volume Ω. We divide the system in blocks of linear size $\xi(\theta)$ and volume $\xi(\theta)^d$ (see figure (5.1)), and assume that each of these blocks of size $\xi(\theta)^d$ can be seen as an individual unit, since they fluctuate coherently (by the definition of correlation length ξ).

Taking completely seriously this statement, and adopting it for evaluating the fluctuations of the individual blocks, is the so called *hyperscaling*

hypothesis. If we use this hypothesis to estimate the singular part of the free energy density we find that:

$$\boxed{f_{\text{singular}}(\theta) = \frac{F_{\text{singular}}(\theta)}{\Omega} \sim \frac{1}{\Omega}\left[\frac{\Omega}{\xi(\theta)^d}\right] \sim \theta^{\nu d} \ ,}$$

where the ratio in square brackets is the number of regions in a given volume Ω, and the validity of this relation, again, is based on the validity of the hyperscaling hypothesis. Now we consider for example the specific heat:

$$C(\theta) \sim \frac{\partial^2 f}{\partial \theta^2} \sim \theta^{\nu d - 2} \sim \theta^{-\alpha} \ ,$$

where the last relation comes from the definition we have already given in former parts of these notes of the exponent that characterizes the singularity of the specific heat (the specific heat diverges for $\alpha > 0$, while it exhibits a cusp if $\alpha < 0$). So, we have been able to derive a first implication of the scaling and hyperscaling hypothesis:

$$\alpha = 2 - \nu d \ . \tag{5.7}$$

When does this result, based on the hyperscaling hypothesis, hold? The two dimensional Ising model has a logarithmic singularity in the specific heat, that translates in $\alpha = 0$. That makes us to expect from (5.7) $\nu = \frac{2-\alpha}{d} = 1$, that is true. Also the exact computation for the $O(n)$ spin model in the limit $n \to \infty$ confirms the validity of hyperscaling.

In mean field theory, on the contrary, there is a problem. Here things do not work, since for all values of d $\alpha_{MF} = 0$ and $\nu_{MF} = \frac{1}{2}$: exponents do not depend on d, and the relation (5.7) is not satisfied in $d > d_c^u$ where the mean field picture is valid. Thinking about it, this failure turns out to be deeply justified. Indeed the construction of the mean field has been based on neglecting fluctuations, so that using their amplitude to determine criticality appears unreasonable. It is nice, on the contrary, that we can turn this failure in a success by using the point to estimate the value of the upper critical dimension d_c^u (in a different way from the usual one based on the linear response theory). We just notice that d_c^u is the value of the space dimension where fluctuations start to be relevant: for $d > d_c^u$ they are irrelevant, while at d_c^u they start to be important and change the critical

behavior of the system. So we can define $d > d_c^u$as the dimension where in mean field hyperscaling holds. That gives

$$\alpha_{MF} = 2 - \nu_{MF} d_c^u \ ,$$

i.e. correctly $d_c^u = 4$, the same result we have obtained when looking at the point where corrections to mean field explode (the Landau-Ginzburg criterion).

5.3 Relations Among Critical Exponents

We will now use these ideas to determine a set of relations among different critical exponents. We will succeed, starting from our hypothesis, in deriving regular relations that connect singular quantities. Notice that in the following discussion we will assume that the singularities of the different quantities are the same above and below T_c. We start by considering the susceptibility χ:

$$\chi = \frac{\partial^2 F}{\partial h^2} \propto \int d^d x \ G(x) \ ,$$

that because of (5.5) gives

$$\chi \simeq \int_{|\vec{x}|<\xi(\theta)} d^d x \ \frac{1}{|\vec{x}|^{d-2+\eta}} \sim \xi(\theta)^{2-\eta} \sim \theta^{-\nu(2-\eta)} \ ,$$

that since

$$\chi(\theta) \simeq \theta^{-\gamma} \tag{5.8}$$

gives

$$\gamma = \nu(2-\eta) \ . \tag{5.9}$$

For continuing our discussion we consider the spontaneous magnetization $m(\theta) \equiv \lim_{h\to 0^+} m(h,\theta)$ when we approach the critical temperature from below. We have a critical behavior:

$$m(\theta) \sim (-\theta)^{\beta} \text{ for } \theta \longrightarrow 0^- \ , \tag{5.10}$$

and for example in mean field one finds $\beta_{MF} = \frac{1}{2}$. Exactly at $\theta = 0$ we have

$$m(h, \theta = 0) \sim_{h\to 0} h^{\frac{1}{\delta}} \ . \tag{5.11}$$

The relations (5.10) and (5.11) tell us that h, m^δ and $(-\theta)^{\beta\delta}$ scale homogeneously. Because of that we can write

$$h = m^\delta \; f_h\left(\theta \; m^{-\frac{1}{\beta}}\right) \; , \tag{5.12}$$

where f_h is an adimensional scaling function. We also notice that, for $\theta \longrightarrow 0^+$,

$$m(\theta) = \chi(\theta) \; h \; . \tag{5.13}$$

We call z the adimensional variable we have used in (5.12), by defining $z \equiv \theta m^{-\frac{1}{\beta}}$, i.e. $m^\delta = z^{-\beta\delta} \; \theta^{\beta\delta}$. Now equation (5.12) becomes

$$h = z^{-\beta\delta} \; \theta^{\beta\delta} f_h(z) \; , \tag{5.14}$$

and (5.13) becomes

$$h = \chi^{-1}(\theta) \; z^{-\beta} \; \theta^{\beta} \; . \tag{5.15}$$

From (5.14) and (5.15) we have that

$$f_h^{-1}(z) \; \chi^{-1}(\theta) = z^{-\beta(\delta-1)} \; \theta^{\beta(\delta-1)} \; . \tag{5.16}$$

From (5.8) this implies that

$$\gamma = \beta(\delta - 1) \; , \tag{5.17}$$

and looking at the z-dependent part we have that

$$f_h(z) \simeq z^{-\beta(\delta-1)} \; .$$

We want to derive a last interesting relation. Since we have defined the exponent α by $C_V \sim \theta^{-\alpha}$ the singular part of the free energy density will scale as $f_{\text{singular}} \sim \theta^{2-\alpha}$. Now we notice that $\chi = \frac{\partial^2 f}{\partial h^2} \sim \theta^{-\gamma}$, and since each derivative with respect to h adds an exponent of $(-\beta \; \delta)$ we find

$$2 - \alpha - 2\beta \; \delta = -\gamma \; . \tag{5.18}$$

So we have derived four relations among six critical exponents: we have relations (5.7), (5.9), (5.17) and (5.18) among the exponents η, ν, α, γ, β and δ. If we decide to express everything as a function of η and ν with a very simple algebra we find that

$$\left\{ \begin{array}{lcl} \alpha & = & 2 - \nu d \; ; \\ \beta & = & \frac{\nu}{2}(d - 2 + \eta) \; ; \\ \gamma & = & \nu(2 - \eta) \; ; \\ \delta & = & \frac{d+2-\eta}{d-2+\eta} \; . \end{array} \right. \tag{5.19}$$

Thanks to (5.19) we are able to determine all critical exponents from the knowledge of η, the exponent of the anomalous scaling of the Green function at the critical point, and of ν, the exponent characterizing the divergence of the correlation length when $T \to T_c$. ν and η, together with the scaling and hyperscaling hypothesis, allow to determine all the critical exponents.

As we have already discussed finite size scaling is a phenomenological theory, where one never needs to say how things work, but starts from a set of plausible hypothesis and, by using mathematical consistence, derives relevant consequences. The theory of renormalization group, that we will not discuss in this notes (see for example [1, 2, 3] for relevant texts) gives foundation to these relations, and is a tool to compute critical exponents and other universal quantities. Other important instruments to do detailed computations are among others high and low T expansions and Monte Carlo methods.

Chapter 6

Lee-Yang Theorem

We will now discuss and prove Lee-Yang theorem [18]: again, we have postponed the discussion of this fundamental tool, that has been already crucial when proving, for example, Griffiths theorem in chapter (4). The discussion here will again inspired by the one in [2] (that is in turn inspired by the treatment of Ruelle [19]): the text of Huang [4] also contains a relevant introductory discussion of the subject.

6.1 A Proof of Lee-Yang Theorem

The approach that we will discuss now is based on determining the position of the zeroes of the partition function. We have already discussed how the presence of singularities in the derivatives of the free energy (i.e. critical phenomena) in the infinite volume limit are connected to zeroes of the partition function for real values of the external parameters of the Hamiltonian. In their seminal paper [18] Lee and Yang established some crucial facts about the location of these zeroes.

We will deal with an Ising model on a generic lattice structure. We will focus on the properties connected to the behavior of the system under the influence of an external magnetic field h. Since we are interested in analyticity properties of the thermodynamical functions we will have to consider complex values of h: we will look for zeroes of partition function in the complex h plane.

Let us just repeat the basics we have already discussed in the first chapters of these notes. In finite volume the partition function Z cannot have

zeroes for real values of the parameters (for example temperature T and magnetic field h). There can be zeroes only if the parameters take complex values. Only in the infinite volume limit can these zeroes migrate on the real axis, creating singularities in the thermodynamical functions. Here we will investigate where the zeroes are located in finite volume in the complex h plane.

The starting point is the key observation that, *for real values of the inverse temperature β, but for a non vanishing multiplying factor, the partition function Z in finite volume is a polynomial in the activity*

$$\rho \equiv e^{-2\beta h} \ . \tag{6.1}$$

All that said, we can start by stating *Lee-Yang Theorem*:

the zeroes of the partition function Z lie on the unitary circle of the plane of complex activity ρ, or, in other words, on the axis of pure imaginary magnetic field h.

We can explain better, in these terms, the mechanism that is behind the birth of a phase transition in the infinite volume limit (but we stress that the Lee-Yang theorem in itself says nothing about the potential existence of a phase transition in the infinite volume limit: Lee-Yang theorem only tells about the locus of the zeroes for finite volume and in the infinite volume limit). So, in a finite volume V the partition function Z is, apart from irrelevant factors, a polynomial in ρ: this implies that there is a finite number of zeroes in the ρ complex plane. The number of zeroes increases with V. When $V \to \infty$ the zeroes "condense" on some subset of the unitary circle.

Now, with an eye to figure (6.1), let us thing about what happens at high T, i.e. for $\beta < \beta_c$ (figure (6.1.a)): in this case the region close to $h = 0$, i.e. to $\rho = 1$ is left free of zeroes in the infinite volume limit. The free energy density is analytic in the thermodynamical limit. When β increases and approaches β_c the gap closes, while the zeroes approach the real axis (figure (6.1.b)). At β_c the gap closes (figure (6.1.c)), and it forbids analytic continuation from the $h > 0$ to the $h < 0$ region: we have a phase transition at $h = 0$. Following this way of thinking one can also see that the transition is of first order, with a jump in the magnetization $\delta m \equiv m^+ - m^- \equiv 2m_s$ (as we have discussed in detail in the first part of these notes).

We will consider here the Ising model (with spin variables $\sigma_i = \pm 1$) on an arbitrary graph $\mathcal{G}$ with N vertices. There are N spins σ_i that seat on the N vertices. Each couple of vertices is connected at most by one link: in

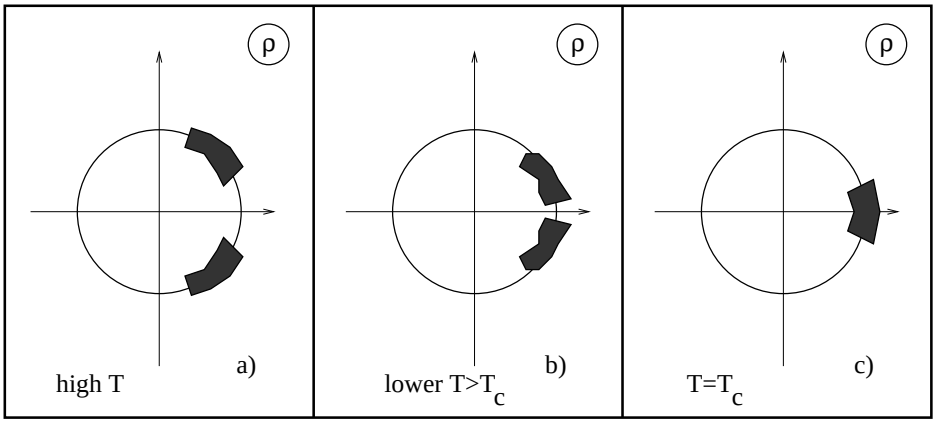

Figure 6.1: Location of the zeroes in the complex ρ plane. **a)** is for high T, where the zeroes lie far away from the real axis. **b)** is for lower T values, always with $T > T_c$: the zeroes are approaching the real axis. **c)** at $T = T_c$ in the infinite volume the zeroes pinch the real axis, creating a singularity in the thermodynamical functions. We only draw what is typically happening in the $\Re(\rho) > 0$ region.

this case we call the two vertices *first neighbors*. We call L the total number of links of the graph (for example in the case of a simple cubic lattice in d dimensions we have that $L = dN$). Let also be h_i the magnetic field on site i, and

$$\rho_i \equiv e^{-2\beta h_i} \quad , \quad \tau \equiv e^{-2\beta} \ . \tag{6.2}$$

The partition function on $\mathcal{G}$ is

$$\begin{aligned} Z_N &\equiv \sum_{\{\sigma_i=\pm 1\}} e^{\beta \sum' \sigma_i\sigma_j + \beta \sum_i h_i\sigma_i} \\ &= e^{\beta L + \beta \sum_i h_i} \sum_{\{\sigma_i=\pm 1\}} e^{\beta \sum'(\sigma_i\sigma_j - 1) + \beta \sum_i h_i(\sigma_i - 1)} \\ &= e^{\beta L + \beta \sum_i h_i} \ P(\tau, \{\rho_i\}) \ , \end{aligned}$$

where the sum $\sum'$ runs over all the couples of first neighbors of $\mathcal{G}$, and

$$P(\tau, \{\rho_i\}) \equiv \sum_{\{\sigma_i=\pm 1\}} e^{\beta \sum'(\sigma_i\sigma_j - 1) + \beta \sum_i h_i(\sigma_i - 1)} \tag{6.3}$$

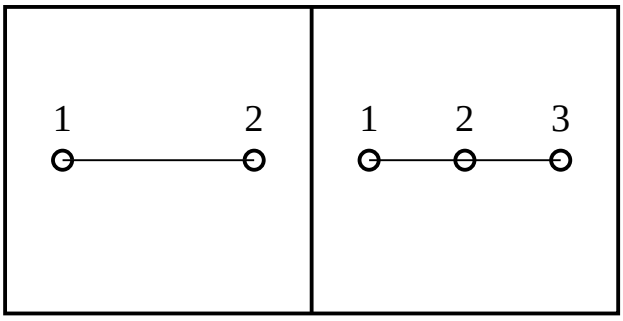

Figure 6.2: The very simple graphs with two points (on the left) and with three points (on the right).

is *a polynomial in τ and in all ρ_i*.

Obviously for τ and $\rho_i \in \mathbb{R}^+$ the polynomial P is positive definite, and it cannot have zeroes. We will assume in the following that we are in the ferromagnetic region, i.e. that $\beta \in \mathbb{R}^+$, that in turn implies that $0 < \tau < 1$.

We will start by computing exactly the partition function (or better its polynomial part P) on very small graphs. Let us start by the two point graph sketched in the left part of figure (6.2). Here we have that

$$\begin{aligned} P_{12} &= \sum_{\sigma_1,\sigma_2=\pm 1} e^{\beta[(\sigma_1\sigma_2-1)+h_1\sigma_1+h_2\sigma_2-h_1-h_2]} \\ &= \sum_{\sigma_2=\pm 1} \left(e^{\beta[(\sigma_2-1)+h_2\sigma_2-h_2]} + e^{\beta[(-\sigma_2-1)-h_1+h_2\sigma_2-h_1-h_2]}\right) \\ &= 1 + e^{-2\beta-2\beta h_1} + e^{-2\beta-2\beta h_2} + e^{-2\beta h_1-2\beta h_2} \ , \end{aligned}$$

i.e. that

$$P_{12} = 1 + \tau(\rho_1 + \rho_2) + \rho_1\rho_2 \ . \tag{6.4}$$

Now the same computation for the graph with three sites, in the right part of figure (6.2). First we sum over σ_1 and then over σ_2:

$$\begin{aligned} P_{123} &= \sum_{\sigma_1,\sigma_2,\sigma_3=\pm 1} e^{\beta[(\sigma_1\sigma_2-1)+(\sigma_2\sigma_3-1)+h_1\sigma_1+h_2\sigma_2+h_3\sigma_3-h_1-h_2-h_3]} \\ &= \sum_{\sigma_2,\sigma_3=\pm 1} \left(e^{\beta[(\sigma_2-1)+(\sigma_2\sigma_3-1)+h_2\sigma_2-h_2+h_3\sigma_3-h_3]}\right. \\ &+ \left. e^{\beta[(-\sigma_2-1)+(\sigma_2\sigma_3-1)-2h_1+h_2\sigma_2-h_2+h_3\sigma_3-h_3]}\right) \end{aligned}$$

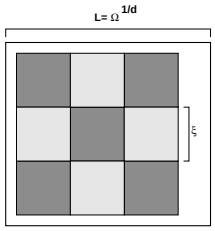

Figure 6.3: caption

$$= \sum_{\sigma_3=\pm 1} \left(e^{\beta[(\sigma_3-1)+h_3\sigma_3-h_3]} + e^{-2\beta+\beta[(\sigma_3-1)-2h_1+h_3\sigma_3-h_3]} \right.$$
$$+ \left. e^{-2\beta+\beta[(-\sigma_3-1)-2h_2+h_3\sigma_3-h_3]} + e^{\beta[(-\sigma_3-1)-2h_1-2h_2+h_3\sigma_3-h_3]} \right) .$$

At last we sum over σ_3, and substituting for τ and ρ_i we find

$$\begin{aligned} P_{123} = 1 &+ e^{-2\beta-2\beta h_1} + e^{-4\beta-2\beta h_2} + e^{-2\beta-2\beta h_1-2\beta h_2} + e^{-2\beta-2\beta h_3} \\ &+ e^{-4\beta-2\beta h_1-2\beta h_3} + e^{-2\beta-2\beta h_2-2\beta h_3} + e^{-2\beta h_1-2\beta h_2-2\beta h_3} \\ = 1 &+ \tau\rho_1 + \tau^2\rho_2 + \tau\rho_1\rho_2 + \tau\rho_3 + \tau^2\rho_1\rho_3 + \tau^2\rho_2\rho_3 + \tau\rho_1\rho_2\rho_3 \ . \end{aligned}$$

Simplifying we have that

$$P_{123} = (1+\rho_1\tau)(1+\rho_3\tau) + \rho_2(\tau+\rho_1)(\tau+\rho_3) \ , \tag{6.5}$$

that we will use in the following together with (6.4) to check our general result.

Let us repeat again that but for a regular function the free energy F is the logarithm of P: the only potential singularities of the free energy are the zeroes of the polynomial we are computing.

We will introduce now a systematic procedure to build the polynomial $P_{\mathcal{G}}$ merging together two smaller graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. In this way we will eventually be able to build step by step the polynomial connected to a generic graph $\mathcal{G}$. The theorem will be proved by detecting and analyzing some important properties of this constructive procedure. We start by considering the two disjoint graphs $\mathcal{A}$ and $\mathcal{B}$ of figure (6.3): when the consider the graph obtained as the sum of the two subgraphs (without identifying any couple of points) the polynomial related to the sum of the two graphs is the product of the two original polynomials:

$$P = P_{\mathcal{A}} P_{\mathcal{B}} \ .$$

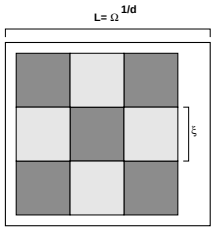

Figure 6.4: caption

By using the procedure described in figure (6.4) we generate now a new graph $a{=}b$ by selecting two sites a (in the graph $\mathcal{A}$) and b (in the graph $\mathcal{B}$) and identifying them (the same exact procedure can also be used to create a new graph by identifying two points of the same graph). The polynomial $P_{\mathcal{A}}$ is linear in ρ_a, and the polynomial $P_{\mathcal{B}}$ is linear in ρ_b (see for example the detailed computation that has led us to equations (6.4) and (6.5)). In other terms we have that

$$\begin{aligned} P_{\mathcal{A}} &= A_+ + \rho_a \; A_- \\ P_{\mathcal{B}} &= B_+ + \rho_b \; B_- \; , \end{aligned}$$

where A_+ is the contribution obtained when $\sigma_a = +1$ (σ_a is the spin defined on site a), while A_- is the contribution obtained when $\sigma_a = -1$, and the same for $B_{+,-}$ and σ_b.

Now we identify sites a and b, and we impose that $\sigma_a = \sigma_b \equiv \sigma_{ab}$, i.e. that the two spins coincide, and that there is a single magnetic field variable h_{ab}: now on the new site $a = b$ there will be a single variable ρ_{ab}. As far as the dependence over sites a and b is concerned the polynomial related to the simple sum over the two disjoint graphs has the form

$$P = P_{\mathcal{A}} P_{\mathcal{B}} = A_+ B_+ + \rho_a \; A_- B_+ + \rho_b \; A_+ B_- + \rho_a \rho_b \; A_- B_- \; ,$$

while now, after contracting a and b, since $\sigma_a = \sigma_b \equiv \sigma_{ab}$, we have that

$$P_{a=b} = A_+ B_+ + \rho_{ab} \; A_- B_- \; . \tag{6.6}$$

This is the summing rule we were looking for. We can use it to contract arbitrary parts of graphs to build up larger graphs. We can check that the procedure works in the simple examples we have analyzed before: we contract

two graphs done of two points to obtain a three point graph. Consider the 2 point graph of equation (6.4), and write its polynomial for the couple of points $(1, 2')$ and for the couple of points $(2'', 3)$. By extracting the dependence over $\rho_{2'}$ and over $\rho_{2''}$ we have that

$$\begin{aligned} P_{12'} &= (1 + \tau\rho_1) + \rho_{2'} (\tau + \rho_1) \ , \\ P_{2''3} &= (1 + \tau\rho_3) + \rho_{2''} (\tau + \rho_3) \ . \end{aligned}$$

Now identifying points $2'$ and $2''$ (and calling the new site 2) and using our rule (6.6) we find that

$$P_{123} = (1 + \tau\rho_1)(1 + \tau\rho_3) + \rho_2 (\tau + \rho_1)(\tau + \rho_3) \ ,$$

that coincides with (6.5). The procedure works. This process can be used to build an arbitrary graph starting from the two point graph.

Let us now analyze in better detail the properties of the polynomial related to the two point graph, P_{12} (6.4). The polynomial has zeroes when

$$1 + \tau (\rho_1 + \rho_2) + \rho_1\rho_2 = 0 \ ,$$

i.e. when

$$\boxed{\rho_1 = -\tfrac{1+\tau\rho_2}{\tau+\rho_2} \ .} \tag{6.7}$$

We can look at (6.7) as at a one to one conformal mapping from the complex plane ρ_2 to the complex plane ρ_1: each complex value of ρ_2 is transformed in a complex value of ρ_1. We notice two important facts.

- For real values of τ the mapping (6.7) leaves invariant the unitary circle (i.e. the locus of ρ complex values such that $|\rho| = 1$). Indeed, on the unitary circle we can define $\rho = e^{i\phi}$, and we have that

$$\left| -\frac{1 + \tau\rho_2}{\tau + \rho_2} \right|^2 = \left| -\frac{1 + 2\tau\cos\phi + \tau^2}{\tau^2 + 2\tau\cos\phi + 1} \right| = 1 \ .$$

- If, as in our hypothesis, $0 < \tau < 1$, it is easy to verify that the mapping (6.7) swaps the interior and the exterior of the unitary circle: the internal region is mapped in the external region and vice-versa. In other terms we have that $|\rho_2| < 1 \Longrightarrow |\rho_1| > 1$ and $|\rho_2| > 1 \Longrightarrow |\rho_1| < 1$.

So, we have shown that couple of values (ρ_1, ρ_2) such that $P_{12}(\rho_1, \rho_2) = 0$ are or both on the unitary circle of the complex plane or one inside and one outside it.

The next and last step consists in showing that if for an arbitrary graph $\mathcal{G}$ all the ρ_i are inside or all the ρ_i are outside the unitary circle than $P_{\mathcal{G}}$ is different from zero. This proves Lee-Yang Theorem, and we will do that by proving that this property survives the contraction rule that we have given. If this is true we are all set: we have proven that the property is valid for the simple two points graph, we have a contraction rule that we can use to build every graph starting from the two point one, and we know that the contraction rule preserves the property, so that all graphs $\mathcal{G}$ will enjoy it.

To prove this last point (i.e. that the property of being non-zero survives the contraction procedure) we consider a couple of graphs for which we know that $P(\{\rho_i\}) \neq 0$ when $|\rho_i| < 1 \forall i$. We make explicit the dependence over the two sites a and b that we want to contract. As we have already discussed the non-contracted graph sum of the two elementary graph is characterized by a factorized polynomial, product of the two elementary polynomial. Here:

$$P = A_{++} + A_{-+}\rho_a + A_{+-}\rho_b + A_{--}\rho_a\rho_b \ , \tag{6.8}$$

while we had seen that for the contracted graph $P_{a=b} = A_{++} + A_{--}\rho_{ab}$. Under our hypothesis $P \neq 0$ if $|\rho_a| < 1$ and $|\rho_b| < 1$, and we wonder now about what happens for $P_{a=b}$. We are interested to the case of constant magnetic field, so we can put $\rho_a = \rho_b \equiv \rho$. The roots of the equation $P = 0$ are

$$\rho_0 = \frac{-\left(A_{-+} + A_{+-}\right) \pm \sqrt{\left(A_{-+} + A_{+-}\right)^2 - 4A_{++}A_{--}}}{2A_{--}} \ ,$$

and our hypothesis is that $|\rho_0| \geq 1$. With some computations one sees that this implies for us that

$$|A_{++}| > |A_{--}| \ . \tag{6.9}$$

But so if we look again at the contracted graph, we find that because of (6.9) its characteristic polynomial $P_{a=b} = A_{++} + A_{--}\rho_{ab}$ cannot be zero for $|\rho_{ab}| < 1$, that proves our statement: after the contraction the property that the original graphs were enjoying is still valid for the resulting graph.

We have proved that P cannot have zeroes for $|\rho| < 1$ (we are using at this point an uniform magnetic field, $h_i = h \ \forall i$, and so $\rho_i = \rho \forall i$. Under the

transformation $h \longrightarrow -h$ the quantity ρ goes to ρ^{-1}. The explicit symmetry under inversion of the magnetic field implies for the partition function that

$$Z(h) = Z(-h) \ ,$$

i.e. that

$$e^{\beta N h} \ P(\tau, \rho) = e^{-\beta N h} \ P(\tau, \frac{1}{\rho}) \ . \tag{6.10}$$

From equation (6.10) it is clear that our result, i.e. the fact that $P \neq 0$ for $|\rho| < 1$ also implies that $P \neq 0$ for $|\rho| > 1$. **Lee-Yang Theorem follows:** the partition function can only have zeroes on the unitary circle in the complex plane, $|\rho| = 1$.

6.2 Two Simple Limiting Cases

It is worth to discuss two simple and interesting limiting cases.

The first is the one of very high temperature:

$$T = \infty \Longrightarrow \tau = e^{-2\beta} = 1 \ .$$

In this case the partition function is proportional to

$$P(1, \rho) = \sum_{\{\sigma_i = \pm 1\}} e^{\beta h \sum_i (\sigma_i - 1)} = \left(1 + e^{-2\beta h}\right)^N = (1 + \rho)^N \ .$$

The second interesting limiting case is at zero temperature:

$$T = 0 \Longrightarrow \tau = 0 \ .$$

Here the term $e^{\beta \sum (\sigma_i \sigma_j - 1)}$ of $P(0, \rho)$, in the limit $\beta \longrightarrow \infty$, forces all the spin to be equal, all $+1$ or all -1. We get

$$P(0, \rho) = \sum_{\{\sigma_i = \pm 1\}} e^{\beta h \sum_i (\sigma_i - 1)} \left(\delta_{\sum \sigma = N} + \delta_{\sum \sigma = -N}\right) = 1^N + e^{-2\beta h N} = 1 + \rho^N \ .$$

At infinite temperature (large τ, close to 1) we have a single zero with very high degeneracy equal to N, at $\rho = -1$. When T is lowered the zero split. At $T = 0$ we have an uniform distribution of simple zeroes

$$\rho_0^{(k)} = e^{\frac{i\pi(2k+1)}{N}} \ .$$

For finite N there cannot be zeroes at $h = 0$, i.e. at $\rho = 1$. What happens in the infinite volume limit is a far more complex issue: the existence or the absence of a phase transition cannot be decided on the basis of the issues discussed in this chapter.

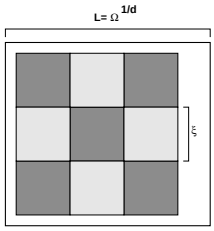

Figure 6.5: caption

6.3 The Transfer Matrix: the $1d$ Ising Model

We will solve here exactly the one dimensional Ising model, and discuss about the zeroes of the partition function. In order to do that we will introduce and use the technique of the **transfer matrix**: this is a crucial technique in Statistical Physics and in the study of disordered systems, and we consider its introduction as one of the main reasons for this diversion. The transfer matrix technique is very direct, as we will see, in one dimensional problems, but it is also very important when generalized to higher dimensional cases: in two dimensions it is a very effective technique, while in higher space dimensions it becomes more complex and time consuming.

We consider a one dimensional linear chain with periodic boundary conditions (see figure (6.5). Inspired from figure (6.5) we write the partition function by dividing its contribution in different pieces:

$$\begin{aligned} Z \simeq \sum_{\{\sigma\}} \quad & \left(e^{\beta(\sigma_1\sigma_2-1)+\frac{\beta h}{2}(\sigma_1-1)+\frac{\beta h}{2}(\sigma_2-1)}\right) \\ & \left(e^{\beta(\sigma_2\sigma_3-1)+\frac{\beta h}{2}(\sigma_2-1)+\frac{\beta h}{2}(\sigma_3-1)}\right) \\ & \left(e^{\beta(\sigma_3\sigma_4-1)+\frac{\beta h}{2}(\sigma_3-1)+\frac{\beta h}{2}(\sigma_4-1)}\right) \dots \ , \end{aligned} \tag{6.11}$$

where we have assigned the different contributions to the links of the chain (by dividing the magnetic field contribution from a given site among the two different links that start from the site). Let us look to one of these contributions, for example to

$$e^{\beta(\sigma_1\sigma_2-1)+\frac{\beta h}{2}(\sigma_1-1)+\frac{\beta h}{2}(\sigma_2-1)} \ .$$

We write a $2 \cdot 2$ matrix that contains the values that the contribution 6.3 takes when respectively $\sigma_1 = +1$ or -1 and when $\sigma_2 = +1$ or -1. The matrix $\mathbf{T}$ will have the form

	σ_1	$+1$	-1
σ_2			
$+1$		$\mathbf{T_{11}}$	$\mathbf{T_{12}}$
-1		$\mathbf{T_{21}}$	$\mathbf{T_{22}}$

i.e.

$$\mathbf{T} = \begin{pmatrix} 1 & e^{-2\beta-\beta h} \\ e^{-2\beta-\beta h} & e^{-2\beta h} \end{pmatrix} = \begin{pmatrix} 1 & \tau\rho^{\frac{1}{2}} \\ \tau\rho^{\frac{1}{2}} & \rho \end{pmatrix} , \tag{6.12}$$

that is the **transfer matrix** of our problem. The crucial point is that now it is very easy to get convinced that

$$Z_N = \left(\frac{1}{2}e^{\beta+\beta h}\right)^N P_N = \left(\frac{1}{2}e^{\beta+\beta h}\right)^N \mathrm{Tr}\left(\mathbf{T}^N\right) , \tag{6.13}$$

i.e. $\mathbf{T}_{ij}\,\mathbf{T}_{jk}\,\mathbf{T}_{kl}\,\ldots\,\mathbf{T}_{mi}$. This is thanks to the periodic boundary conditions. Indeed we start from the link 12, with a term $\mathbf{T}_{12}$, then we use the link 23 by multiplying times $\mathbf{T}_{23}$, and so on till the last links that closes the chain by contributing a factor $\mathbf{T}_{N1}$, and building the Trace operator.

To make the procedure more clear let us be pictorial on what happens when we contract the first (12) and the second (23) links. We have that

$$\left(\begin{array}{cc|cc} \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{2} \\ +1 & +1 & +1 & -1 \\ \hline \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{2} \\ -1 & +1 & -1 & -1 \end{array}\right) \left(\begin{array}{cc|cc} \mathbf{2} & \mathbf{3} & \mathbf{2} & \mathbf{3} \\ +1 & +1 & +1 & -1 \\ \hline \mathbf{2} & \mathbf{3} & \mathbf{2} & \mathbf{3} \\ -1 & +1 & -1 & -1 \end{array}\right) =$$

$$\left(\begin{array}{ccccccc|ccccccc} \mathbf{1} & \mathbf{(22)} & \mathbf{3} & & \mathbf{1} & \mathbf{(22)} & \mathbf{3} & \mathbf{1} & \mathbf{(22)} & \mathbf{3} & & \mathbf{1} & \mathbf{(22)} & \mathbf{3} \\ +1 & (+1\,+1) & +1 & + & +1 & (-1\,-1) & +1 & +1 & (+1\,+1) & -1 & + & +1 & (-1\,-1) & -1 \\ \hline \mathbf{1} & \mathbf{(22)} & \mathbf{3} & & \mathbf{1} & \mathbf{(22)} & \mathbf{3} & \mathbf{1} & \mathbf{(22)} & \mathbf{3} & & \mathbf{1} & \mathbf{(22)} & \mathbf{3} \\ -1 & (+1\,+1) & +1 & + & -1 & (-1\,-1) & +1 & -1 & (+1\,+1) & -1 & + & -1 & (-1\,-1) & -1 \end{array}\right) ,$$

where with bold symbols we have indicated the site number, in parenthesis we have the value of the spin **2** that we have summed over with the first multiplication of our transfer matrix. The successive matrix multiplications build step by step the partition function: at the end the trace enforces periodic boundary conditions (since we are summing imposing that the last site coincides with the first one).

Let us call λ_+ and λ_- the two eigenvalues of $\mathbf{T}$:

$$\text{Tr } \mathbf{T}^N = \lambda_+^N + \lambda_-^N \ .$$

Let us compute the two eigenvalues λ. They are the eigenvalues of the transfer matrix (6.12), i.e. the solution of the equation

$$\det(\mathbf{T}) = \lambda^2 - (1+\rho)\lambda + \rho(1-\tau^2) = 0 \ .$$

The solutions are

$$\lambda_\pm = \frac{1}{2}(1+\rho) \pm \sqrt{\frac{1}{4}(1-\rho)^2 + \rho\tau^2} \ .$$

For real values of h (i.e. for $\rho > 0$) and for $0 < \tau < 1$ in the infinite volume limit, i.e. for $N \longrightarrow \infty$ only the largest eigenvalues, λ_+ contributes to the partition function. In this limit the free energy density f is given by

$$f = \lim_{N\longrightarrow\infty} \frac{\log Z_N}{N} = (\beta(1+h) - \log 2) + \log\left(\frac{1+\rho}{2} + \sqrt{\frac{(1-\rho)^2}{4} + \rho\tau^2}\right) \ .$$

This function is singular when the argument of the square root is zero (notice that we have now already taken the $N \longrightarrow \infty$ limit, and we are observing the situation in the infinite volume limit), i.e. for

$$\rho_c^{(\pm)} = \left(1 - 2\tau^2\right) + i\, 2\tau\sqrt{1-\tau^2} \ ,$$

where as usual we are in the region $0 < \tau < 1$. So

$$\left|\rho_c^{(\pm)}\right|^2 = \left(1 + 4\tau^4 - 4\tau^2\right) + 4\tau^2\left(1-\tau^2\right) = 1,$$

i.e.

$$\rho_c^{(\pm)} = e^{\pm i 2h_c} \text{ with } \cos(2h_c) = 1 - 2\tau^2 = 1 - 2e^{-4\beta} \ . \tag{6.14}$$

This is the main result of this exercise: the finite volume zeroes has conjured to form a couple of singularities on the unitary circle in the complex ρ plane. When $T \longrightarrow 0$ the singularities go to $h_c = 0$ ($T = 0$ in the $1d$ Ising model, and more in general at the lower critical dimension is somehow like a critical point).

Chapter 7

$T = 0$ Dynamics and Remanence

The presence of a remanent magnetization and the existence of non-exponential relaxation processes have been some of the first signatures of complex behaviors in disordered systems. Nowadays experiments, that we will discuss in some detail later in these notes, do confirm the importance of these features, by detecting even more impressive phenomena (like for example the so-called *memory* experiments, see chapter (9)).

The main experimental finding that we will discuss here will be the following. When at low values of the temperature we switch off a magnetic field applied to a spin glass (see chapter (9) for a minimal introductory experimental introduction to spin glasses) we find that the magnetization that was present in the field does not disappear: it decreases fast to a finite value, the *remanent magnetization*, and than it decays very slowly, typically according to a logarithmic pace, towards zero.

We will discuss here how remanent magnetization and slow relaxation can be understood in a very simple model, the *one dimensional spin glass* (that is not a frustrated model). It is important that these crucial effects emerge very clearly already in this very simple model, where we will see that exact computations allow us to get very clear results. We will follow here the very nice treatment of [23]: for a more advanced discussion of metastable states see [24].

We will first define the model and discuss general features that will be relevant for us in introducing the dynamical behavior of spin glasses and, more in general, of disordered and complex systems. We will discuss how remanence works, and give a formulation of our problem that leads both to a clear understanding of remanence and to a closed analytic solution. We will

compute in detail the values of the $T = 0$ remanent energy and remanent magnetization. At last we will try to give some arguments to understand better the reason of slow decays: in our phenomenological picture we will be able to explain the low temperature logarithmic decay of magnetization and energy towards their equilibrium value.

7.1 The $1d$ Ising Spin Glass

Here we will study a very simple model. We will consider an Ising spin glass in one space dimension. The Hamiltonian of a system with N spins is

$$H = -\sum_{i=1}^{N} \sigma_i J_i \sigma_{i+1} - h \sum_{i=1}^{N} \sigma_i \ , \tag{7.1}$$

where the Ising spins σ_i can take the two values ± 1, h is the magnetic field, the first sum is over the links of the chain and the second over the sites (they coincide in one dimension, where to each link corresponds a single site, $N_{link} = N_{site}$). The couplings J (that we are labelling with the position of the site from which they originate, pointing in the positive direction of the chain) are quenched random variables: they are assigned a priori according to a probability distribution P and they do not change during the dynamics. We select a Gaussian quenched probability distribution

$$P(J) \propto e^{-\frac{J^2}{2(\Delta J)^2}} \ , \tag{7.2}$$

where ΔJ is the width of the distribution (in the case of the detailed computation we will discuss now this is technically important).

As we are discussing many times in these notes the presence of quenched disorder characterizes crucially the physics of the system: the *traps* cannot change in times, and the configuration of the disorder can have (and has here) a dramatic effect. Indeed the approach we will use here for this detailed computation has the merit to make very transparent the most important processes that determine the physical behavior of the system.

We start by introducing and discussing some quantities that will be important in the following (and will come back frequently in the rest of these notes). Some of these quantities will not be used in this section, but we keep this discussion as a moment in which we first qualify objects that are very

important when one talks about disordered systems and that will be very present in the rest of these notes.

First we introduce a spin-spin correlation function that represents the *overlap* among two spin configurations (in the same quenched noise) at different times:

$$q_i(t) \equiv E\left(\sigma_i(0)\sigma_i(t)\right) \equiv \overline{\langle \sigma_i(0)\sigma_i(t)\rangle} \ , \tag{7.3}$$

where the thermal average $\langle\cdot\rangle$ is taken at temperature T. Since we are in one dimension we know on general grounds that we cannot have a phase transition, and

$$q_i(t) \longrightarrow_{t\to\infty} q_i(t)0 \ .$$

Before continuing to discuss a few general concepts that we want to start to understand we want to discuss the **equilibrium behavior** of the system. Since the system is one dimensional it is easy to compute its thermodynamical properties. We set now $h = 0$, and we consider a one dimensional chain with open boundary conditions. We can easily think about two possible ways to specify the state of the system:

1. we can give the values of all the spins of the chain. The spin defined on the first site is $+1$, the one on the second site is -1, the one on the third site too, and so on;

2. we can give the value of the first spin and the can tell if, in the specified realization of the quenched disorder, the links of the system are satisfied or not. So, assuming for example the case of all couplings equal to $+1$, the state of the system of the former specifications can be given by claiming that the spin defined on the first site is $+1$, the first link is broken, the second one is satisfied and so on.

By using this second way to characterize the system

7.2 Remanence

7.3 Remanent Energy

7.4 Remanent Magnetization

7.5 An Estimate of the Magnetization Non-Exponential Decay

Chapter 8

The Random Field Ising Model

In the following sections we will discuss about the Random Field Ising Model (RFIM). We will mainly focus on the "simplest" issue, i.e. the existence of a phase transition in dimension D. Somehow the Random Field Ising Model plays a special role among disordered systems, and its nature is not yet completely clear: it has somehow an intermediate nature among the simple, diluted models, and the complex, frustrated spin glasses. Even if, at this point, we know in which conditions a phase transition is present, the possibility of the presence of more than one phases (for example one ferromagnetic phase at low T and one spin glass phase at intermediate T values) has not yet been confirmed or excluded (and we will not discuss this issue here).

We will first discuss the original *Imry and Ma* argument (in section 8.1). We will than discuss in section 8.3 the *Parisi-Sourlas* theorem (a very direct proof of perturbative dimensional reduction valid at all loops), that we will also use as a reason to introduce Grassmann algebras and fermionic variables. We will try here to give all the needed details to follow the computation. We will end the chapter, in section 8.4, by only sketching the main lines of the theorem by which Imbrie has settled in a rigorous way the dispute about the $3D$ model, proving that in $3D$ the RFIM exhibits long range order at $T = 0$.

8.1 The Imry and Ma Argument

In the pure Ising model, with no disorder, there is a low T ferromagnetic phase in all dimensions larger than 1: the lower critical dimension here is

$D_c^L = 1$ (where we know in any case that on very general basis there cannot be phase transitions for systems with finite range interactions).

In the Random Field Ising Model we add to the Hamiltonian of the usual Ising model a term containing a magnetic field that is a quenched, site dependent random variable:

$$H_{RFIM} \equiv -J \sum_{}^{'} \sigma_i \sigma_j - \sum_i h_i \sigma_i \ , \tag{8.1}$$

where

$$\overline{h_i} = 0, \quad \overline{h_i h_j} = h^2 \delta_{i,j} \ ,$$

where, as usual, we have denoted by an upper bar the average over the quenched disorder.

The term containing the random magnetic field disorders the system: now it is more difficult to get an ordered state. The energy is done of two term, whose magnitude is controlled by the parameters J and h: when h becomes very large as compared to J the system becomes disordered even at low T, since the random field decides about the spin alignment.

Here we want to discuss what happens at very low T and $h \ll J$ (i.e. what happens when the start a random field of very small magnitude).

Imry and Ma first have developed in [20] the most direct and intuitive energetic argument, that we will analyze now. The *Imry and Ma argument* says that:

> the ferromagnetic ground state becomes unstable with respect to domain creation in all dimensions $D \leq 2$ (but it is stable in $D = 3$).

In other terms the system has (at least) a phase transition to a low T ferromagnetic states in $D \geq 3$: the lower critical dimension $D_c^L(RFIM) = 2$ (i.e. is equal to one plus the one of the pure Ising model).

The argument is very straightforward. Consider the ground state of a spin system, and turn a domain of radius R and volume R^D. The energetic cost one has to pay for doing that is proportional to the surface of the domain, i.e.

$$\Delta E \sim J R^{D-1} \ .$$

In a ferromagnetic system this is the only effect that plays a role. If now we also have an energetic term related to the presence of the quenched random

field, as in the Hamiltonian we have defined in 8.1, we have also to consider that we can also gain some energy, thanks to the presence of the random magnetic field: by rearranging a domain we pay some energy because of flipping the boundary, but we can gain some energy in the bulk. How do we estimate this bulk energy that we can gain? We first notice that even if $\overline{h_i} = 0$ the total value of the magnetic field summed over all sites in a region of size R will fluctuate (it is itself a random variable). Central limit theorem tells us that

$$|E_{RF}| \simeq hR^{\frac{D}{2}} \ .$$

In other terms we are arguing that when we divide the system in domains of size R^D we have to pay a surface energy of order $O(R^{D-1})$ per domain, while we can gain an energy of order $O(R^{\frac{D}{2}})$ per domain. There is an important point of detail that we have to clarify. The energy E_{RF} can be negative or positive with equal probability: i.e. if we consider a given, precise domain, when flipping it the energy will typically change of a value of order E_{RF}, where one time over two the energy will be gained and one time over two it will be lost. But let us consider one site i : we select the domain we will flip around this site, with fixed surface. When doing that, for R large enough, it is always possible to find a region such that $E_{RF} > 0$: when we flip this region we *gain* an energy of order E_{RF}.

So in total the energetic balance gives:

$$\Delta E(R) \simeq JR^{D-1} - hR^{\frac{D}{2}} \ .$$

So, for large R and $h \ll J$

$$D \geq 2 \Longrightarrow \Delta E > 0 \ , \tag{8.2}$$

$$D < 2 \Longrightarrow \Delta E < 0 \ . \tag{8.3}$$

8.2 Grassmann Algebras

8.3 Parisi-Sourlas Theorem

8.4 The Imbrie Rigorous Result

Chapter 9

Remarkable Experiments on Spin Glasses

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