A categorical proof of the Löwenheim-Skolem theorem

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Abstract. We present a categorical proof of the Löwenheim-Skolem theorem through the use of methods in categorical logic, particularly a functorial characterization, mainly due to A. Joyal, of Boolean models.

Mathematics Subject Classification (2010). Primary 18C10; Secondary 03G30.

Keywords. Functorial semantics, Categorical model theory, Löwenheim-Skolem theorem.

1. Introduction

In 1915 Leopold Löwenheim gave an incomplete proof of the fact that for every countable signature, a sentence which is satisfiable is already satisfiable in a countable model ([6]). In 1920 Thoralf Skolem gave a new argument filling the gap in Löwenheim's proof and generalizing the result to countable infinite sets of sentences ([11]; for a translation of both articles, see [2]):

Theorem 1.1. Löwenheim-Skolem theorem. *Every model $\mathcal{M}$ of a first order theory T with countable signature has an elementary submodel $\mathcal{N}$ which is at most countable.*

Skolem introduced for this purpose what would be later called *Skolem normal form* of first order formulas, linked to the so called *Skolem functions*. The axiom of choice played a crucial rôle in the proof, as noted by himself, although he was able to obtain a weaker version of the theorem by dropping the condition that the countable model for T be an elementary submodel of $\mathcal{M}$.

The purpose of this article is to give a categorical proof of Löwenheim-Skolem theorem making use of functorial semantics and a functorial characterization of models given by André Joyal. The proof does not use Skolem functions and follows instead some ideas introduced by Joyal in the categorical proof of Gödel completeness theorem, presented in a series of unpublished lectures in Montreal in 1978.

Sections 2 and 3 expose well known lemmas and results which we reproduce here for the sake of completeness. In section 2 we use functorial semantics to consider models of first order theories and deduce a characterization of Boolean models. The construction of an explicit type of model for first order theories is done in section 3, following Joyal's ideas. Section 4 is new; we combine here the results from previous sections with some new ideas to give a proof of the Löwenheim-Skolem theorem.

2. Boolean models

The tradition, initiated by William Lawvere ([5]), of considering syntactic categories to describe theories (see, for instance, [4], D1.4), and functors from these to the category of sets to describe models, allows us to translate in categorical language classical results from model theory. We will be interested particularly in Boolean categories (see [10], ch. 1), which describe those theories within first order classical logic, and Boolean functors, which represent their models. We recall here the definitions:

Definition 2.1. A regular category is a category having the following three properties:
1) It has all finite limits.
2) Every arrow $f : A \to B$ can be factored as $f : A \twoheadrightarrow C \rightarrowtail B$, where C, called the image of f, is the least subobject of B through which f can factor. The arrow $f : A \twoheadrightarrow C$ not factoring through any proper subobject of C is called a cover.
3) Images are stable under base change (i.e., pullbacks preserve covers).
A regular category is said to be Boolean if it also satisfies the following two conditions:
4) The poset $\mathcal{S}ub(X)$ of subobjects of a given object X has finite unions and these are stable under pullbacks.
5) Every subobject A in the poset $\mathcal{S}ub(X)$ has a complement, i.e., there exists a subobject B such that the intersection $A \wedge B$ is initial in $\mathcal{S}ub(X)$ and $A \vee B = X$ (in particular, $\mathcal{S}ub(X)$ is a Boolean algebra, and we denote $B = \neg A$).

Definition 2.2. A functor between regular categories is regular provided it preserves finite limits and images factorizations. A regular functor between Boolean categories is called Boolean if it also preserves unions and complements.

A reformulation, due mainly to Joyal, of existing results on limit ultrapowers (see [10], ch. 4) allows us to characterize all Boolean models of a first order theory. We start with the following:

Definition 2.3. A functor $h : \mathcal{C} \to \mathcal{S}$ is said to be ultra-representable if there is $C \in \mathcal{C}_\omega$ such that it can be expressed as a (filtered) colimit $h = \varinjlim_{A \in \Phi} [A, -]$ for some ultrafilter Φ in $\mathcal{S}ub(C)$.

We have now:

Theorem 2.4. *Every model of a first order theory T is given by (the image of) a functor $\mathcal{M} : \mathcal{C}_T \to \mathcal{S}et$, where $\mathcal{C}_T$ is the syntactic category of T and $\mathcal{M} : \mathcal{C}_T \to \mathcal{S}et$ is a filtered colimit of ultra-representable functors.*

Proof. Suppose that $\mathcal{M} : \mathcal{C}_T \to \mathcal{S}et$ is a Boolean functor corresponding to a model of T. Define the category $\mathcal{X}$ whose objects are pairs (A, ξ) where A is an object of $\mathcal{C}_T$ and $\xi \in \mathcal{M}(A)$, and whose morphisms $(A, \xi) \to (B, \eta)$ are given by those arrows $f : A \to B$ in $\mathcal{C}_T$ such that $\eta = \mathcal{M}(f)(\xi)$. Because $\mathcal{C}_T$ has finite limits and $\mathcal{M}$ preserves them, $\mathcal{X}$ will have finite limits. Therefore, its dual $\mathcal{W} = \mathcal{X}^{op}$ is a (small) filtered category. We shall now define a functor H from $\mathcal{W}$ to the category of ultra-representable functors. For each object $(A, \xi) \in \mathcal{W}$, let $\Phi(A, \xi)$ be the set of all subobjects $C \rightarrowtail A$ such that $\xi \in \mathcal{M}(C)$. Because $\mathcal{M}$ is Boolean, it is easy to check that $\Phi(A, \xi)$ is an ultrafilter in $\mathcal{S}ub(A)$. Define then $H((A, \xi)) = h_{\Phi(A,\xi)}$, the ultra-representable functor corresponding to $\Phi(A, \xi)$. Given an arrow $f : (A, \xi) \to (B, \eta)$, the mapping $C \mapsto f^{-1}(C)$, defined for each $C \in \Phi(B, \eta)$, determines a natural transformation $H(f^{op}) : h_{\Phi(B,\eta)} \to h_{\Phi(A,\xi)}$ in the following way: for each representative $a : C \to X$ in $h_{\Phi(B,\eta)}(X)$ we let $H(f^{op})_X([a]) = [af']$, where $f' : f^{-1}(C) \to C$ is the arrow arising from the pullback of $C \rightarrowtail B$ along f. This application is well defined and makes H a functor.

We shall prove that $\varinjlim_{(A,\xi) \in \mathcal{W}} h_{\Phi(A,\xi)} \cong \mathcal{M}$. To define an isomorphism $K : \varinjlim_{(A,\xi) \in \mathcal{W}} h_{\Phi(A,\xi)} \to \mathcal{M}$ it suffices to define natural transformations $\psi_{(A,\xi)} : h_{\Phi(A,\xi)} \to \mathcal{M}$ which will induce the required morphism. This can be done by setting $(\psi_{(A,\xi)})_X([f]) = \mathcal{M}(f)(\xi)$, where $f : C \to X$ and $C \in \Phi(A, \xi)$. It can be easily checked that the definition does not depend on the representative of the class $[f]$; also, note that $C \in \Phi(A, \xi)$ implies that $\xi \in \mathcal{M}(C)$, and therefore $\mathcal{M}(f)(\xi) \in \mathcal{M}(X)$ and $(\psi_{(A,\xi)})_X$ is well defined.

Let us first prove that K is a monomorphism. For this it is enough to verify that each $\psi_{(A,\xi)} : h_{\Phi(A,\xi)} \to \mathcal{M}$ is monic, for which it suffices in turn to check that each $(\psi_{(A,\xi)})_X$ is injective. So suppose that we have arrows $f : C \to X, g : C' \to X$ such that $(\psi_{(A,\xi)})_X([f]) = (\psi_{(A,\xi)})_X([g])$. Then, $\mathcal{M}(f)(\xi) = \mathcal{M}(g)(\xi)$. Take the intersection $C \wedge C'$ in $\mathcal{S}ub(A)$, which gives monics $a : C \wedge C' \rightarrowtail C$ and $b : C \wedge C' \rightarrowtail C'$. Consider the equalizer E of fa and gb. Since $\mathcal{M}$ preserves equalizers and $\xi \in \mathcal{M}(C \wedge C')$, then $\xi \in \mathcal{M}(E)$, that is, $E \in \Phi(A, \xi)$. Therefore, f and g belong to the same class, i.e., $[f] = [g]$.

Finally, let us prove that K is an epimorphism, for which it suffices to show that each $K(D) : \varinjlim_{(A,\xi) \in \mathcal{W}} h_{\Phi(A,\xi)}(D) \to \mathcal{M}(D)$ is surjective. Given an element $\chi \in \mathcal{M}(D)$, consider the ultrafilter $\Phi(D, \chi)$, that contains D. Since $(\psi_{(D,\chi)})_D : h_{\Phi(D,\chi)}(D) \to \mathcal{M}(D)$ satisfies $(\psi_{(D,\chi)})_D([Id_D]) = \mathcal{M}(Id_D)(\chi) =$

χ, we conclude that the family $\{(\psi_{(A,\xi)})_D/(A,\xi) \in \mathcal{W}\}$ is jointly epic, from which we can deduce that $K(D)$ is necessarily surjective. □

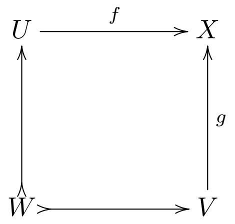
According to [4], D1.2, there is an equivalence between the category of models of a first order theory, where the morphisms are given by elementary embeddings, and the full subcategory of $\mathcal{S}et^{\mathcal{C}_T}$ consisting of Boolean functors. Suppose now that a theory T has countable signature. We know from theorem 2.4 that every model of T can be expressed as a filtered colimit of ultra-representable functors defined on the syntactic category of T. Since any of these functors must have a countable image in $\mathcal{S}et$, it would give the required countable elementary submodel provided it is Boolean. However, this is generally not the case, and therefore we will need to enlarge the category $\mathcal{C}_T$ so that some ultra-representable functor h_Φ in the colimit diagram becomes Boolean.

3. Joyal's construction

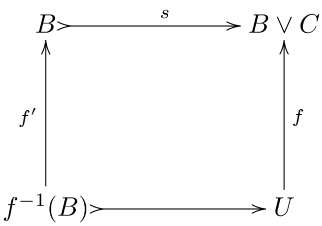
For an ultra-representable functor to be Boolean it is necessary that it preserves finite limits, covers and unions in the subobjects posets. Because filtered colimits commute with finite limits in $\mathcal{S}et$, it is clear that each ultra-representable functor preserves finite limits, since representable functors do. The following lemma shows that they also preserve unions:

Lemma 3.1. *Every ultra-representable functor $h_\Phi : \mathcal{C}_T \to \mathcal{S}et$ preserve unions of subobjects.*

Proof. We need to prove that given B, C subobjects of D, we have $h(B \vee C) = h(B) \vee h(C)$. Clearly, $h(B) \vee h(C) \leq h(B \vee C)$, since h preserves finite limits, and therefore monomorphisms. To prove the converse inequality, note that, according to the usual construction of filtered colimits in $\mathcal{S}et$, we have $h(X) = \coprod_{A \in \Phi}[A, X]/\sim$, where $\sim$ is the equivalence relation which identifies $f : U \to X$ with $g : V \to X$ if and only if there exists some $W \in \Phi$ such that the following square commutes:



Take (some representative of) an arrow $f : U \to B \vee C$, for some $U \in \Phi$. Since unions are stable under pullback, we have $U = f^{-1}(B) \vee f^{-1}(C)$. Now, since Φ is an ultrafilter, either $f^{-1}(B)$ or $f^{-1}(C)$ is in Φ; suppose without loss of generality $f^{-1}(B) \in \Phi$. Then the following pullback:

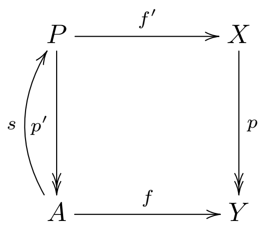


shows that f and sf' are in the same class in $h(B \vee C)$. We may therefore assign to each $f \in h(B \vee C)$ either an arrow $f' \in h(B)$ or an arrow $f'' \in h(C)$, from which we conclude that $h(B \vee C) \leq h(B) \vee h(C)$. Hence, we must have $h(B \vee C) = h(B) \vee h(C)$ in $\mathcal{S}et$. □

It is not generally true that ultra-representable functors preserve covers. In fact, note that since the injections into the colimit form a jointly regular epimorphic family and this property is preserved by h, it is mapped onto a jointly epic family in $\mathcal{S}et$ (where covers are surjections), and hence h will preserve covers provided each representable functor $[A, -]$ does. Now, for a representable functor $[A, -]$ to preserve covers it is necessary that A be projective with respect to covers (i.e., to be cover-projective), since covers in $\mathcal{S}et$ are precisely the surjections. For a regular category, we can find an equivalent condition:

Lemma 3.2. *The representable functor $[A, -]$ preserves covers (i.e., A is cover-projective) if and only if every cover $p : X \twoheadrightarrow A$ has a section.*

Proof. If $[A, -]$ is cover-projective, given a cover $p : X \twoheadrightarrow A$ we can take the factorization of $Id_A : A \to A$ through X, which provides a section for p. Conversely, suppose that every cover over A has a section. Given a morphism $f : A \to Y$ and a cover $p : X \twoheadrightarrow Y$, form the pullback P of p along f:



Then p' must be a cover over A, and if $s : A \to P$ is a section, we get the factorization $f = pf's$. □

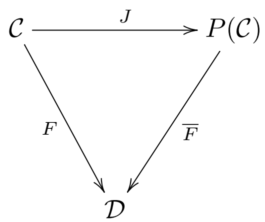
As a consequence of lemma 3.2, we seek to enlarge the syntactic category $\mathcal{C}_T$ including sections for every cover over the subobjects of the ultrafilter considered. When the ultrafilter is selected in $\mathcal{S}ub(1)$, there is a known construction that makes the terminal object 1 cover-projective. If in addition we start from a category with finite disjoint coproducts, this construction also makes every subobject of $S \rightarrowtail 1$ cover-projective.

Before proceeding to describe Joyal's construction, we first expose results on the positivization of a Boolean category and also on bicolimits in $\mathcal{C}at$.

Definition 3.3. A Boolean category $\mathcal{C}$ is said to be positive if it has finite disjoint coproducts.

Following [4], A1.4, we have:

Lemma 3.4. *If $\mathcal{C}$ is a Boolean category, there exists a Boolean embedding $J : \mathcal{C} \to P(\mathcal{C})$, where $P(\mathcal{C})$ is a positive Boolean category that can be constructed as follows:*
1) Its objects are n-tuples $(A_1, ..., A_n)$ of objects of $\mathcal{C}$.
2) A morphism between $(A_1, ..., A_n)$ and $(B_1, ..., B_m)$ is specified by the following:
i) A m-fold decomposition of each A_i, that is, a set of m pairwise disjoint subobjects $A_{i_1}, ..., A_{i_m}$ of A_i such that $\bigvee_{j=1}^{m} A_{i_j} = A_i$ (we allow some of the subobjects to be the initial subobject of A_i).
ii) A set of $\mathcal{C}$-morphisms $f_{ij} : A_{i_j} \to B_j$ for each $i = 1, ..., n, j = 1, ..., m$.
3) The identity morphism $Id_{(A_1,...,A_n)}$ is given by trivial decompositions of each A_i and the arrows $f_{ii} = Id_{A_i}$, f_{ij} initial for $i \neq j$.
4) Given morphisms $F : (A_1, ..., A_n) \to (B_1, ..., B_m)$ associated to arrows $f_{ij} : A_{i_j} \to B_j$ and $G : (B_1, ..., B_m) \to (C_1, ..., C_p)$ associated to arrows $g_{jk} : B_{j_k} \to C_k$, define the composite $GF : (A_1, ..., A_n) \to (C_1, ..., C_p)$ to be the morphism given by the arrows $h_{ik} = (\exists_{g_{1k} f_{i1}}, ..., \exists_{g_{mk} f_{im}}) : \bigvee_{j=1}^{n} (f_{ij}^{-1}(B_{j_k}) \wedge A_{i_j}) \to C_k$.
Moreover, $P(\mathcal{C})$ satisfies the following universal property: for every Boolean functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is a positive Boolean category, there exists a Boolean functor $\overline{F} : P(\mathcal{C}) \to \mathcal{D}$ (which necessarily preserves coproducts) satisfying $\overline{F} J = F$:



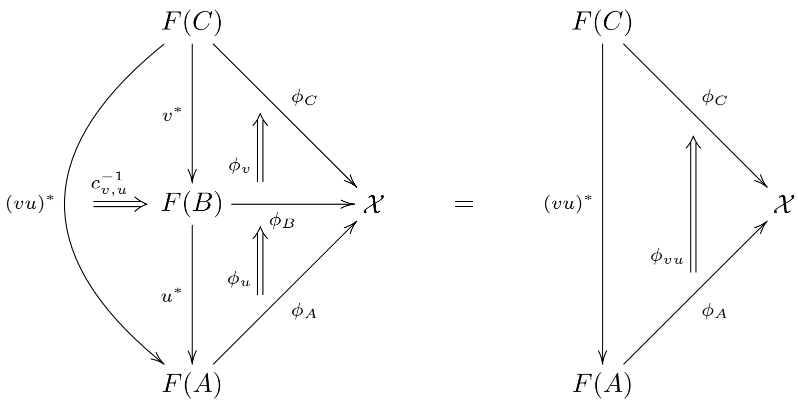
We recall now the notion of pseudofunctor, following [1]:

Definition 3.5. Given a category $\mathcal{D}$, a (normalized) pseudofunctor $F : \mathcal{D}^{op} \to \mathcal{C}at$ consists of the following:
a) A function $\mathcal{O}b(F) : \mathcal{O}b(\mathcal{D}) \to \mathcal{O}b(\mathcal{C}at)$ (for convenience we shall refer to $F(D)$ for the category corresponding to the object D).
b) An application $\mathcal{A}r(F) : \mathcal{A}r(\mathcal{D}) \to \mathcal{A}r(\mathcal{C}at)$ which assigns to every arrow $f : C \to D$ in $\mathcal{D}$ a functor $f^* : F(D) \to F(C)$.
c) An application c defined in $\mathcal{A}r(\mathcal{D})^2$ which assigns to each pair (f, g) of

arrows of $\mathcal{D}$ a natural isomorphism $c_{f,g} : g^* f^* \to (fg)^*$.
Furthermore, the following properties hold:
1) For every object C in $\mathcal{D}$ we have $(Id_C)^* = Id_{F(C)}$.
2) For every arrow $f : C \to D$ in $\mathcal{D}$, we have $c_{f,Id_C} = Id_{f^*}$ and $c_{Id_D,g} = Id_{f^*}$.
3) For a triple of composable arrows $f : C \to D$, $g : D \to E$ and $h : E \to G$, we have $c_{f,gh}(\xi) \circ c_{g,h}(f^*(\xi)) = c_{fg,h}(\xi) \circ h^*(c_{f,g}(\xi))$.

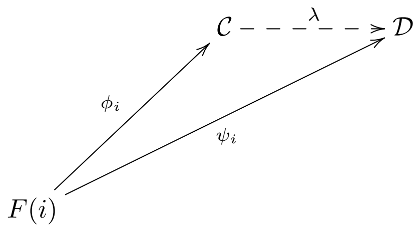
In the special case when $c_{f,g} = Id_{(fg)^*}$ the pseudofunctor reduces to a functor.

Definition 3.6. Given a pseudofunctor $F : \mathcal{D}^{op} \to \mathcal{C}at$, a pseudococone with vertex at the category $\mathcal{X}$ consists of a family of functors $\{\phi_A : F(A) \to \mathcal{X} \ / \ A \in \mathcal{D}\}$ and a family of natural isomorphisms $\{\phi_u : \phi_A \circ u^* \to \phi_B \ / \ (u : A \to B) \in \mathcal{D}\}$ that satisfy the following conditions:
a) $\phi_{Id_A} = Id_{\phi_A}$.
b) For $u : A \to B$ and $v : B \to C$, we have $\phi_{vu} = \phi_v \circ \phi_u Id_{v^*} \circ (Id_{\phi_A} \circ c^{-1}_{v,u})$:



Pseudococones allow us to consider a variation of the colimit notion that we shall call bicolimit, introduced in [12], Ex. VI 6.4.0, under the notation "$\underrightarrow{Lim}$" (with capital L). We have:

Definition 3.7. Given a pseudofunctor $F : I^{op} \to \mathcal{C}at$, the bicolimit $\mathcal{C} = \varinjlim_{i \in I^{op}} F(i)$ is the universal pseudococone associated to F. In other words, it is a pseudococone $\phi : F \Rightarrow \mathcal{C}$ such that for every pseudococone $\psi : F \Rightarrow \mathcal{D}$ there is a unique functor $\lambda : \mathcal{C} \to \mathcal{D}$ such that $\psi_i = \lambda \phi_i$ for every $i \in I$:



In [12], Ex VI 6.8, it is mentioned that for filtered diagrams the bicolimit and the colimit are equivalent categories, and a construction of the bicolimit in the case where I^{op} is filtered is described by means of categories of fractions. We mention here a well known equivalent construction:

Lemma 3.8. *The bicolimit* $\mathcal{C} = \varinjlim_{i \in I^{op}} F(i)$*, where* I^{op} *is filtered, is a category specified by the following:*
1) Objects are pairs (C, i)*, where* C *is an object of* $F(i)$
2) Morphisms between two objects (C, i) *and* (D, j) *are triples* (u, v, f)*, where* $u : i \to k$*,* $v : j \to k$ *are arrows of* I^{op} *and* $f : F(u)(C) \to F(v)(D)$ *is a morphism in* $F(k)$*, divided by the following equivalence relation:* $(u, v, f) \sim (u', v', f')$ *if and only if there are arrows* $x : k \to l$*,* $y : k' \to l$ *such that* $xu = yu'$*,* $xv = yv'$ *and* $F(x)(f) = F(y)(f')$*.*
3) The identity morphism on (C, i) *is represented by* (Id_i, Id_i, Id_C)*.*
4) To define the composite of two morphisms represented by $(u, v, f) : (C, i) \to (D, j)$ *and* $(u', v', g) : (D, j) \to (E, l)$*, let* w *in* I^{op} *be such that we have arrows* $a : k \to w$*,* $b : k' \to w$ *and let* $c : w \to z$ *in* I^{op} *be such that* $cav = cbu'$ *(we use here the filteredness of* I^{op}*). Then the composite* $(u', v', g) \circ (u, v, f)$ *is defined as the class of the triple* $(cau, cbv', F(cb)(g) \circ F(ca)(f))$ *(once more filteredness shows that this composite does not depend on the chosen* w *nor on the representatives).*

Using the construction mentioned in the preceding lemma, it is well known we can prove that, in general, all finitary constructions in the categories of a filtered diagram that are preserved by the transition functors are inherited by the bicolimit. In our case, we can state, more precisely, the following result:

Lemma 3.9. *Let* I *be a cofiltered category and* $F : I^{op} \to \mathcal{C}at$ *a pseudofunctor such that each* $F(i)$ *is a regular (resp. Boolean) category with finite disjoint coproducts and each transition functor* $F(f) : F(j) \to F(i)$ *(for* $f : i \to j$ *in* I*) is regular (resp. Boolean) and preserves coproducts. Then the bicolimit* $\mathcal{C} = \varinjlim_{i \in I^{op}} F(i)$ *is a regular (resp. Boolean) category with finite disjoint coproducts. Moreover, for each* i *in* I*, the injection into the bicolimit* $I_i : F(i) \to \mathcal{C}$ *is a regular (resp. Boolean) functor that preserves coproducts.*

Define now $\mathcal{C}_0 = P(\mathcal{C}_T)$. We need to add to $\mathcal{C}_0$ a section for every cover on the terminal object, i.e., our goal is to make the terminal object cover-projective. We shall do so by constructing succesive categories $\{\mathcal{C}_n / n \in \mathbf{N}\}$, each one embedded in the next, such that the terminal object of $\mathcal{C}_n$ is cover-projective for all covers that are images of covers in $\mathcal{C}_{n-1}$.

First, let us describe an embedding $I_1 : \mathcal{C}_0 \to \mathcal{C}_1$ that has this property for all covers in $\mathcal{C}_0$. Let Γ be the indexing set of all such covers $\{A_i \twoheadrightarrow 1 / i \in \Gamma\}$. Consider, for each finite $F \subseteq \Gamma$, the set of covers $\{A_i \twoheadrightarrow 1 / i \in F\}$ together with the canonical projections $\pi_{FG} : \prod_{i \in G} A_i \twoheadrightarrow \prod_{i \in F} A_i$ for $F \subseteq G$. Define the category I whose objects are all finite products of objects $P_F = \prod_{i \in F} A_i$, $F \subseteq \Gamma$, and whose arrows are given by the corresponding canonical (induced) morphisms π_{FG} between such products. Note that even if the products are not canonical, the morphisms are nevertheless canonical (even those morphisms between isomorphic products). Then I^{op} is clearly filtered, since between any two objects there is at most one morphism, and for a pair of objects P_F, P_G we can consider the object $P_{F\cup G}$. Define the pseudofunctor $F : I^{op} \to \mathcal{C}at$ as follows: define F on an object P_F as the slice category $\mathcal{C}/P_F$; for each identity arrow Id_{P_F} define F as the corresponding identity $Id_{\mathcal{C}/P_F}$, and for each arrow π_{FG} of I select a fixed pullback functor π^*_{FG} and set it as the value of F on such an arrow (note that we are using here the axiom of choice). Then, if for arrows f, g of I we define $c_{f,g}$ as the corresponding induced natural isomorphism between pullbacks, F becomes a pseudofunctor, as can be easily checked. We have now:

Lemma 3.10. *The bicolimit $\mathcal{C}_1 = \varinjlim_{P_F \in I^{op}} \mathcal{C}_0/P_F$ is a Boolean category with finite disjoint coproducts. Moreover, there is a Boolean functor $I_1 : \mathcal{C}_0 \to \mathcal{C}_1$ that preserves coproducts such that every cover $I_1(A) \twoheadrightarrow I_1(1)$ has a section.*

Proof. Since Booleanness is preserved under slicing, each category $\mathcal{C}_0/P_F$ is Boolean. The pullback functors π^*_{FG} are Boolean, and moreover, because injections into the coproduct are monomorphisms whose intersection is initial, the coproduct is their union and it is thus preserved by these pullback functors. Therefore, by lemma 3.9, $\mathcal{C}_1$ is a Boolean category with finite disjoint coproducts. The functor I_1 can be defined as the injection into the bicolimit, sending the object A into the object $(A, 0)$, and the arrow f into the arrow $(Id_{F(0)}, Id_{F(0)}, f)$. Finally, given a cover $p : A \twoheadrightarrow 1$ in $\mathcal{C}$, the morphism represented by $(\pi^*_{\emptyset,\{A\}}, \pi^*_{\emptyset,\{A\}}, \Delta) : (1, 0) \to (A, 0)$ (i.e., the diagonal $\Delta : A \to A \times A$ regarded as a morphism in $\mathcal{C}_0/A$) provides a section for $I_1(p)$. □

Note that the procedure described above allows to have a section for each cover $I_1(A_i) \twoheadrightarrow 1_{\mathcal{C}_1}$, where $A_i \twoheadrightarrow 1$ is in the set of covers of $\mathcal{C}$. The next step to take is to extend this property to every cover $A \twoheadrightarrow 1_{\mathcal{C}_1}$ in $\mathcal{C}_1$. But the process to follow now is clear: repeating the whole construction above for $\mathcal{C}_1$ instead of $\mathcal{C}$ we get an embedding $I_2 : \mathcal{C}_1 \to \mathcal{C}_2$ preserving all the structure, such that in the new category $\mathcal{C}_2$ every cover $I_2(A) \twoheadrightarrow 1_{\mathcal{C}_2}$ has a section for each A in the set of covers $A \twoheadrightarrow 1_{\mathcal{C}_1}$ of $\mathcal{C}_1$. Iterating this construction, we can obtain a sequence of Boolean embeddings preserving coproducts, $I_n : \mathcal{C}_{n-1} \to \mathcal{C}_n$. This amounts to having a functor $F : \omega \to \mathcal{C}at$, or, formally dualizing, a functor $F : \omega^{op} \to \mathcal{C}at$. It is now easy to verify that $\mathcal{C}_\omega$ is the category we need, as shown in the following:

Theorem 3.11. *There is a Boolean functor $I_0 : \mathcal{C}_0 \to \mathcal{C}_\omega$, where $\mathcal{C}_\omega$ is a Boolean category with finite disjoint coproducts such that every cover over the terminal object 1_ω has a section.*

Proof. The fact that $\mathcal{C}_\omega$ is a Boolean category with finite disjoint coproducts and the induced functors $\mathcal{C}_n \to \mathcal{C}_\omega$ are Boolean and coproduct preserving follows as before from lemma 3.9. Finally, given a cover $p : A \twoheadrightarrow 1$ in $\mathcal{C}_\omega$, there exists some $n \in \mathbf{N}$ such that p has a representative that lies in some

$\mathcal{C}_n$, and, by construction, in $\mathcal{C}_{n+1}$ we can find a representative of a section $1 \to A$. □

We have thus obtained a Boolean category $\mathcal{C}_\omega$ where the terminal object is cover-projective, that is, where the representable functor $[1, -]$ preserves covers. The fact that $\mathcal{C}_\omega$ has finite disjoint coproducts allows to deduce a better result:

Lemma 3.12. *Every subobject of* 1 *in* $\mathcal{C}_\omega$ *is cover-projective.*

Proof. Let $S \rightarrowtail 1$ be a subobject and S' its complement in $\mathcal{S}ub(1)$. If $\pi : A \twoheadrightarrow S$ is a cover, then we also have a cover $\pi \coprod Id_{S'} : A \coprod S' \twoheadrightarrow S \coprod S' \cong 1$. Since 1 is cover-projective, this cover splits, and because coproducts are disjoint, its section must map S into A, which yields a section for π. □

Corollary 3.13. *In* $\mathcal{C}_\omega$*, every ultra-representable functor* h_Φ*, for ultrafilters* Φ *in* $\mathcal{S}ub(1)$*, is Boolean.*

Remark 3.14. Because the choice of finite limits in $\mathcal{C}_1$ is not generally canonical, the iteration of the process that constructs that category forces us to make use of the axiom of choice if we wish to construct $\mathcal{C}_2$, $\mathcal{C}_3$, etc. One way to avoid this appeal to the axiom of choice is to use Grothendieck's construction of the bicolimit with categories of fractions. However, since our goal is to prove the Löwenheim-Skolem theorem, we will eventually need to use some form of choice.

4. The Löwenheim-Skolem theorem

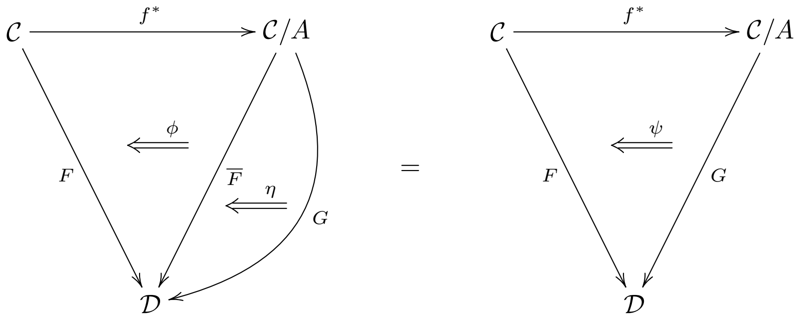
The idea of the proof of Löwenheim-Skolem theorem can be now understood by putting all the pieces together. Given a model $F : \mathcal{C}_T \to \mathcal{S}et$, we will find a factorization through $I = I_0 J : \mathcal{C}_T \to \mathcal{C}_0 \to \mathcal{C}_\omega$ obtaining a Boolean functor $\overline{F} : \mathcal{C}_\omega \to \mathcal{S}et$ such that $F = \overline{F} I$. Now, we know from theorem 2.4 that $\overline{F}$ must be a filtered colimit of ultra-representable functors; if we can find one of these functors in the colimit diagram corresponding to an ultrafilter in $\mathcal{S}ub(1)$, we know from corollary 3.13 that it will necessarily be Boolean. This will allow to find an elementary submodel of F in the form of a Boolean subfunctor. The difficulty in this process lies precisely in finding the extension $\overline{F}$. This will be done by using the axiom of choice in an essential way.

It should also be noted that although we mention the large category $\mathcal{S}et$, it is only as a convenient way of stating the results, and the whole argument can be formalized entirely within ZFC.

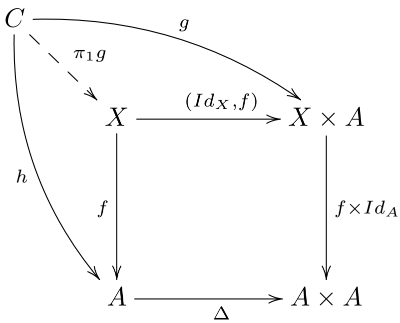
We start with the following:

Lemma 4.1. *Let* $\mathcal{C}$ *be a regular (resp. Boolean) category and let* A *be an object with full support (i.e. such that* $f : A \twoheadrightarrow 1$ *is a cover); let* $F : \mathcal{C} \to \mathcal{D}$ *be a regular (resp. Boolean) functor and let* s *be a section of* $F(A) \twoheadrightarrow F(1)$ *in the regular (resp. Boolean) category* $\mathcal{D}$*. Then, for each fixed pullback functor* f^**, there is a regular (resp. Boolean) functor* $\overline{F} : \mathcal{C}/A \to \mathcal{D}$ *satisfying* $\overline{F}(\Delta) = s$

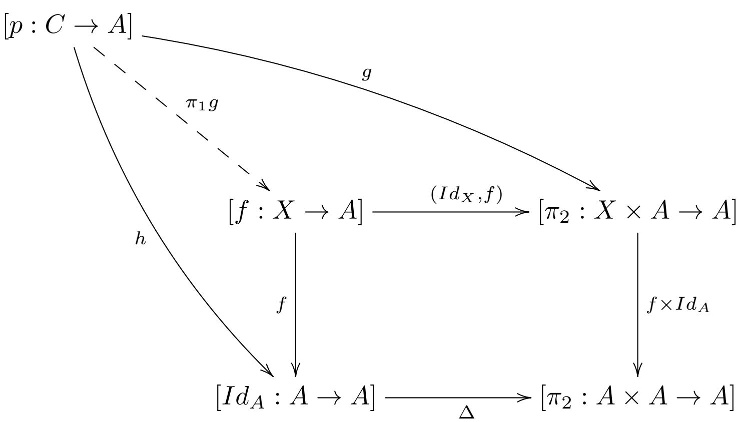
and a natural isomorphism $\phi : \overline{F} f^* \Rightarrow F$*, such that for every regular (resp. Boolean) functor* $G : \mathcal{C}/A \to \mathcal{D}$ *satisfying* $G(\Delta) = s$ *and natural isomorphism* $\psi : Gf^* \Rightarrow F$*, there exists a unique natural isomorphism* $\eta : G \Rightarrow \overline{F}$ *such that* $\phi \circ \eta Id_{f^*} = \psi$*:*



Proof. The square of the diagram below is a pullback in $\mathcal{C}$:



which implies that the following square is a pullback in $\mathcal{C}/A$:

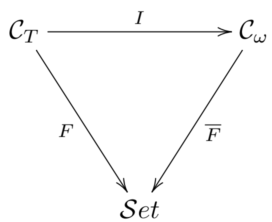


This leads to define $\overline{F}$ on $[f : X \to A]$ as the pullback of $F(f) : F(X) \to F(A)$

along $s : 1 \to F(A)$, in order to preserve the pullback above; as for arrows, we define $\overline{F}$ in the obvious way using the induced arrows between pullback diagrams. It can now be shown that the functor so defined enjoys all the required properties. Indeed, since $f^*(C)$ is the arrow $\pi_2 : C \times A \to A$, one pullback of $\pi_2 : F(C \times A) \to F(A)$ along $s : 1 \to F(A)$ is precisely the arrow $F(C) \to 1$, which gives, by the universal property of the pullback, morphisms $\phi_C : \overline{F} f^* \to F$. Moreover, if there is another extension of F, G, with the stated properties, then there are canonical isomorphisms $\eta_C : G(C) \to \overline{F}(C)$, induced again by the universal property of the pullback. Given a morphism $f : C \to D$, the arrows $\overline{F}(f)\phi_C$ and $\phi_D G(f)$ would be two induced morphisms between the pullbacks $G(C)$ and $\overline{F}(D)$; therefore, they must coincide, and then the isomorphisms η_C define a natural isomorphism $\eta : G \Rightarrow \overline{F}$, as stated. Similarly, the isomorphisms ϕ_C define a natural isomorphism $\phi : \overline{F} f^* \Rightarrow F$ and by uniqueness, we must have $\phi \circ \eta Id_{f^*} = \psi$. □

As an application of the preceding lemma we have the following:

Lemma 4.2. *For any Boolean functor $F : \mathcal{C}_T \to \mathcal{S}et$ there exists a Boolean functor $\overline{F} : \mathcal{C}_\omega \to \mathcal{S}et$ such that $\overline{F} I = F$, where I is the composition $I_0 J : \mathcal{C}_T \to \mathcal{C}_\omega$ and I_0, J are the embeddings defined in lemmas* 3.11 *and* 3.4 *respectively:*

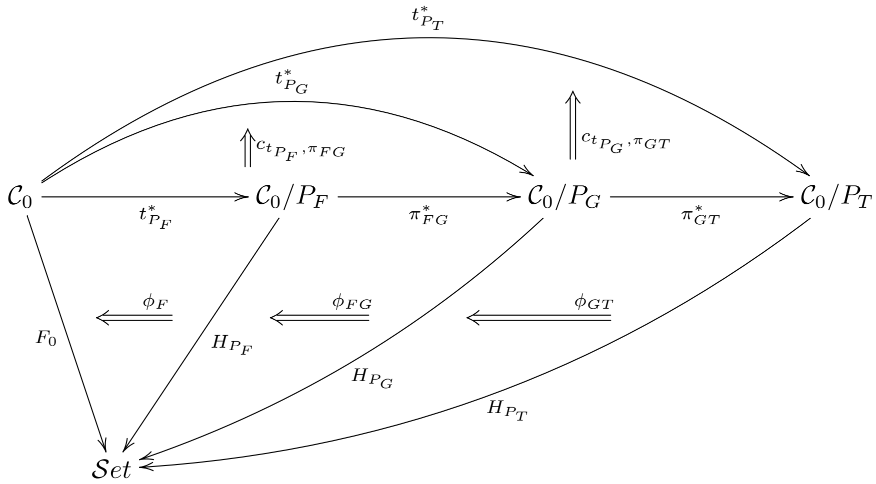


Proof. Because of lemma 3.4, we can choose a functor $F_0 : \mathcal{C}_0 = P(\mathcal{C}) \to \mathcal{S}et$ such that $F_0 J = F$. We shall use the axiom of choice to define succesive functors $F_i : \mathcal{C}_i \to \mathcal{S}et$ that form a pseudococone (see definition 3.6). Then, the universal property of the bicolimit $\mathcal{C}_\omega$ will give the desired functor $\overline{F}$. In fact, this is the idea that will be used to get F_1 from F_0 (we only show here this case since the others are similar).

Consider then the set of all finite products of covers $\{t_{P_F} : P_F = \prod_{i \in F} A_i \twoheadrightarrow 1 \ / \ F \subseteq \Gamma, finite\}$ in $\mathcal{C}_0$ (as in the considerations preceding lemma 3.10), which are mapped by F_0 into corresponding surjections $F_0(t_{P_F}) : F_0(P_F) \twoheadrightarrow F_0(1) = 1$. Use the axiom of choice to select a section s_{P_F} for each one of the surjections. Let π^*_{FG} be the pullback corresponding to π_{FG} through the pseudofunctor considered there. By lemma 4.1, F_0 provides Boolean functors $H_{P_i} : \mathcal{C}_0/P_i \to \mathcal{S}$ and natural isomorphisms $\phi_i : H_{P_i} t^*_{P_i} \Rightarrow F_0$, where each $t^*_{P_i}$ is the pullback selected by the pseudofunctor.

We shall now prove that the functors H_{P_i} form a pseudococone diagram in $\mathcal{C}at$. To do this we need to define natural isomorphisms $\phi_{FG} : H_{P_G} \pi^*_{FG} \Rightarrow$

H_{P_F}, where $F \subseteq G$, for each morphism π^*_{FG}. Now, note that we have a natural isomorphism given by $\phi_G \circ Id_{H_{P_G}} c_{t_{P_F}, \pi_{FG}} : H_{P_G} \pi^*_{FG} t^*_{P_F} \Rightarrow F_0$, and since H_{P_F} satisfies the universal property stated in lemma 4.1 with respect to the triangular diagram below with vertices $\mathcal{C}_0, \mathcal{C}_0/P_F, \mathcal{S}et$, there is a unique natural isomorphism $\phi_{FG} : H_{P_G} \pi^*_{FG} \Rightarrow H_{P_F}$ such that $\phi_F \circ \phi_{FG} Id_{t^*_{P_F}} = \phi_G \circ Id_{H_{P_G}} c_{t_{P_F}, \pi_{FG}}$.



If for $F \subseteq G \subseteq T$ we define similarly ϕ_{GT} and ϕ_{FT}, we just need to verify that with these natural isomorphisms the diagram becomes a pseudococone, which reduces in turn to verify condition b) of definition 3.6. But we can see that this is again a consequence of the universal property of lemma 4.1. Indeed, the natural isomorphism $\phi_{FG} \circ \phi_{GT} Id_{\pi^*_{FG}} \circ Id_{H_{P_T}} c^{-1}_{\pi_{FG}, \pi_{GT}} : H_{P_T} \pi^*_{FT} \Rightarrow H_{P_F}$ satisfies:

$$\phi_F \circ (\phi_{FG} \circ \phi_{GT} Id_{\pi^*_{FG}} \circ Id_{H_{P_T}} c^{-1}_{\pi_{FG}, \pi_{GT}}) Id_{t^*_{P_F}}$$

$$= (\phi_F \circ \phi_{FG} Id_{t^*_{P_F}}) \circ (\phi_{GT} Id_{\pi^*_{FG}} Id_{t^*_{P_F}}) \circ (Id_{H_{P_T}} c^{-1}_{\pi_{FG}, \pi_{GT}} Id_{t^*_{P_F}})$$

$$= [\phi_G \circ (Id_{H_{P_G}} c_{t_{P_F}, \pi_{FG}}) \circ (\phi_{GT} Id_{\pi^*_{FG}} Id_{t^*_{P_F}})] \circ (Id_{H_{P_T}} c^{-1}_{\pi_{FG}, \pi_{GT}} Id_{t^*_{P_F}})$$

$$= [\phi_G \circ (\phi_{GT} Id_{t^*_{P_G}}) \circ (Id_{H_{P_T}} Id_{\pi^*_{GT}} c_{t_{P_F}, \pi_{FG}})] \circ (Id_{H_{P_T}} c^{-1}_{\pi_{FG}, \pi_{GT}} Id_{t^*_{P_F}})$$

$$= [(\phi_T \circ Id_{H_{P_T}} c_{t_{P_G}, \pi_{GT}}) \circ (Id_{H_{P_T}} Id_{\pi^*_{GT}} c_{t_{P_F}, \pi_{FG}})] \circ (Id_{H_{P_T}} c^{-1}_{\pi_{FG}, \pi_{GT}} Id_{t^*_{P_F}})$$

$$= \phi_T \circ Id_{H_{P_T}} (c_{t_{P_G}, \pi_{GT}} \circ Id_{\pi^*_{GT}} c_{t_{P_F}, \pi_{FG}} \circ c^{-1}_{\pi_{FG}, \pi_{GT}} Id_{t^*_{P_F}}) = \phi_T \circ Id_{H_{P_T}} c_{t_{P_F}, \pi_{FT}}$$

where the last equality follows from property 3) of definition 3.5. Then, because of the uniqueness of ϕ_{FT}, we must have $\phi_{FT} = \phi_{FG} \circ \phi_{GT} Id_{\pi^*_{FG}} \circ Id_{H_{P_T}} c^{-1}_{\pi_{FG}, \pi_{GT}}$, as we wanted to prove. Therefore, the diagram is a pseudococone and the required functor $F_1 : \mathcal{C}_1 \to \mathcal{S}et$ is induced.

Finally, because the functors H_{P_F} in the pseudococone diagram are Boolean, the construction of the bicolimit in $\mathcal{C}at$ for a filtered diagram shows, after a straightforward verification, that F_1 preserves finite limits, covers, unions and disjoint coproducts; in particular, it is Boolean, which finishes the proof. □

Theorem 4.3. Löwenheim-Skolem theorem. *Every model $\mathcal{M}$ of a first order theory T with countable signature has an elementary submodel $\mathcal{N}$ which is at most countable.*

Proof. Suppose that T is a first order theory with countable signature that has a model $\mathcal{M}' : \mathcal{C}_T \to \mathcal{S}et$. By theorem 2.4, $\mathcal{M}'$ has an extension $\mathcal{M} : \mathcal{C}_\omega \to \mathcal{S}et$ that is a filtered colimit of ultra-representable functors. Furthermore, since $\mathcal{M}$ is not trivial, we see from the proof of that theorem that $(1, *)$ is an object in $\mathcal{W}$, and hence the corresponding ultrafilter $\Phi(1, *)$ in $\mathcal{S}ub(1)$ (defined as $\{S \in \mathcal{S}ub(1) \ / \ M(S) \neq \emptyset\}$) gives an ultra-representable functor $h_{\Phi(1,*)}$ that belongs to the colimit diagram for $\mathcal{M}$. But then it is easy to see that $h_{\Phi(1,*)}I$ is an elementary submodel for T which is at most countable, which finishes the proof. □

Remark 4.4. Even if one employs Grothendieck's construction of the bicolimit, the use of the axiom of choice throughout this section is not entirely avoidable, in the sense that some form of choice is needed to deduce theorem 2.4. For suppose we could prove in ZF that M' has an extension $M : \mathcal{C}_\omega \to \mathcal{S}et$; then, the Löwenheim-Skolem theorem would be derivable in ZF, while it is known to be unprovable there ([3]). As a consequence, the existence of the Boolean extension M must as well be unprovable in ZF.

Acknowledgment

I would like to thank Prof. E. Dubuc for his valuable suggestions and guidance.

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