PEANO ARITHMETIC & ITS NONSTANDARD MODELS

We look at a (first-order) language whose vocabulary consists of a single name letter (constant) "1", a 1-place function letter S, two 2-place function letters $\oplus$ and $\otimes$, and a 2-place predicate letter **<**

Let $\prod$ be the theorems of arithmetic; i.e., the (closed) deductive consequences of the following sentences (the axioms of Peano arithmetic).

1. $\bigwedge x\ [S(x) \neq 1]$
2. $\bigwedge x \bigwedge y[S(x)=S(y) \rightarrow x=y]$
3. $\bigwedge x[x \oplus 1=S(x)]$
4. $\bigwedge x \bigwedge y[x \oplus S(y)=S(x \oplus y)]$
5. $\bigwedge x[x \otimes 1=x]$
6. $\bigwedge x \bigwedge y[x \otimes S(y)=(x \otimes y) \oplus x]$
7. $\bigwedge x \bigwedge y[x < y \leftrightarrow \bigvee z(y=x \oplus z)]$
8. For every formula $\phi(x)$ of the language that contains only x free,

$$[\phi(1) \wedge \bigwedge x[\phi(x) \rightarrow \phi(S(x))] \rightarrow \bigwedge x \phi(x)$$

Add to the vocabulary of $\prod$ a new name letter α and, for every n, let α_n be the sentence defined as follows. α_1 is"$1<\alpha$" and if α_n is "$x<\alpha$" then α_{n+1} is "$S(x)<\alpha$". This yields : $1<\alpha$, $S(1)<\alpha$, $SS(1)<\alpha$, ... Thus α_n says "α is greater than the (n-1)th successor of 1"; i.e., that α is greater than n.

Consider the sentences $\prod^* = \prod \cup \{\alpha_1, \alpha_2, ..., \alpha_n, ...\}$.

Every finite subset of $\prod^*$ has a model. For every finite subset Σ of $\prod^*$ includes at most finitely many α_n. Look at the largest n such that α_n is in Σ and choose any number greater than that n as the extension of α. This choice for α will satisfy all the $\alpha_1, \alpha_2, ..., \alpha_n$ sentences and then all of Σ will be satisfied by the arithmetic of the natural numbers {1, 2, ... } under the usual conventions regarding successor, addition, multiplication, and the less-than relation.

By compactness it follows that $\prod^*$ has a model. Call it M*. All the theorems of arithmetic are satisfied in M* and, according to the Löwenheim-Skolem theorem, we can assume that the universe of M* is countably infinite. M* has extensions for 1, S(1), SS(1), etc. Let's call them 1, 2, 3 ... as usual. And similarly we can call the extensions in M* of $\oplus$, $\otimes$ and **<** by the usual names of +, . , and <. But M* also contains an extension for the new constant α. Let's call it β.

M* is usually referred to as a "nonstandard" model of arithmetic, and β as a nonstandard number. (The standard model of arithmetic, consisting of the standard numbers 1, 2, 3 ... under the usual operations and relations, is also in M*.) We are going to try to work out what the universe of a nonstandard model, like M*, looks like.

1. Although the universe of M* is countable there is no way to make it correspond one-to-one with the standard numbers in a way the preserves order, since β has to be greater than every standard number. So the standard and nonstandard models are not isomorphic.

2. Starting with β we can taking successors to get: $\beta<\beta+1<\beta+2, \ldots$. Since it is a theorem of arithmetic that every number except 1 is the successor of another number, β will be the successor of another number, call it β-1, and that the successor of another, etc. Thus we have in M*

$$\ldots,\beta-n,\ \ldots,\ \beta-2,\ \beta-1,\ \beta,\ \beta+1,\ \beta+2,\ \ldots\beta+n,\ \ldots$$

This is an infinite sequence, in order, that ascends and descends from β -- just as the integers ascend and descend from 0. Each number in this "cluster" around β is nonstandard; i.e., greater than all the standard numbers. (For if any number in the cluster were equal to a standard number, then β would either be the successor or the descendent of a standard number -- and not greater than all of them.)

3. Consider now $\beta+\beta$. Since it is a theorem of arithmetic that adding a number to itself produces a bigger number, $\beta<\beta+\beta$. As above, we can construct the infinite cluster $(\beta+\beta) \pm n$ around $(\beta+\beta)$. None of the numbers in this cluster are standard and none occur in the $\beta\pm n$ cluster, for either situation would make β standard. (For example, if $\beta+m=\beta+\beta$ then $\beta=m$.) Hence they are all greater than β. Thus adding nonstandard numbers gives us new and bigger ones and each such number is surrounded by a cluster of its successors and predecessors. So there isn't any biggest cluster.

4. Moreover there isn't any smallest cluster of nonstandard numbers either. For it is a theorem of arithmetic that for any number $x > 1$, there is a smaller number y such that either $2y=x$ or $2y=x+1$. So if β is nonstandard then there is a $\gamma<\beta$ such that either $\gamma+\gamma=\beta$ or $\gamma+\gamma=\beta+1$. In either case if γ were standard, β would be too. Moreover γ can't differ from β by any standard number without that forcing γ to be standard. For example of $\gamma+n=\beta$ and $\gamma+\gamma=\beta$, then $\gamma=n$. In the alternative, $\gamma+\gamma=\beta+1$, but then $\gamma+\gamma=(\gamma+n)+1$ and so $\gamma=n+1$.

5. It gets stranger. Suppose $\omega_1 < \omega_2$ are nonstandard numbers where $\omega_1 \neq \omega_2 \pm m$, for any standard m. (E.g., like β and $\beta+\beta$ in (3) above.) Then there is a nonstandard number γ, $\omega_1 < \gamma < \omega_2$, where $\gamma \neq \omega_1 \pm n$, for any standard n, and similarly $\gamma \neq \omega_2 \pm n$, for any standard n. For example we can get such a γ by starting with the sum $(\omega_1 + \omega_2)$. Either $(\omega_1 + \omega_2)=2\gamma$ or $2\gamma+1$ for some number γ (i.e., the sum is either even or odd). Suppose the former. Then the mean $(\omega_1 + \omega_2)/2$ is a whole number γ. But the mean lies between ω_1 and ω_2; that is, $\omega_1 <\gamma< \omega_2$. [**PROVE IT**!] So γ must be an infinite, nonstandard number, since it is larger than the nonstandard ω_1. Moreover, γ belongs neither to the ω_1-cluster, nor to the ω_2-cluster. For instance if it were the case that $\gamma= \omega_2 \pm n$, then $(\omega_1 + \omega_2)=2\gamma=2\omega_2 \pm 2n$. But then subtract off ω_2 and we would have that $\omega_1= \omega_2 \pm 2n$, contrary to the choice of ω_1 and ω_2 as belonging to different clusters. A similar argument hold in the case where $(\omega_1 + \omega_2)=2\gamma+1$. Since γ has its own cluster, this means that between any two clusters there is a third.

So then what does the universe of M* look like? It consists, first, of all the standard numbers in order. They are followed (in the "<" order) by an array of nonstandard clusters arranged in an order like that of the rational numbers, with neither a smallest nor a largest cluster, and with clusters occuring between any two.

Nevertheless, it follows from compactness that this entire nonstandard structure satisfies all the axioms of Peano arithmetic; i.e., that all arithmetical theorems are true in this structure.