The structure of the escaping set of a transcendental entire function

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Basic definitions

- $f : \mathbb{C} \to \mathbb{C}$ is analytic
- f^n is the nth iterate of f

Definition

The **Fatou set** (or stable set) is

$$F(f) = \{z : (f^n) \text{ is equicontinuous in some neighbourhood of } z\}.$$

The Fatou set is *open* and $z \in F(f) \iff f(z) \in F(f)$.

Definition

The **Julia set** (or chaotic set) is

$$J(f) = \mathbb{C} \setminus F(f).$$

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For polynomials:

- $I(f)$ is a neighbourhood of ∞;
- points in $I(f)$ escape at same rate;
- $I(f) \subset F(f)$;
- $J(f) = \partial I(f)$.

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For transcendental functions:

- $I(f)$ is *not* a neighbourhood of ∞;
- points in $I(f)$ escape at different rates;
- $I(f)$ must meet $J(f)$ and may meet $F(f)$.

Eremenko's conjectures

Theorem (Eremenko, 1989)

If f is transcendental entire then

- $J(f) \cap I(f) \neq \emptyset$;
- $J(f) = \partial I(f)$;
- *all components of* $\overline{I(f)}$ *are unbounded.*

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Conjecture 2 holds for many functions in class $\mathcal{B}$ but fails for others in class $\mathcal{B}$.

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Theorem (R+S, 2017)

$I(f)$ is connected or, for large $R > 0$, $I(f) \cap \{z : |z| \geq R\}$ has uncountably many unbounded components.

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Theorem (R+S, 2005)

For large $R > 0$, all the components of $A_R(f)$ are unbounded.

Examples

Exponential functions - disconnected escaping set

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- $A_R(f)$ is an uncountable union of curves, for large $R > 0$

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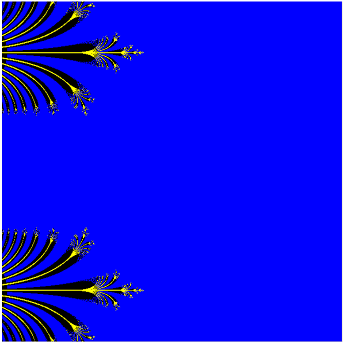
Fatou's function - connected escaping set

$$f(z) = z + 1 + e^{-z}$$



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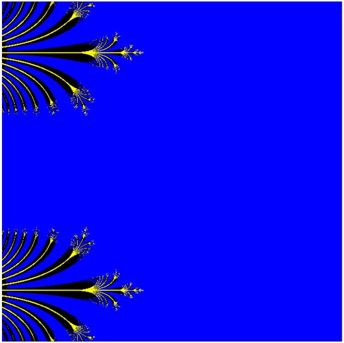

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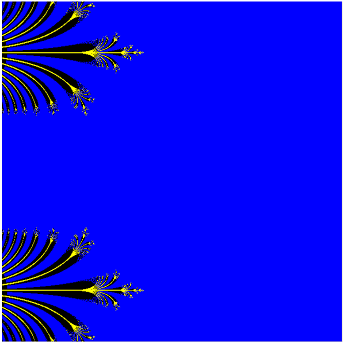

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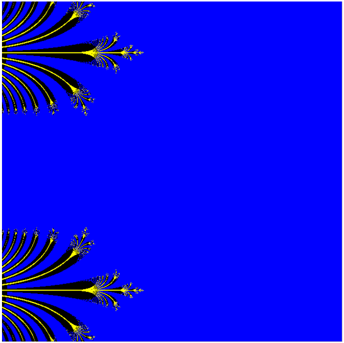

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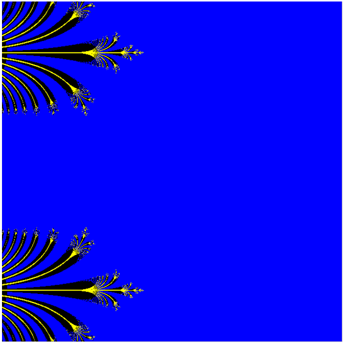

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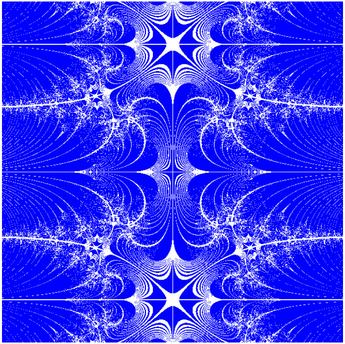

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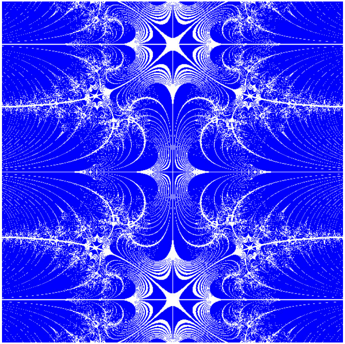

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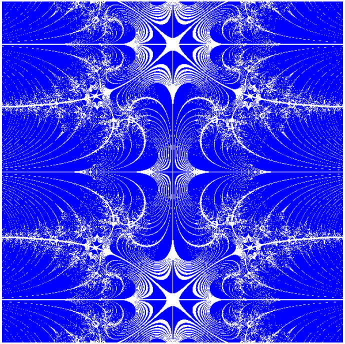

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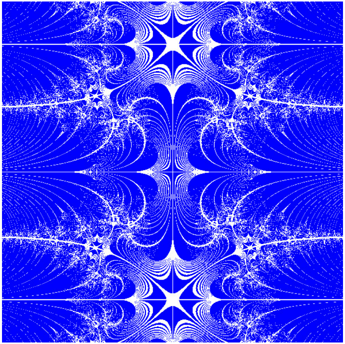

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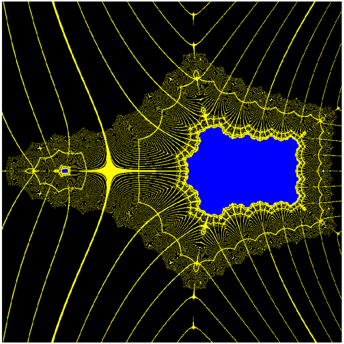
$$f(z) = \frac{1}{2}(\cos z^{1/4} + \cosh z^{1/4})$$



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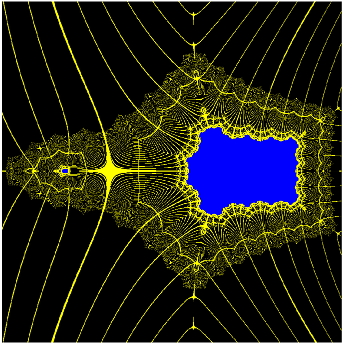

Definition

E is a **spider's web** if

- E is connected;
- there is a sequence of bounded simply connected domains G_n with

$$\partial G_n \subset E, \ G_{n+1} \supset G_n,$$

$$\bigcup_{n \in \mathbb{N}} G_n = \mathbb{C}.$$

Each of $I(f)$, $A(f)$ and $A_R(f)$ is connected and is a spider's web.

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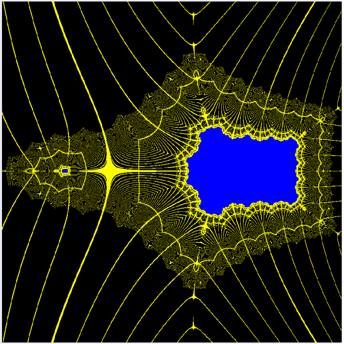

$A_R(f)$ *is a spider's web.*

Sketch proof

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Step 2 Refine Eremenko's method to construct uncountably many points in $A_R(f)$.
Step 3 Show that, if two of these points are in the same component of $A_R(f)$, then $A_R(f)$ is a spider's web.

Thanks for your attention!