A REMARK ON PFAFFIAN SURFACES AND ACM BUNDLES

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ABSTRACT. We prove that a general surface of degree d is the Pfaffian of a square matrix with (almost) quadratic entries if and only if $d \leq 15$.

1. INTRODUCTION

Given a sheaf $\mathscr{E}$ on a projective variety Y polarized by $\mathscr{O}_Y(1)$, we consider the cohomology modules:

$$\mathrm{H}^p_*(Y, \mathscr{E}) = \bigoplus_{t \in \mathbb{Z}} \mathrm{H}^p(Y, \mathscr{E} \otimes \mathscr{O}_Y(t)).$$

Here we will focus on those sheaves $\mathscr{E}$ that satisfy $\mathrm{H}^p_*(Y, \mathscr{E}) = 0$ for all $0 < p < \dim(Y)$. These are called aCM sheaves, standing for arithmetically Cohen-Macaulay, indeed $\mathrm{H}^0_*(Y, \mathscr{E})$ is a Cohen-Macaulay module over the coordinate ring of Y iff $\mathscr{E}$ is an aCM sheaf.

It is possible to classify all aCM bundles on projective spaces, (Horrocks, [Hor64]), quadrics (Knörrer, [Knö87]) and few other varieties, see [BGS87] and [EH88]. On the other hand, a detailed study of the families aCM bundles of low rank has been carried out in some cases, for instance some Fano threefolds (see e.g. [Mad02], [AC00], [AF06]) and Grassmannians, [AG99]. An even richer literature is devoted to aCM bundles of rank 2 on hypersurfaces Y_d of degree d in $\mathbb{P}^n$. If $n \geq 4$, and Y_d is general, the classification is complete, as it results from the papers [Kle78], [CM00], [CM04], [CM05], [KRR05], [KRR06].

On the other hand, for $n = 3$, the classification has been completed only up to $d \leq 5$, see [Fae05], [CF06], while only partial results are available for higher d. One of them is due to Beauville and Schreyer ([Bea00]), and states that Y_d can be written as a linear Pfaffian if and only if Y_d supports a certain aCM 2-bundle $\mathscr{E}$ with $\det(\mathscr{E}) \cong \mathscr{O}_{Y_d}(d-1)$, and this happens for general Y_d if and only if $d \leq 15$,

In this short note we prove that a general surface Y_d of degree d in $\mathbb{P}^3$ supports an aCM bundle $\mathscr{E}$ of rank 2 with $\det(\mathscr{E}) \cong \mathscr{O}_{Y_d}(d-2)$ if and only if $d \leq 15$. This amounts to writing the equation of Y_d as the Pfaffian of a certain skew-symmetric matrix. Part of the proof relies on a computation done with the computer algebra package `Macaulay 2`.

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2. Quadratic Pfaffian Surfaces

We will assume that the underlying field $\mathbf{k}$ is algebraically closed of characteristic zero. Recall that a torsionfree sheaf $\mathscr{E}$ on a polarized variety Y is called *initialized* if $\mathrm{H}^0(Y, \mathscr{E}) \neq 0$, and $\mathrm{H}^0(Y, \mathscr{E}(-1)) = 0$.

Let us introduce some notation. Given a projective variety $Y \subset \mathbb{P}^n$, polarized by $\mathscr{O}_Y(1)$, we write h_Y for the Hilbert function of Y, and $R(Y)$ for the coordinate ring of Y, so that $h_Y(t) = \dim_{\mathbf{k}}(R(Y)_t)$. We will write R for the coordinate ring of $\mathbb{P}^3$.

Given a smooth projective surface Y, polarized by $H_Y = c_1(\mathscr{O}_Y(1))$, and given an integer r and the Chern classes (c_1, c_2), we denote by $\mathsf{M}_Y(r, c_1, c_2)$ the moduli space of Gieseker-semistable sheaves with respect to H_Y, of rank r with Chern classes c_1, c_2. We will often denote the Chern classes by a pair integers: this stands for c_1 times H_Y and c_2 times the class of a point in Y.

Recall that the vanishing locus Z of a nonzero global section of a rank 2 initialized bundle $\mathscr{E}$ on a surface Y is arithmetically Gorenstein (i.e. R_Z is a Gorenstein ring) if and only if $\mathscr{E}$ is aCM. The *index* i_Z of a zero-dimensional aG subscheme Z is the largest integer c such that $h_Z(c) < \mathrm{len}(Z)$.

For basic material on aCM bundles and aG subschemes we refer to [IK99], [Die96], [Kle98]. In particular we recall the notation $\mathscr{G}_h(i, m, d)$, see [CF06, Section 3]. We will make use of the computer algebra package `Macaulay 2`, see [GS].

We will prove that a general surface Y_d of degree d is the Pfaffian of a skew-symmetric matrix with quadratic entries if and only if $d \leq 15$. This sentence makes sense only if d is an even number, so we will look for *almost quadratic* matrices. Namely, we consider a matrix of the form:

$$\mathscr{O}_{\mathbb{P}}(-2)^d \oplus \mathscr{O}_{\mathbb{P}}(-1)^\epsilon \to \mathscr{O}_{\mathbb{P}}^d \oplus \mathscr{O}_{\mathbb{P}}(-1)^\epsilon, \tag{2.1}$$

where ϵ is the remainder of the divison of d by 2. A surface Y_d can be written as an (almost) quadratic Pfaffian if and only if there is an aCM initialized rank 2 bundle $\mathscr{E}$ on Y_d with $c_1(\mathscr{E}) = d - 2$.

By [CF06, Proposition 4.1], we always have $c_1(\mathscr{E}) \leq d-1$. The case $c_1(\mathscr{E}) = d-1$ corresponds to matrices of size $2\,d$ whose entries are linear forms. This case was addressed by Beauville and Schreyer, who proved a general surface Y_d is the Pfaffian of a matrix of this form if and only if $d \leq 15$.

Recall by [CF06, Proposition 4.1] that $c_2(\mathscr{E}) = d - 2$ implies $c_2(\mathscr{E}) = d\,(d-1)\,(d-2)/3$.

Theorem 2.1. *On a general surface $Y_d \subset \mathbb{P} = \mathbb{P}^3$, it is defined a rank 2 initialized aCM bundle $\mathscr{E}$ with:*

$$c_1(\mathscr{E}) = d - 2, \qquad c_2(\mathscr{E}) = \frac{d\,(d-1)\,(d-2)}{3},$$

if and only if $d \leq 15$.

Proof. Note that the aCM bundle $\mathscr{E}$ is defined on a surface Y_d if and only if Y_d contains an aG subscheme Z of length $m = d\,(d-1)\,(d-2)/3$, and index $i = 2\,d-6$. This means that the function h_Z must agree with $h_{\mathbb{P}}$ up to degree $d-3$ and symmetric around $d-2$. In particular h_Z is uniquely determined.

To compute the dimension of the component $\mathscr{G}_{\mathrm{h}_Z}(i,m,d)$ of the scheme $\mathscr{G}(i,m,d)$ we may choose a subscheme Z having a minimal graded resolution of the form:

$$0 \to \mathscr{O}_{\mathbb{P}}(-2\,d+2) \to \begin{array}{c} \mathscr{O}_{\mathbb{P}}(-d)^{d-1} \\ \oplus \\ \mathscr{O}_{\mathbb{P}}(-d+1)^{\epsilon} \end{array} \to \begin{array}{c} \mathscr{O}_{\mathbb{P}}(-d+2)^{d-1} \\ \oplus \\ \mathscr{O}_{\mathbb{P}}(-d+1)^{\epsilon} \end{array} \to J_{Z,\mathbb{P}} \to 0.$$

Then the dimension of this component equals $4\,d^2-4\,d-1$, see [Kle98, Theorem 2.3]. Therefore, given a surface Y_d in the image of $p_{m,i,d}$ we have:

$$\begin{aligned} \dim(\mathrm{Im}(p_{m,i,d})) &\leq 4\,d^2-4\,d-1-\dim(p_{m,i,d}^{-1}(Y_d)) \leq \\ &\leq 4\,d^2-4\,d-1-d+1-\dim(\mathsf{M}_{Y_d}(2,d-2,m)) \leq \\ &\leq 4\,d^2-5\,d+\frac{d^2-18\,d+41}{6}. \end{aligned}$$

It is easy it see that this quantity is strictly less than $\mathrm{h}^0(\mathbb{P},\mathscr{O}_{\mathbb{P}}(d))-1$ for $d \geq 16$. So the map $p_{m,i,d}$ cannot be dominant for $d \geq 16$.

To prove the converse, we use the package `Macaulay 2`. We distinguish two cases according to the parity of d, and we let f be a generic mapping of the form (2.1).

In both cases, we consider the map Pf which associates to a skew-symmetric matrix the square root of its determinant. We would like to prove that Pf is dominant at the point represented by the matrix f, for each $d \leq 15$. For $d=1,2$, the assertion is trivial, while the case $d=3$ is clear by [Fae05].

For $d \geq 4$, we consider the the ideal J generated by the Pfaffians of order $d-2$ and degree $d-2$. Let $\mathfrak{m}$ be the ideal generated by the four variables of R, and define the ideal:

$$\mathcal{J} = \mathfrak{m}^2 \cdot J.$$

By Adler's method (see for instance the appendix of [Bea00]), our claim takes place if we show the equality:

$$\dim_{\mathbf{k}}(R/\mathcal{J}_d) = 0.$$

For each $4 \leq d \leq 15$, our claim can be checked by the `Macaulay 2` script:

```
isPrime(32003)
kk = ZZ/32003
R = kk[x_0..x_3];
almostQuadratic = (e1,e2,R) -> (
  -- a random almost quadratic skew-symmetric
  -- matrix on R of order e1+e2
  e:=e1+e2;
  N1:=binomial(e1,2);
  N2:=binomial(e2,2);
  N12:=e1*e2;
  N:=binomial(e,2);
  S:=kk[t_0..t_(N-1)];
  G:=genericSkewMatrix(S,t_0,e);
  substitute(G,random(R^{0},R^{N1:0,N12:-1,N2:-2}))
  )
quadraticAdler:=(M,d)->(
  --- returns the ideal generated by Pfaffians of degree d-2
  --- and by all polynomials of degree 2
  I := pfaffians((rank (source(M))-2),M);
```

```
  minI := mingens(I);
  mi := (min(degrees source minI))_0;
  va := ideal(vars(R));
  ideal(submatrix(minI,
   (toList select(0..(rank(source(minI))-1),i->(degree (minI)_i_0)_0==mi))
   ))*(va^(2)))
isDominant = (d)->(
  M := almostQuadratic((d-2*floor(d/2)),d,R);
  PF := quadraticAdler(M,d);
  (0 == hilbertFunction(d,R/PF))
  )
for d from 4 to 15 do print (d,isDominant(d))
```

This returns the value `true` for each $4 \leq d \leq 15$, in the approximate time of four hours on a personal computer. □

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