# Epipolar Geometry

Slides mostly from Hartley-Zisserman
CVPR'99 tutorial

Presented by: Varsha Hedau
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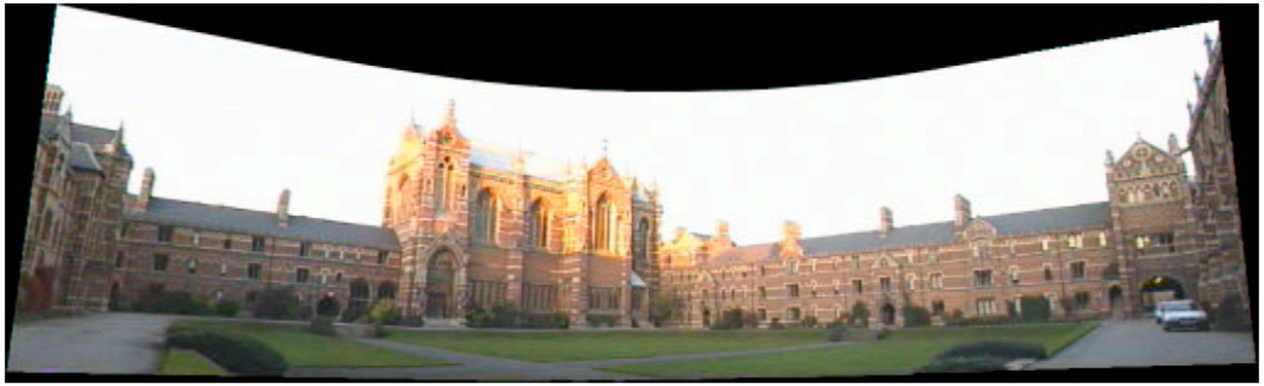
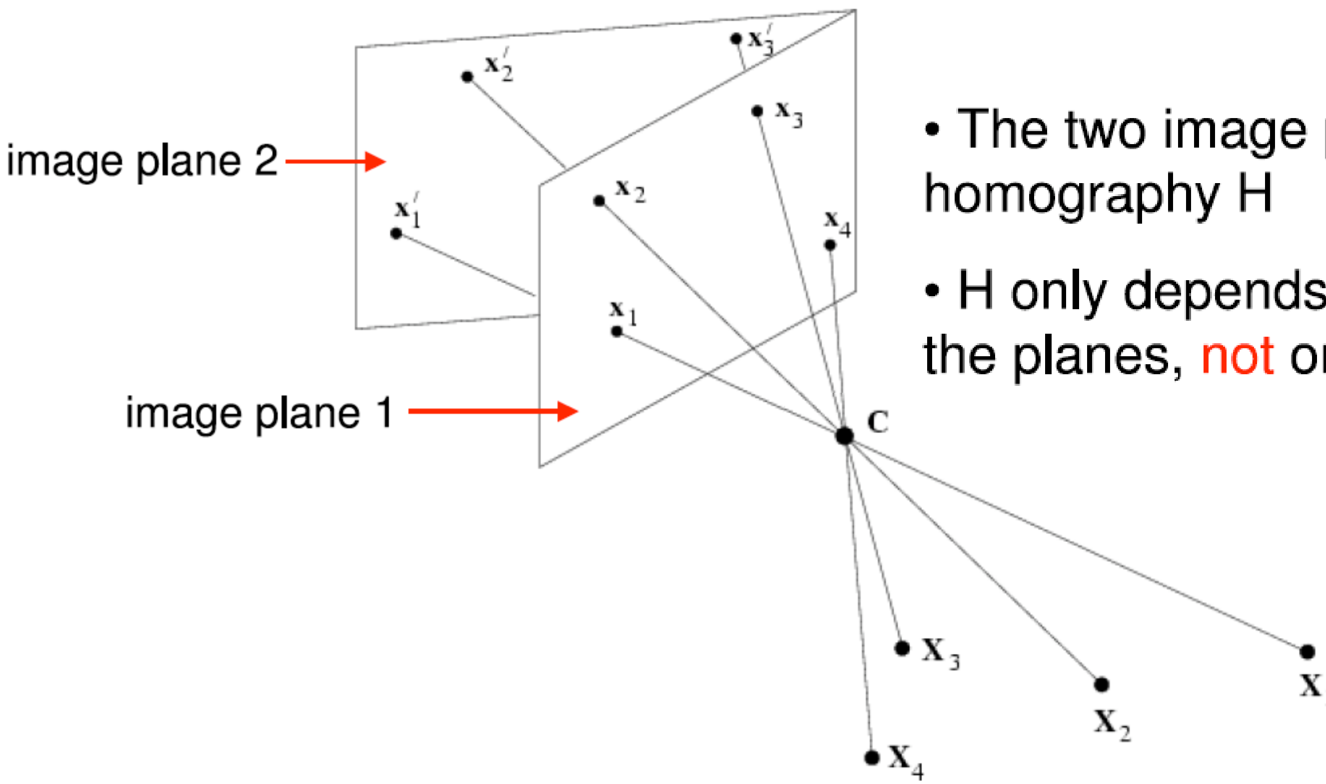
# Last Class





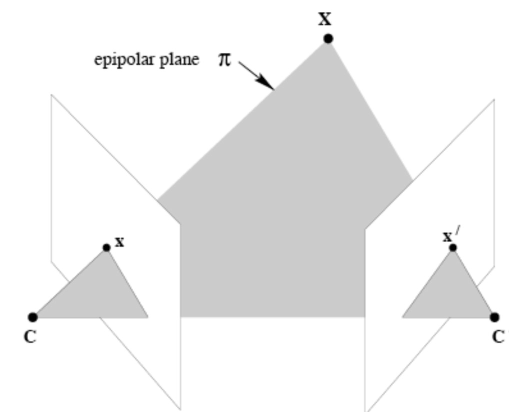

- The two image planes are related by a homography H
- H only depends on the camera centre, C, and the planes, not on the 3D structure

# Two View Geometry

- Cameras $\mathtt{P}$ and $\mathtt{P}'$ such that

$$\mathbf{x} = \mathtt{P}\mathbf{X} \qquad \mathbf{x}' = \mathtt{P}'\mathbf{X}$$

- Baseline between the cameras is non-zero.



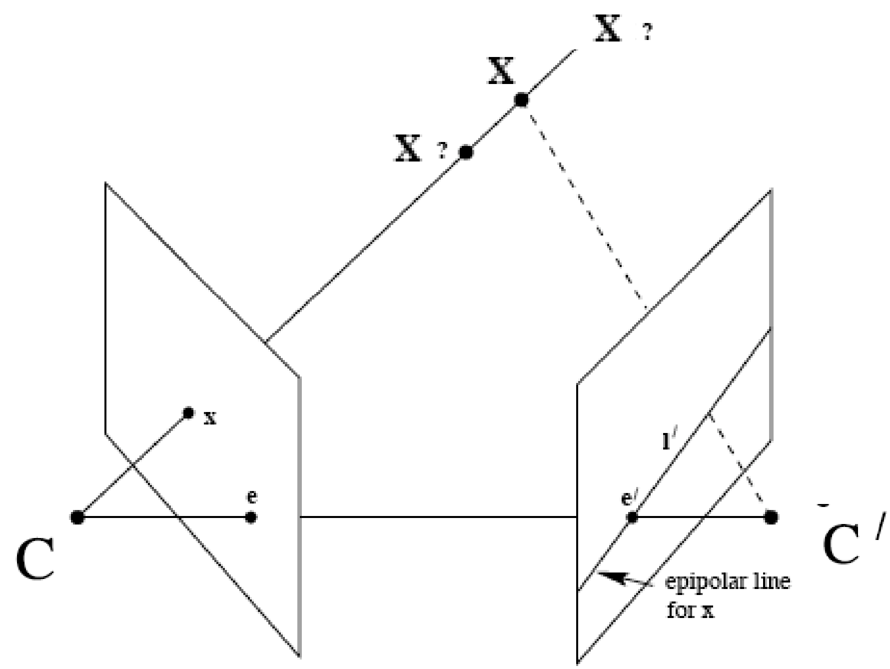

Given an image point in the first view, where is the corresponding point in the second view?

What is the relative position of the cameras?

What is the 3D geometry of the scene?

# Correspondence Geometry

Given the image of a point in one view, what can we say about its position in another?



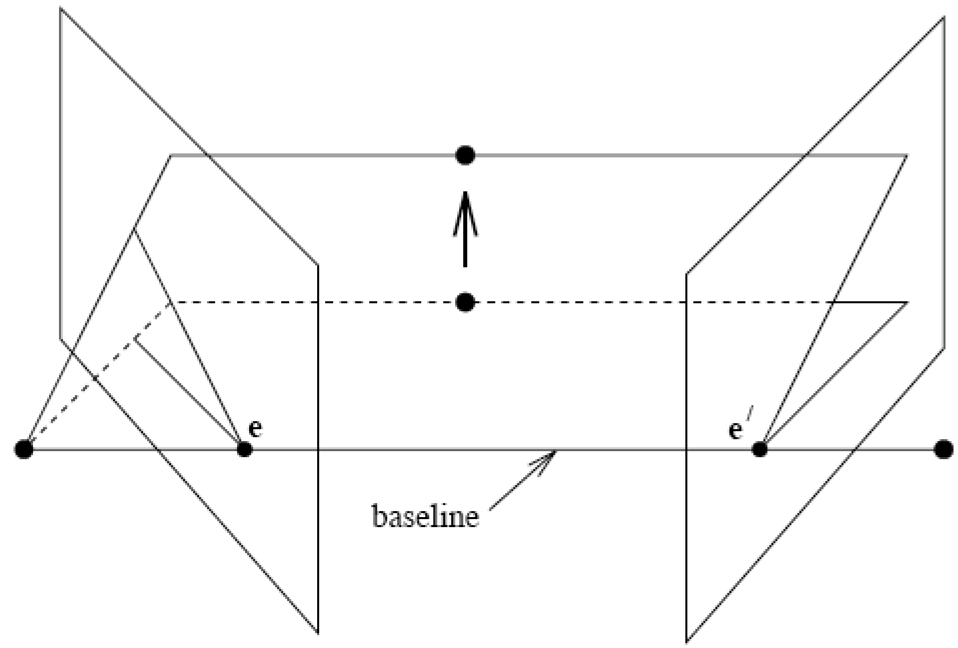

- A point in one image "generates" a line in the other image.
- This line is known as an epipolar line, and the geometry which gives rise to it is known as epipolar geometry.

# Epipolar line

Epipolar constraint

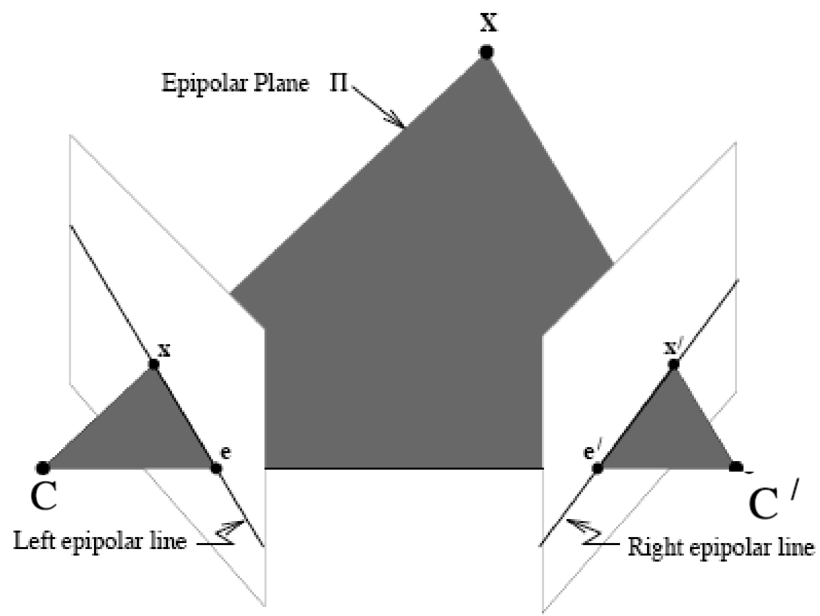
- Reduces correspondence problem to 1D search along an epipolar line

# Epipolar pencil



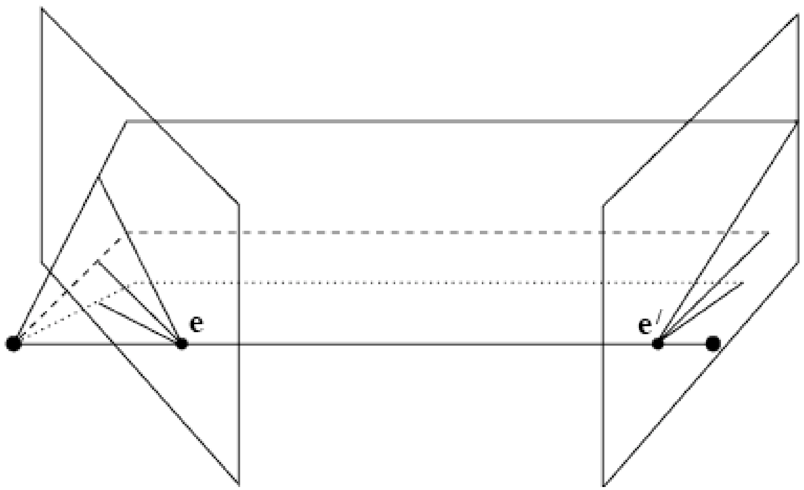

As the position of the 3D point $\mathbf{X}$ varies, the epipolar planes "rotate" about the baseline. This family of planes is known as an epipolar pencil. All epipolar lines intersect at the epipole.

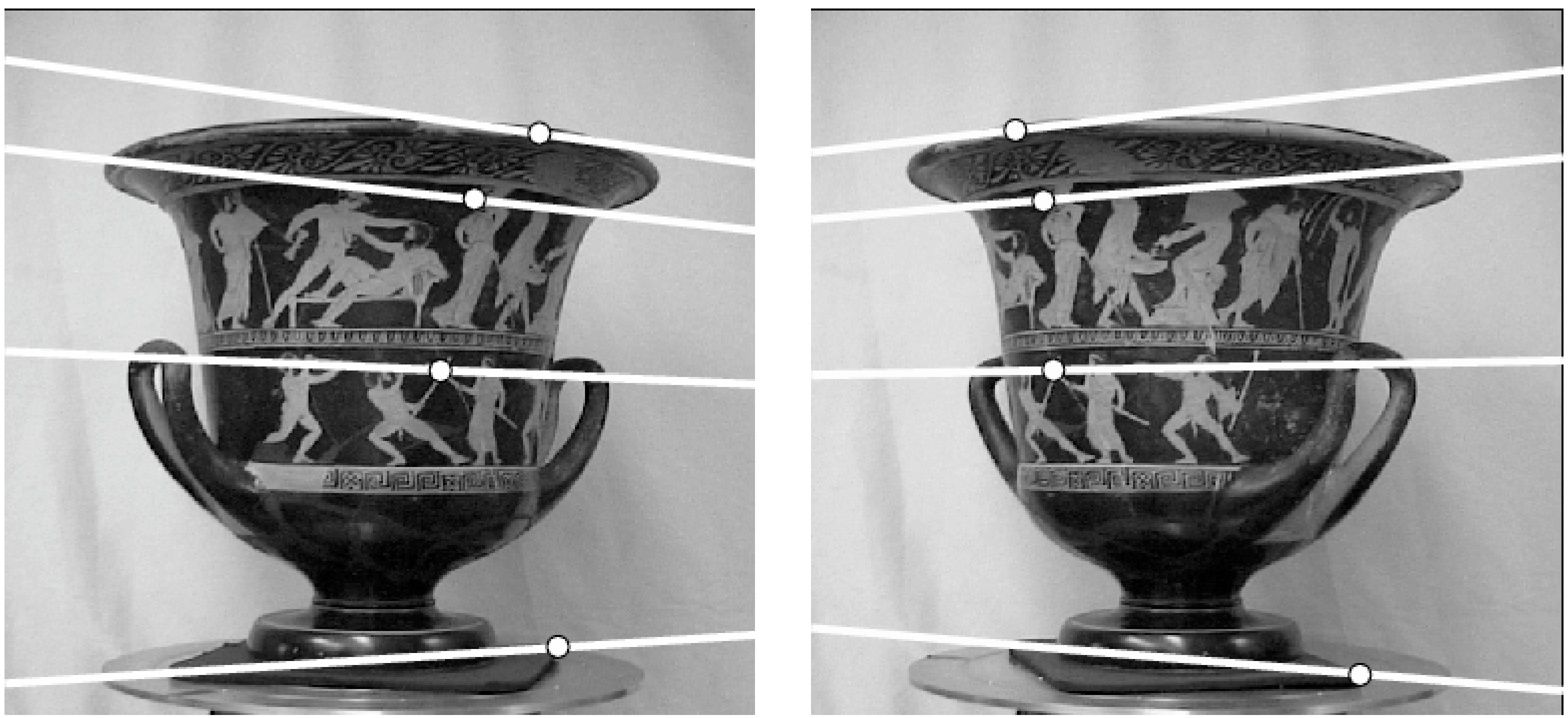
# Epipolar Geometry



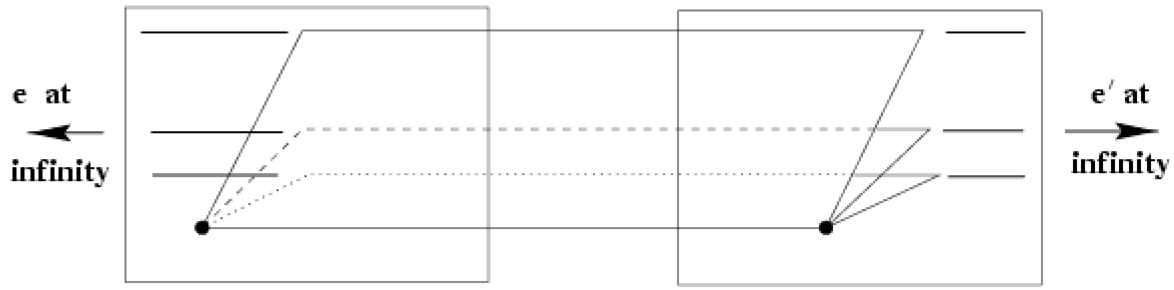

- The epipolar line $\mathbf{l}'$ is the image of the ray through $\mathbf{x}$.
- The epipole $\mathbf{e}'$ is the point of intersection of the line joining the camera centres—the baseline—with the image plane.
- The epipole is also the image in one camera of the centre of the other camera.
- All epipolar lines intersect in the epipole.

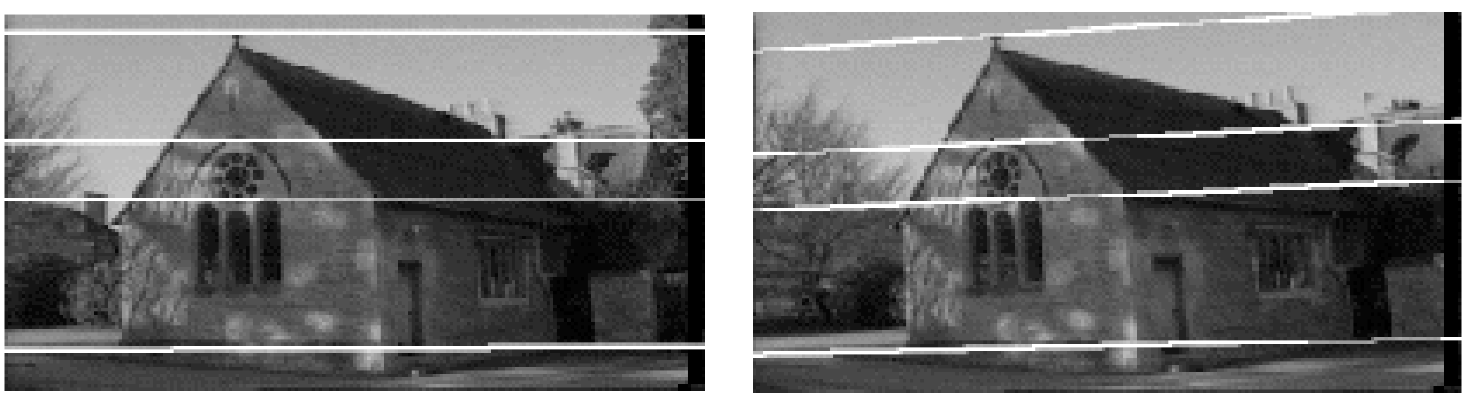
# Example: converging cameras



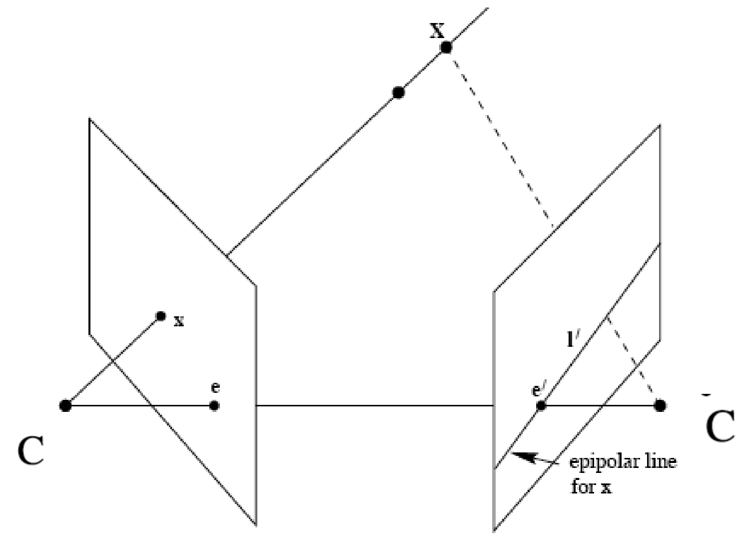
# Example: motion parallel to image plane



# Fundamental Matrix

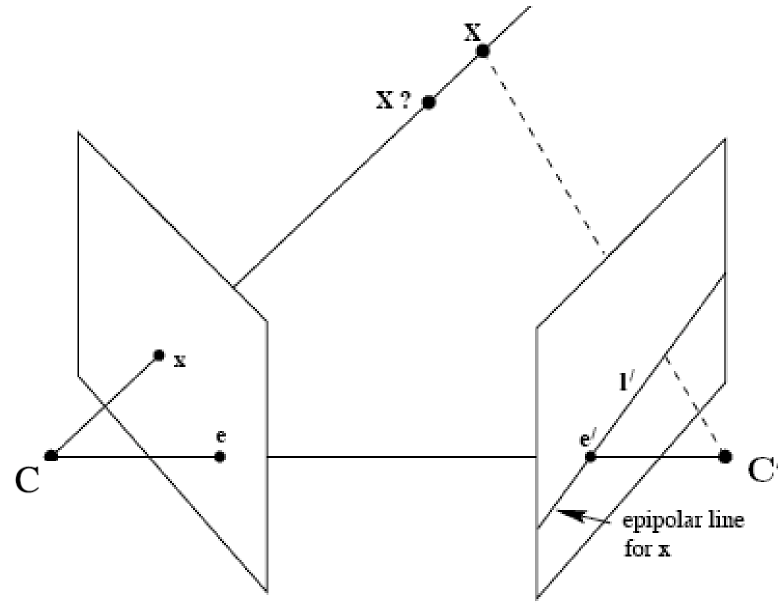
- $\mathrm{x} <-> \mathbf{x}'$
- $\mathbf{x}'^{\top}\mathbf{l}'(\mathbf{x}) = 0$
- $\mathbf{l}'(\mathbf{x}) = \mathrm{F}\mathbf{x}$
- $\mathbf{x}'^{\top}\mathrm{F}\mathbf{x} = 0.$
- $\mathrm{F}$ is 3X3 matrix of rank 2
- det( $\mathrm{F}$ )=0



## Properties of $\mathtt{F}$

- $\mathtt{F}$ is a rank 2 homogeneous matrix with 7 degrees of freedom.
- **Point correspondence**: If $\mathbf{x}$ and $\mathbf{x}'$ are corresponding image points, then $\mathbf{x}'^{\top}\mathtt{F}\mathbf{x} = 0$.
- **Epipolar lines**:
  - ◇ $\mathbf{l}' = \mathtt{F}\mathbf{x}$ is the epipolar line corresponding to $\mathbf{x}$.
  - ◇ $\mathbf{l} = \mathtt{F}^{\top}\mathbf{x}'$ is the epipolar line corresponding to $\mathbf{x}'$.
- **Epipoles**:
  - ◇ $\mathtt{F}\mathbf{e} = \mathbf{0}$ $\qquad$ $\mathtt{F}^{\top}\mathbf{e}' = \mathbf{0}$

# Fundamental Matrix in terms of camera matrices




- $\mathbf{l}'$ is projection of ray $CX$ in camera $C'$
- $\mathbf{l}' = (P'C) \times (P'P^{+}\mathbf{x})$ $PP^{+} = I$
- $\mathbf{l}' = F\mathbf{x}$ $F = [\mathbf{e}']_{\times} P'P^{+}$ $\mathbf{e}' = P'C$

# Matrix notation for vector product

The vector product $\mathbf{v} \times \mathbf{x}$ can be represented as a matrix multiplication

$$\mathbf{v} \times \mathbf{x} = [\mathbf{v}]_{\times} \mathbf{x}$$

where

$$[\mathbf{v}]_{\times} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}$$

# Projective Reconstruction from 2 views

Given

Corresponding points $\mathrm{x}_i \leftrightarrow \mathrm{x}'_i$ in two images.

Find

Cameras $\mathtt{P}$ and $\mathtt{P}'$ and 3D points $\mathrm{X}_i$ such that

$$\mathbf{x}_i = \mathtt{P}\mathbf{X}_i \quad ; \quad \mathbf{x}'_i = \mathtt{P}'\mathbf{X}_i$$

# Reconstruction Ambiguity

Given: image point correspondences $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$,
compute a reconstruction:

$$\{\mathrm{P},\ \mathrm{P}',\ \mathbf{X}_i\} \quad \text{with} \quad \mathbf{x}_i = \mathrm{P}\mathbf{X}_i \quad \mathbf{x}'_i = \mathrm{P}'\mathbf{X}_i$$

Ambiguity

$$\mathbf{x}_i = \mathrm{P}\mathbf{X}_i = \mathrm{P}\ \mathrm{H}(\mathrm{H})^{-1}\ \mathbf{X}_i = \tilde{\mathrm{P}}\tilde{\mathbf{X}}_i$$

$$\mathbf{x}'_i = \mathrm{P}'\mathbf{X}_i = \mathrm{P}'\ \mathrm{H}(\mathrm{H})^{-1}\ \mathbf{X}_i = \tilde{\mathrm{P}}'\tilde{\mathbf{X}}_i$$

$\{\tilde{\mathrm{P}},\ \tilde{\mathrm{P}}',\ \tilde{\mathbf{X}}_i\}$ is an equivalent Projective Reconstruction.

Reconstruction takes place in following steps

- Compute fundamental matrix F from point correspondences
- Decompose F to get camera projection matrices
- Compute points in 3D by triangulation

# Camera projection matrix from F

- P and P'can be obtained upto projective transformation due to projective ambiguity
- $\mathrm{F} = [\mathbf{e}']_\times \mathrm{P}'\mathrm{P}^+$
- Canonical pair of cameras $\mathrm{P} = [\mathrm{I} \mid \mathbf{0}]$ and $\mathrm{P}' = [\mathrm{M} \mid \mathbf{t}]$
- Factor the fundamental matrix F as $\mathrm{F} = [\mathbf{t}]_\times \mathrm{M}$ $\mathbf{t} = \mathbf{e}'$
- Get $\mathbf{e}'$ from svd(F)

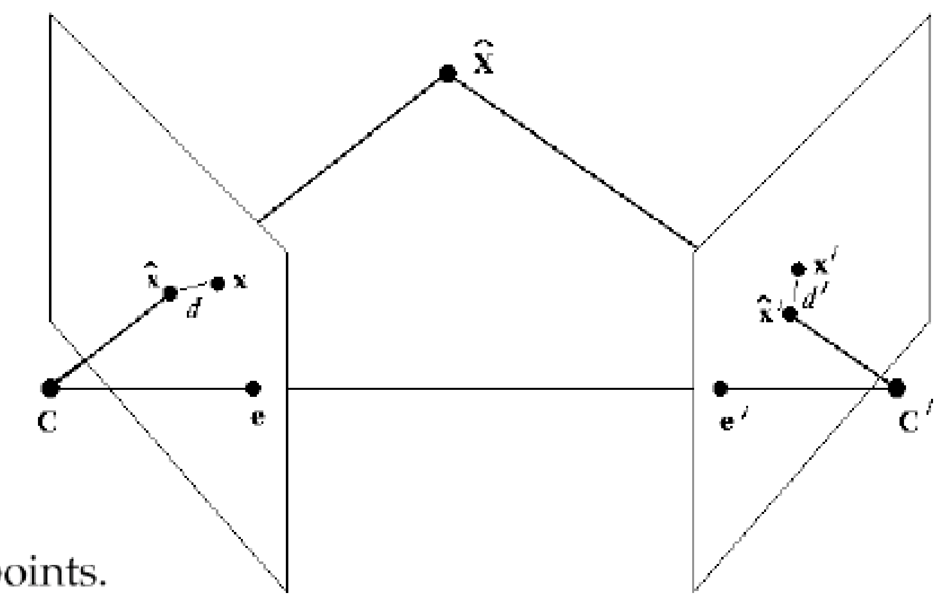
$\mathrm{F}^\top \mathbf{e}' = \mathbf{0}$ $\mathbf{e}'$ eigenvector with minimum eigen value

# Reconstructing the points in 3D

- Back project rays and compute intersection
- Rays do not intersect in presence of noise
- Estimate $\hat{\mathrm{X}}$ by minimizing projection error

$$\hat{\mathrm{x}} = \mathrm{P}\hat{\mathrm{X}} \qquad \hat{\mathrm{x}}' = \mathrm{P}'\hat{\mathrm{X}}$$

$$\min_{\hat{\mathrm{X}}} \quad \mathcal{C}(\mathbf{x}, \mathbf{x}') = d(\mathbf{x}, \hat{\mathbf{x}})^2 + d(\mathbf{x}', \hat{\mathbf{x}}')^2$$




where $d(*, *)$ is the Euclidean distance between the points.

# Computation of the Fundamental Matrix

# Basic equations

Given a correspondence

$$\mathbf{x} \leftrightarrow \mathbf{x}'$$

The basic incidence relation is

$$\mathbf{x}'^{\top}\mathtt{F}\mathbf{x} = 0$$

May be written

$$x'xf_{11} + x'yf_{12} + x'f_{13} + y'xf_{21} + y'yf_{22} + y'f_{23} + xf_{31} + yf_{32} + f_{33} = 0 \quad .$$

# Single point equation - Fundamental matrix

Gives an equation :

$$(x'x, x'y, x', y'x, y'y, y', x, y, 1) \begin{pmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{pmatrix} = 0$$

where

$$\mathbf{f} = (f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33})^\top$$

holds the entries of the Fundamental matrix

# Total set of equations

$$\mathtt{Af} = \begin{bmatrix} x'_1x_1 & x'_1y_1 & x'_1 & y'_1x_1 & y'_1y_1 & y'_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ x'_nx_n & x'_ny_n & x'_n & y'_nx_n & y'_ny_n & y'_n & x_n & y_n & 1 \end{bmatrix} \begin{pmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{pmatrix} = \mathbf{0}$$

- F has 9 entries but defined upto scale
- Singularity constraint det(F)=0
- F has 7 degrees of freedom
- 7 point algorithm - nonlinear equations
- 8 point algorithm -linear solution

  \- constraint enforcement

| Sample size | Proportion of outliers $\epsilon$ | | | | | | |
|---|---|---|---|---|---|---|---|
| $s$ | 5% | 10% | 20% | 25% | 30% | 40% | 50% |
| 2 | 2 | 2 | 3 | 4 | 5 | 7 | 11 |
| 3 | 2 | 3 | 5 | 6 | 8 | 13 | 23 |
| 4 | 2 | 3 | 6 | 8 | 11 | 22 | 47 |
| 5 | 3 | 4 | 8 | 12 | 17 | 38 | 95 |
| 6 | 3 | 4 | 10 | 16 | 24 | 63 | 191 |
| 7 | 3 | 5 | 13 | 21 | 35 | 106 | 382 |
| 8 | 3 | 6 | 17 | 29 | 51 | 177 | 766 |

- In presence of noise 7 point algorithm could be more efficient

# Computing F from 7 points

- F has only 7 degrees of freedom.
- It is possible to solve for F from just 7 point correspondences.

# 7-point algorithm

## Computation of $\mathtt{F}$ from 7 point correspondences

(i) Form the $7 \times 9$ set of equations $\mathtt{A}\mathbf{f} = 0$.

(ii) System has a 2-dimensional solution set.

(iii) General solution (use SVD) has form

$$\mathbf{f} = \lambda \mathbf{f}_0 + \mu \mathbf{f}_1$$

(iv) In matrix terms

$$\mathtt{F} = \lambda \mathtt{F}_0 + \mu \mathtt{F}_1$$

(v) Condition $\det \mathtt{F} = 0$ gives cubic equation in $\lambda$ and $\mu$.

(vi) Either one or three real solutions for ratio $\lambda : \mu$.

# Complete 8-point algorithm

8 point algorithm has two steps :

(i) **Linear solution.** Solve $\mathtt{A}\mathtt{f} = 0$ to find $\mathtt{F}$.

(ii) **Constraint enforcement.** Replace $\mathtt{F}$ by $\mathtt{F}'$.

# 8 Point Algorithm

- 8 points $\Rightarrow$ unique solution
- $> 8$ points $\Rightarrow$ least-squares solution.

## Least-squares solution

(i) Form equations $\mathtt{A}\mathbf{f} = \mathbf{0}$.

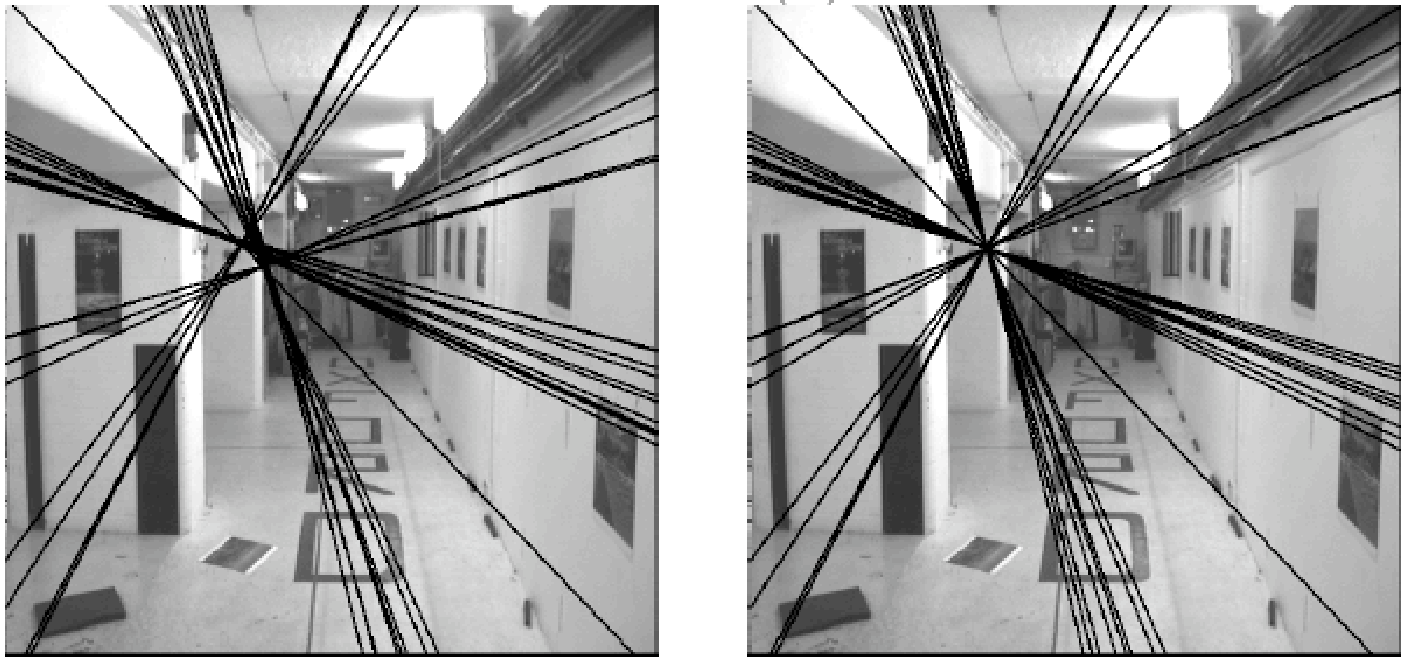
(ii) Take SVD : $\mathtt{A} = \mathtt{UDV}^\top$.

(iii) Solution is last column of $\mathtt{V}$ (corresp : smallest singular value)

(iv) Minimizes $||\mathtt{A}\mathbf{f}||$ subject to $||\mathbf{f}|| = 1$.

# The singularity constraint

Fundamental matrix has rank 2 : $\det(\mathtt{F}) = 0$.



**Left :** Uncorrected $\mathtt{F}$ – epipolar lines are not coincident.

**Right :** Epipolar lines from corrected $\mathtt{F}$.

## Correcting $\mathtt{F}$ using the Singular Value Decomposition

If $\mathtt{F}$ is computed linearly from 8 or more correspondences, singularity condition does not hold.

### SVD Method

(i) SVD : $\mathtt{F} = \mathtt{UDV}^\top$

(ii) $\mathtt{U}$ and $\mathtt{V}$ are orthogonal, $\mathtt{D} = \mathrm{diag}(r, s, t)$.

(iii) $r \geq s \geq t$.

(iv) Set $\mathtt{F}' = \mathtt{U}\,\mathrm{diag}(r, s, 0)\,\mathtt{V}^\top$.

(v) Resulting $\mathtt{F}'$ is singular.

(vi) Minimizes the Frobenius norm of $\mathtt{F} - \mathtt{F}'$

(vii) $\mathtt{F}'$ is the "closest" singular matrix to $\mathtt{F}$.

# The normalized 8-point algorithm

Raw 8-point algorithm performs badly in presence of noise.

## Normalization of data

- 8-point algorithm is sensitive to origin of coordinates and scale.
- Data must be translated and scaled to "canonical" coordinate frame.
- Normalizing transformation is applied to both images.
- Translate so centroid is at origin
- Scale so that RMS distance of points from origin is $\sqrt{2}$.
- "Average point" is $(1, 1, 1)^\top$.

# Normalized 8-point algorithm

(i) **Normalization:** Transform the image coordinates :

$$\hat{\mathbf{x}}_i = \mathrm{T}\mathbf{x}_i$$
$$\hat{\mathbf{x}}'_i = \mathrm{T}'\mathbf{x}'_i$$

(ii) **Solution:** Compute $\mathtt{F}$ from the matches $\hat{\mathbf{x}}_i \leftrightarrow \hat{\mathbf{x}}'_i$

$$\hat{\mathbf{x}}'^{\top}_i \widehat{\mathrm{F}} \hat{\mathbf{x}}_i = 0$$

(iii) **Singularity constraint :** Find closest singular $\widehat{\mathtt{F}}'$ to $\widehat{\mathtt{F}}$.

(iv) **Denormalization:** $\mathtt{F} = \mathtt{T}'^{\top} \widehat{\mathtt{F}}' \mathtt{T}$.