

## Some Applications of Skew Hermitian Linear Maps

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**Abstract:** If  $M$  be closed subspace of a Hilbert space  $H$ , then there exist a unique pair of Linear mappings  $P$  and  $Q$  such that  $P$  maps  $H$  onto  $M$ ,  $Q$  maps  $H$  onto  $M^\perp$ . And  $x = Px + Qx$  for all  $x \in H$ . In this study we prove some similar results For any Hilbert space  $H$ , the identity map  $I$  and all skew Hermitian maps. Some applications also followed.

**Key words:** Hilbert space, maximally isotropic, uniqueness theorem. 2000 mathematics subject classification; 46C05, 26D15, 35A05

### INTRODUCTION

In view of the decomposition theorem on Hilbert spaces, any Hilbert space can be decomposed as a direct sum of any subspace  $M$  and its orthogonal complement. In this study it is shown that any  $z \in H$  has a unique representation  $z = x + Sx$  for all densely defined skew Hermitian maps and some  $x \in H$ . As a first step, we study some preliminary properties of the graph of a skew Hermitian map. Then, as a useful application, a decomposition theorem was proved. In the next, the theorem is generalized to densely defined skew hermitian maps.

### MAXIMALLY ISOTROPIC SUBSPACE

Let  $(H, \langle, \rangle)$  be a Hilbert space. The componentwise inner product on  $H \oplus H$

is also denoted by  $\langle, \rangle_+$ . Let  $\langle, \rangle_+$  be the pairing on  $H \oplus H$  defined by,

$$\langle (x, y), (x', y') \rangle_+ = \langle x, y' \rangle + \langle y, x' \rangle, \forall x, x', y, y' \in H$$

With the above notations, a maximal isotropic subspace (m.i.s) on  $H$ , is a subspace  $L$  of  $H \oplus H$  such that,

$$\langle (x, y), (x', y') \rangle_+ = 0, \forall (x, y), (x', y') \in L$$

furthermore,  $L$  has no proper extension with the above property.

**Example:** for a Hilbert space  $H$  the Set  $I$  of all isotropic subspaces of  $H \oplus H$  is nonempty and partially ordered by inclusion relation in which every chain has an upper bound, so  $I$  possesses a maximal element by *Zorn's lemma*.

**Example:** let  $S: H \rightarrow H$  be a skew Hermitian linear map and  $L = \text{graph}(S) \subseteq H \oplus H$ .  $L$  is a (m.i.s) on  $H \oplus H$  (Courant, 1990).

**Lemma:** Let  $M \subseteq H \oplus H$  be isotropic. Then  $\overline{M}$  is also isotropic. **PROOF.** Assume that  $(a, b) \in \overline{M}$  and  $(x_n, y_n)$  be a sequence of elements of  $M$  converging to  $(a, b)$ , then,

$$\begin{aligned} \langle (a, b), (x, y) \rangle_+ &= \langle a, y \rangle + \langle b, x \rangle = \langle \lim x_n, y \rangle + \langle \lim y_n, x \rangle = \\ &= \lim \langle (x_n, y_n), (x, y) \rangle_+ = 0 \end{aligned}$$

**Corollary:** Every (m.i.s) on  $H$ , is a Hilbert subspace of  $H \oplus H$

**Theorem:** let  $L$  be a (m.i.s) on  $H$  and  $P_i: H \rightarrow H$  ( $i = 1, 2$ ) be the first and second projections. Then the map  $P_1 + P_2: L \rightarrow H$  is surjective.

**Proof:** Let  $z \in (P_1 + P_2)(L)$ , by the decomposition theorem on Hilbert spaces,

There are  $(x, y), (x', y') \in L$  such that  $(z, z) = (x, y) + (y', x')$ . Therefore,

$$\begin{aligned} 2\|z\|^2 &= \langle (z, z), (z, z) \rangle = \langle (z, z), (x, y) \rangle + \langle (z, z), (y', x') \rangle = \\ &= \langle z, x + y \rangle + \langle z, x' + y' \rangle = 0 \end{aligned}$$

i.e.,  $z = 0$ . Hence  $(P_1 + P_2)(L)$  is dense in  $H$ .  $P_1 + P_2$  is also norm preserving, because for  $(x, y) \in L$ ,

$$\begin{aligned} \|(x, y)\|^2 &= \langle (x, y), (x, y) \rangle = \langle x, x \rangle + \langle y, y \rangle = \|x\|^2 + \|y\|^2 = \\ &= \langle x + y, x + y \rangle = \|x + y\|^2 \end{aligned}$$

consequently if  $(P_1 + P_2)(x_n)$  be a Cauchy sequence in  $H$ , then  $(x_n)$  is also a Cauchy sequence in  $L$ . Hence,  $(P_1 + P_2)(L) = H$ .

**Theorem:** with the above notations, the map  $P_1+P_2 : L \rightarrow H$  is an isomorphism of Hilbert spaces.

**Proof:** By theorem 2.5 and Banach's theorem it is enough to show that the map is injective. Let  $z = (x, y) \in \ker (P_1+P_2)$  then  $x+y = 0$  and  $(x, -y) \in L$ . Therefore,

$$0 = \langle (x, -x), (x, -x) \rangle = -2\|x\|^2$$

i.e.,  $z = 0$ .

**Theorem:** Let  $S : H \rightarrow H$  be a skew Hermitian linear map. Then every  $h \in H$  has a unique representation  $h = x + Sx$  for some  $x \in H$  such that,  $\|h\|^2 = \|x\|^2 + \|Sx\|^2$

**Proof:** Let  $L = \text{graph}(S)$ . By example 2.1  $L$  is a (m.i.s) on  $H$ . Thus by theorem 2.6,  $P_1+P_2$  is an isometry. Hence, for some  $x \in H$  we have  $h = x+Sx$  with required identity on norms.

Some interesting examples will be appearing by the following lemma.

**Lemma:** Let  $H \rightarrow H$  is a densely defined Skew Hermitian linear map. Then  $\text{graph}(s)$  Is closed.

**Proof:** Let  $(x_n, y_n)$  be a sequence in  $\text{graph}(S)$  converging to  $(x, y)$  and  $\mu \in \text{Dom}(S)$ , then,

$$\langle u, -Sx_n \rangle = \langle Su, x_n \rangle$$

Therefore,

$$\langle u, -y \rangle = \langle Su, x \rangle \text{ and } x \in \text{Dom}(S) = \text{Dom}(S^*)$$

Thus  $\langle u, -y \rangle = \langle u, -Sx \rangle$  and  $u \perp y - Sx$  for all  $u \in \text{Dom}(S)$ . Since  $\text{Dom}(S)$  is dense in  $H$ , we have  $y - Sx = 0$  and  $(x, y) \in \text{graph}(S)$ .

**Theorem:** Let  $S : H \rightarrow H$  be a densely defined skew Hermitian linear map. Then  $\text{graph}(S)$  is a (m.i.s) on  $H$ .

**Proof:** by lemma 2.8  $\text{graph}(S)$  is a closed subspace of  $H \oplus H$  Let  $\{(a, b)\} \cup \text{graph}(S)$  be an isotropic subset of  $H \oplus H$  Then we have,

$$\langle (a, b), (x, Sx) \rangle_+ = \langle a, Sx \rangle + \langle b, x \rangle = 0$$

for all  $x \in \text{Dom}(S)$  Furthermore without loss of generality we can assume,

$$\langle a, x \rangle + \langle b, Sx \rangle = 0$$

Consequently,

$$\langle a + b, x + Sx \rangle = \langle a, x \rangle + \langle b, Sx \rangle + \langle a, Sx \rangle + \langle b, x \rangle = 0$$

for all  $x \in \text{Dom}(S)$  Let  $u = a+b$ , then, and  $0 = \langle x + Sx, u \rangle = \langle x, -u \rangle = \langle x, -Su \rangle = \langle x, u \rangle + \langle Sx, u \rangle$ , therefore and  $\mu \in \text{Dom}(S^*) = \text{Dom}(S)$  and  $\langle x, -u \rangle = \langle Sx, u \rangle$  for all  $x \in \text{Dom}(S)$ . Consequently,  $Su = u$ . Now we have,

$$\|u\|^2 = \langle u, u \rangle = \langle Su, u \rangle = \langle u, -Su \rangle = -\|u\|^2$$

or  $u = 0$  and  $a = -b$ . since  $\text{graph}(S) \cup \{(a, b)\}$  is isotropic subspace,  $a = b = 0$  and the  $\text{graph}(S)$  is maximally isotropic subspace of  $H \oplus H$ . i.e.,  $\text{graph}(s)$  is a (m.i.s) on  $H$ .

**Theorem:** Let  $S : H \rightarrow H$  be a densely defined skew Hermitian linear map. Then every  $h \in H$  has a unique representation  $h = x+Sx$ , for some  $h = x+Sx$ , for some  $x \in H$  such that,

$$\|h\|^2 = \|x\|^2 + \|Sx\|^2$$

**Proof:** let  $L = \text{graph}(S)$ . by 2.9  $L$  is a (m.i.s) on  $H$ . Thus  $P_1+P_2 : L \rightarrow H$  is an isometry, by theorem 2.6. Hence,  $h = x+Sx$  for some  $x \in H$  and the above identity on norms.

**Application:** Let  $H = L^2(\mathbb{R})$  and  $S = d/dx$  be differential operator defined on the space  $C_0^1(\mathbb{R}) = H$ , the space of continuously differentiable real functions with compact support. Using integration by parts, it can be shown that  $S$  is skew Hermitian linear map. Thus by theorem 2.10, any  $h \in L^2(\mathbb{R})$  has a unique representation  $h = f + \frac{df}{dx}$  for

some  $f \in C_0^1(\mathbb{R})$ .

## REFERENCES

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