On fuzzy compact-open topology

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Abstract

The concept of fuzzy compact-open topology is introduced and some characterizations of this topology are discussed.

Keywords: Fuzzy locally compact Hausdorff space; Fuzzy product topology; Fuzzy compact-open topology; Fuzzy homeomorphism; Evaluation map; Exponential map.

1. Introduction

Ever since the introduction of fuzzy set by Zadeh [13] and fuzzy topological space by Chang [2], several authors have tried successfully to generalize numerous pivot concepts of general topology to the fuzzy setting. The concept of compact-open topology has a vital role in defining function spaces in general topology. We intend to introduce the concept of fuzzy compact-open topology and contribute some theories and results relating to this concept. The concepts of fuzzy local compactness and fuzzy product topology play a vital role in the theory that we have developed in this paper. We have used the fuzzy locally compactness notion due to Wong [11], Christoph [3] and fuzzy product topology due to Wong [12].

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2. Preliminaries

The concepts of fuzzy topologies are standard by now and can be referred from [2, 8, 13]. For the definitions and various results of fuzzy topology, fuzzy continuity, fuzzy open map, fuzzy compactness we refer to [2]. For the definitions of fuzzy point and fuzzy neighborhood of a fuzzy point we refer to [8].

So far as the notation is concerned, we denote fuzzy sets by letters such as A, B, C, U, V, W, etc. I^X denotes the set of all fuzzy sets on a nonempty set X. 0_X and 1_X denote, respectively, the constant fuzzy sets taking the values 0 and 1 on X. $\bar{A}$, A°, A' will denote the fuzzy closure, fuzzy interior and the fuzzy complement of $A \in I^X$, respectively.

We need the following definitions and results for our subsequent use.

Theorem 2.1 (Ganguly and Saha [5]). *A mapping $f: X \to Y$ from an fts X to an fts Y is said to be fuzzy continuous at a fuzzy point x_t of X if and only if for* *every fuzzy neighborhood V of $f(x_t)$, there exists a fuzzy neighborhood U of x_t such that $f(U) \leqslant V$.*

f is said to be fuzzy continuous on X if it is so at each fuzzy point of X.

Definition 2.2. (Chang [2]). A mapping $f: X \to Y$ is said to be a fuzzy homeomorphism if f is bijective, fuzzy continuous and fuzzy open.

Definition 2.3 (Srivastava and Srivastava [10]). An fts (X, T) is called a fuzzy Hausdorff space or T_2-space if for any pair of distinct fuzzy points (i.e., fuzzy points with distinct supports) x_t and y_r, there exist fuzzy open sets U and V such that $x_t \in U$, $y_r \in V$ and $U \wedge V = 0_X$.

Definition 2.4 (Wong [11]). An fts (X, T) is said to be fuzzy locally compact if and only if for every fuzzy point x_t in X there exists a fuzzy open set $U \in T$ such that $x_t \in U$ and U is fuzzy compact, i.e., each fuzzy open cover of U has a finite subcover.

Note 2.5. Each fuzzy compact space is fuzzy locally compact.

Proposition 2.6. *In an Hausdorff fts X, the following conditions are equivalent:*

(a) *X is fuzzy locally compact.*

(b) *For each fuzzy point x_t in X, there exists a fuzzy open set U in X such that $x_t \in U$ and $\bar{U}$ is fuzzy compact.*

Proof. (a) $\Rightarrow$ (b): By hypothesis for each fuzzy point x_t in X there exists a fuzzy open set U which is fuzzy compact. Since X is fuzzy Hausdorff (a fuzzy compact subspace of a fuzzy Hausdorff space is fuzzy closed [6]) U is fuzzy closed: thus $U = \bar{U}$. Hence $x_t \in U$ and $\bar{U}$ is fuzzy compact.

(b) $\Rightarrow$ (a): Obvious. □

Proposition 2.7. *Let X be a Hausdorff fts. Then X is fuzzy locally compact at a fuzzy point x_t in X if and only if for every f-open set U containing x_t, there exists an f-open set V such that $x_t \in V$, $\bar{V}$ is fuzzy compact and $\bar{V} \leqslant U$.*

Proof. First suppose that X is fuzzy locally compact at an f-point x_t. By Definition 2.4 there exists a fuzzy open set U such that $x_t \in U$ and U is fuzzy compact. Since X is fuzzy Hausdorff, by Theorem 4.13 of [6], U is fuzzy closed; thus $U = \bar{U}$. Consider a fuzzy point $y_r \in (1_X - U)$. Since X is fuzzy Hausdorff, by Definition 2.3 there exist open sets C and D such that $x_t \in C$ and $y_r \in D$ and $C \wedge D = 0_X$. Let $V = C \wedge U$. Hence $V \leqslant U$ implies $\bar{V} \leqslant \bar{U} = U$. Since $\bar{V}$ is f-closed and U is fuzzy compact, by Theorem 4.2 of [6], it follows that $\bar{V}$ is fuzzy compact. Thus $x_t \in \bar{V} \leqslant U$ and $\bar{U}$ is fuzzy compact. The converse follows from Proposition 2.6(b). □

Definition 2.8 (Katsara and Lin [7]). The product of two fuzzy sets A and B in an fts (X, T) is defined as $(A \times B)(x, y) = \min(A(x), B(y))$ for all $(x, y) \in X \times Y$.

Definition 2.9 (Azad [1]). If $f_i: X_i \to Y_i$, $i = 1,2$, then $f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for each (x_1, x_2) $X_1 \times X_2$.

Theorem 2.10 (Azad [1]). *For any two fuzzy sets A and B in I^X, $(A \times B)' = (A' \times 1_X) \vee (1_X \times B')$. Similarly, if $f_i: X_i \to Y_i$, $i = 1, 2$, then $(f_1 \times f_2)^{-1}(B_1 \times B_2) = f_1^{-1}(B_1) \times f_2^{-1}(B_2)$ for all $B_1 \times B_2 \subset X_2 \times Y_2$.*

Definition 2.11. Let X and Y be two fts's. The function $T: X \times Y \to Y \times X$ defined by $T(x, y) = (y, x)$ for each $(x, y) \in X \times Y$ is called a switching map.

The proof of the following lemma is easy.

Lemma 2.12. *The switching map $T: X \times Y \to Y \times X$ defined as above is fuzzy continuous.*

3. Fuzzy compact-open topology

We now introduce the concept of fuzzy compact-open topology in the set of all fuzzy continuous functions from an fts X to an fts Y.

Definition 3.1. Let X and Y be two fts's and let

$Y^X = \{f: X \to Y \mid f \text{ is fuzzy continuous}\}$.

We give this class Y^X a topology called the fuzzy compact-open topology as follows: Let

$$\mathscr{K} = \{K \in I^X: K \text{ is fuzzy compact in } X\}$$

and

$$\mathscr{V} = \{V \in I^Y: V \text{ is fuzzy open in } Y\}.$$

For any $K \in \mathscr{K}$ and $V \in \mathscr{V}$, let

$$N_{K,V} = \{f \in Y^X: f(K) \leqslant V\}.$$

The collection $\{N_{K,V}: K \in \mathscr{K},\ V \in \mathscr{V}\}$ can be used as a fuzzy subbase [4] to generate a fuzzy topology on the class Y^X, called the fuzzy compact-open topology. The class Y^X with this topology is called a fuzzy compact-open topological space. Unless otherwise stated, Y^X will always have the fuzzy compact-open topology.

4. Evaluation map

We now consider the fuzzy product topological space $Y^X \times X$ and define a fuzzy continuous map from $Y^X \times X$ into Y.

Definition 4.1. The mapping $e: Y^X \times X \to Y$ defined by $e(f, x_t) = f(x_t)$ for each f-point $x_t \in X$ and $f \in Y^X$ is called the fuzzy evaluation map.

Theorem 4.2. *Let X be a fuzzy locally compact Hausdorff space. Then the fuzzy evaluation map $e: Y^X \times X \to Y$ is fuzzy continuous.*

Proof. Consider $(f, x_t) \in Y^X \times X$ where $f \in Y^X$ and $x_t \in X$. Let V be a fuzzy open set containing $f(x_t) = e(f, x_t)$ in Y. Since X is fuzzy locally compact and f is fuzzy continuous, by Proposition 2.7, there exists a fuzzy open set U in X such that $x_t \in U$, $\bar{U}$ is fuzzy compact and $f(\bar{U}) \leqslant V$.

Consider the fuzzy open set $N_{\bar{U},V} \times U$ in $Y^X \times X$. Clearly $(f, x_t) \in N_{\bar{U},V} \times U$. Let $(g, x_r) \in N_{\bar{U},V} \times U$ be arbitrary; thus $g(\bar{U}) \leqslant V$. Since $x_r \in U$, we have $g(x_r) \in V$ and hence $e(g, x_r) = g(x_r) \in V$. Thus $e(N_{\bar{U},V} \times U) \leqslant V$, showing e to be fuzzy continuous. □

We now consider the induced map of a given function $f: Z \times X \to Y$.

Definition 4.3. Let X, Y, Z be fts's and $f: Z \times X \to Y$ be any function. Then the induced map $\hat{f}: X \to Y^Z$ is defined by $(\hat{f}(x_t))(z_r) = f(z_r, x_t)$ for f-points $x_t \in X$ and $z_r \in Z$. Conversely, given a function $\hat{f}: X \to Y^Z$, a corresponding function f can also be defined by the same rule.

The continuity of $\hat{f}$ can be characterized in terms of the continuity of f and vice versa. We need the following result for this purpose.

Proposition 4.4. *Let X and Y be two fts's with Y fuzzy compact. Let x_t be any fuzzy point in X and N be a fuzzy open set in the fuzzy product space $X \times Y$ containing $\{x_t\} \times Y$. Then there exists some fuzzy neighborhood W of x_t in X such that $\{x_t\} \times Y \leqslant W \times Y \leqslant N$.*

Proof. It is clear that $\{x_t\} \times Y$ is fuzzy homeomorphic [2] to Y and hence $\{x_t\} \times Y$ is fuzzy compact [9]. We cover $\{x_t\} \times Y$ by the basis elements $\{U \times V\}$ (for the fuzzy topology of $X \times Y$) lying in N. Since $\{x_t\} \times Y$ is fuzzy compact, $\{U \times V\}$ has a finite subcover, say, a finite number of basis elements $U_1 \times V_1, \ldots, U_n \times V_n$. Without loss of generality we assume that $x_t \in U_i$ for each $i = 1, 2, \ldots, n$; since otherwise the basis elements would be superfluous. Let

$$W = \bigwedge_{i=1}^{n} U_i.$$

Clearly W is fuzzy open and $x_t \in W$. We show that

$$W \times Y \leqslant \bigvee_{i=1}^{n} (U_i \times V_i).$$

Let (x_r, y_s) be any fuzzy point in $W \times Y$. We consider the fuzzy point (x_t, y_s). Now $(x_t, y_s) \in U_i \times V_i$ for some i; thus $y_s \in V_i$. But $x_r \in U_j$ for every $j = 1, 2, \ldots, n$ (because $x_r \in W$). Therefore $(x_r, y_s) \in U_i \times V_i$, as desired. But $U_i \times V_i \leqslant N$ for all $i = 1, 2, \ldots, n$; and

$$W \times Y \leqslant \bigvee_{i=1}^{n} (U_i \times V_i).$$

Therefore $W \times Y \leqslant N$. □

Theorem 4.5. *Let Z be a fuzzy locally compact Hausdorff space and X, Y be arbitrary fuzzy topological spaces. Then a map $f: Z \times X \to Y$ is fuzzy continuous if and only if $\hat{f}: X \to Y^Z$ is fuzzy continuous, where $\hat{f}$ is defined by the rule*

$$(\hat{f}(x_t))(z_s) = f(z_s, x_t).$$

Proof. Suppose that $\hat{f}$ is fuzzy continuous. Consider the functions

$$Z \times X \xrightarrow{i_Z \times \hat{f}} Z \times Y^Z \xrightarrow{t} Y^Z \times Z \xrightarrow{e} Y,$$

where i_Z denotes the fuzzy identity function on Z, t denotes the switching map and e denotes the evaluation map. Since $et(i_Z \times \hat{f})(z_s, x_t) = et(z_s, \hat{f}(x_t)) = e(\hat{f}(x_t), z_s) = \hat{f}(x_t)(z_s) = f(z_s, x_t)$, it follows that $f = et(i_Z \times \hat{f})$; and f being the composition of fuzzy continuous functions is itself fuzzy continuous.

Conversely suppose that f is fuzzy continuous. Let x_k be any arbitrary fuzzy point in X. We have $\hat{f}(x_k) \in Y^Z$. Consider a subbasis element $N_{K,U} = \{g \in Y^Z$: $g(K) \leqslant U$, $K \in I^Z$ is fuzzy compact and $U \in I^Y$ is fuzzy open$\}$ containing $\hat{f}(x_k)$. We need to find a fuzzy neighborhood [8] W of x_k such that $\hat{f}(W) \leqslant N_{K,U}$; this will suffice to prove $\hat{f}$ to be a fuzzy continuous map.

For any fuzzy point z_u in K, we have $(\hat{f}(x_k))(z_u) = f(z_u, x_k) \in U$; thus $f(K \times \{x_k\}) \leqslant U$, i.e., $K \times \{x_k\} \leqslant f^{-1}(U)$. Since f is fuzzy continuous, $f^{-1}(U)$ is a fuzzy open set in $Z \times X$. Thus $f^{-1}(U)$ is a fuzzy open set in $Z \times X$ containing $K \times \{x_k\}$. Hence by Proposition 4.4, there exists a fuzzy neighborhood W of x_k in X such that $K \times \{x_k\} \leqslant K \times W \leqslant f^{-1}(U)$. Therefore $f(K \times W) \leqslant U$. Now for any $x_r \in W$ and $z_v \in K$, $f(z_v, x_r) = (\hat{f}(x_r))(z_v) \in U$. Therefore $(\hat{f}(x_r))(K) \leqslant U$ for all $x_r \in W$, i.e., $\hat{f}(x_r) \in N_{K,U}$ for all $x_r \in W$. Hence $\hat{f}(W) \leqslant N_{K,U}$ as desired. □

5. Exponential map

We define exponential law by using induced maps (cf. Definition 4.3) and study some of its properties.

Theorem 5.1 (The exponential law). *Let X and Z be fuzzy locally compact Hausdorff spaces. Then for any fuzzy topological space Y, the function*

$$E: Y^{Z \times X} \to (Y^Z)^X$$

defined by $E(f) = \hat{f}$ (i.e., $E(f)(x_t)(z_u) = f(z_u, x_t) = (\hat{f}(x_t))(z_u)$) for all $f: Z \times X \to Y$ is a fuzzy homeomorphism.

Proof. (a) Clearly E is onto.

(b) For E to be injective, let $E(f) = E(g)$ for $f, g: Z \times X \to Y$. Thus $\hat{f} = \hat{g}$, where $\hat{f}$ and $\hat{g}$ are the induced maps of f and g, respectively. Now for any fuzzy point x_t in X and any fuzzy point z_u in Z, $f(z_u, x_t) = (\hat{f}(x_t)(z_u)) = (\hat{g}(x_t)(z_u)) = g(z_u, x_t)$; thus $f = g$.

(c) For proving the fuzzy continuity of E, consider any fuzzy subbasis neighborhood V of $\hat{f}$ in $(Y^Z)^X$, i.e., V is of the form $N_{K,W}$ where K is a fuzzy compact subset of X and W is fuzzy open in Y^Z. Without loss of generality we may assume that $W = N_{L,U}$ where L is a fuzzy compact subset of Z and $U \in I^Y$ is fuzzy open. Then $\hat{f}(K) \leqslant N_{L,U} = W$ and this implies that $(\hat{f}(K))(L) \leqslant U$. Thus for any fuzzy point x_t in K and for all fuzzy point z_u in L, we have $(\hat{f}(x_t))(z_u) \in U$, i.e., $f(z_u, x_t) \in U$ and therefore $f(L \times K) \leqslant U$. Now since L is fuzzy compact in Z and K is fuzzy compact in X, $L \times K$ is also fuzzy compact in $Z \times X$ (cf. [9]) and since U is a fuzzy open set in Y we conclude that $f \in N_{L \times K, U} \leqslant Y^{Z \times X}$. We assert that $E(N_{L \times K, U}) \leqslant N_{K,W}$. Let $g \in N_{L \times K, U}$ be arbitrary. Thus $g(L \times K) \leqslant U$, i.e., $g(z_u, x_t) = (\hat{g}(x_t))(z_u) \in U$ for all fuzzy point $z_u \in L \leqslant Z$ and for all fuzzy points $x_t \in K \leqslant X$. So $(\hat{g}(x_t))(L) \leqslant U$ for all fuzzy points $x_t \in K \leqslant X$, i.e., $(\hat{g}(x_t)) \in N_{L,U} = W$ for all fuzzy points $x_t \in K \leqslant X$, i.e., $(\hat{g}(x_t)) \in N_{L,U} = W$ for all fuzzy points $x_t \in K \leqslant X$. Hence we have $\hat{g}(K) \leqslant W$, i.e., $\hat{g} = E(g) \in N_{K,W}$ for any $g \in N_{L \times K, U}$. Thus $E(N_{L \times K, U}) \leqslant N_{K,W}$. This proves that E is fuzzy continuous.

(d) For proving the fuzzy continuity of E^{-1} we consider the following evaluation maps: $e_1: (Y^Z)^X \times X \to Y^Z$ defined by $e_1(\hat{f}, x_t) = \hat{f}(x_t)$ where $\hat{f} \in (Y^Z)^X$ and x_t is any fuzzy point in X and $e_2: Y^Z \times Z \to Y$ defined by $e_2(g, z_u) = g(z_u)$ where $g \in Y^Z$ and z_u is a fuzzy point in Z. Let ψ denote the composition of the following fuzzy continuous functions:

$$(Z\times X)\times(Y^Z)^X \xrightarrow{T} (Y^Z)^X\times(Z\times X)$$

$$\xrightarrow{i\times t} (Y^Z)^X\times(X\times Z)$$

$$\Rightarrow ((Y^Z)^X\times X)\times Z \xrightarrow{e_1\times i_Z} Y^Z\times Z \xrightarrow{e_2} Y,$$

where i, i_Z denote the fuzzy identity maps on $(Y^Z)^X$ and Z respectively and T, t denote the switching maps. Thus $\psi:(Z\times X)\times(Y^Z)^X\to Y$, i.e., $\psi\in Y^{(Z\times X)\times(Y^Z)^X}$. We consider the map

$$\tilde{E}: Y^{(Z\times X)\times(Y^Z)^X} \to (Y^{Z\times X})^{(Y^Z)^X}$$

(as defined in the statement of the theorem, in fact it is E). So

$$\tilde{E}(\psi)\in(Y^{Z\times X})^{(Y^Z)^X},$$

i.e., we have a fuzzy continuous map

$$\tilde{E}(\psi):(Y^Z)^X \to Y^{Z\times X}.$$

Now for any fuzzy points $z_u\in Z$, $x_t\in X$ and $f\in Y^{Z\times X}$, it is a routine matter to check that $(\tilde{E}(\psi)\circ E)(f)(z_u, x_t)=f(z_u, x_t)$; hence $\tilde{E}(\psi)\circ E=$ identity. Similarly for any $\hat{g}\in(Y^Z)^X$ and fuzzy points $x_t\in X$, $z_u\in Z$, it is also a routine matter to check that $(E\circ\tilde{E}(\psi))(\hat{g})(x_t)(z_u)=\hat{g}(x_t)(z_u)$; hence $E\circ\tilde{E}(\psi)=$ identity. This completes the proof that E is a fuzzy homeomorphism. □

Definition 5.2. The map E in Theorem 5.1 is called the exponential map.

An easy consequence of Theorem 5.1 is as follows.

Corollary 5.3. *Let X, Y, Z be fuzzy locally compact Hausdorff spaces. Then the map*

$$S: Y^X\times Z^Y \to Z^X$$

defined by $S(f, g)=g\circ f$ is fuzzy continuous.

Proof. Consider the following compositions:

$$X\times Y^X\times Z^Y \xrightarrow{T} Y^X\times Z^Y\times X \xrightarrow{t\times i_X} Z^Y\times Y^X\times X$$

$$\Rightarrow Z^Y\times(Y^X\times X) \xrightarrow{i\times e_1} Z^Y\times Y \xrightarrow{e_2} Z,$$

where T, t denote the switching maps, i_X and i denote the fuzzy identity functions on X and Z^Y, respectively, and e_1 and e_2 denote the evaluation maps. Let $\varphi=e_2\circ(i\times e_1)\circ(t\times i_X)\circ T$. By Theorem 5.1, we have an exponential map

$$E: Z^{X\times Y^X\times Z^Y} \to (Z^X)^{Y^X\times Z^Y}.$$

Since $\varphi\in Z^{X\times Y^X\times Z^Y}$, $E(\varphi)\in(Z^X)^{Y^X\times Z^Y}$; let $S=E(\varphi)$, i.e., $S: Y^X\times Z^Y\to Z^X$ is fuzzy continuous. For $f\in Y^X$, $g\in Z^Y$ and for any fuzzy point x_t in X, it is easy to see that $S(f, g)(x_t)=g(f(x_t))$. □

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