Bayes Estimator

Given θ, $X_1, \cdots, X_n$, iid $\sim f(x|\theta)$.
prior distribution $g(\theta)$.
joint distribution: $f(x_1, \cdots, x_n|\theta)g(\theta)$
posterior $g(\theta|x_1, \cdots, x_n)$.
Estimator $T = h(X_1, \cdots, X_n)$.
Loss: $L(t, \theta)$
e.g., $L(t, \theta) = (t - \theta)^2, L(t, \theta) = |t - \theta|$.
Risk: $R(T, \theta) = E_\theta L(T, \theta)$

Bayes risk of T is
$R(T, g) = \int_\Theta R(T, \theta) g(\theta) d\theta$.
T is Bayes estimator if
$R(T, g) = \inf_T R(T, g)$.

To minimize Bayes risk, we only need to minimize the conditional expected loss for each x.

Binomial model

$X \sim binom(x|n, \theta)$
prior $g(\theta) \sim unif(0, 1)$.
posterior $g(\theta|x) \sim beta(x + 1, n - x + 1)$.
For squared error loss, the Bayes estimator is
posterior mean $\frac{x+1}{n+2}$.
If X is a random variable, choose c
$\min E(X - c)^2$.
$c = E(X)$

Prior distributions

Where do prior distributions come from?
* a prior knowledge about θ
* population interpretation–(a population of possible θ values).
* mathematical convenience (conjugate prior)
conjugate distribution– the prior and the posterior distribution are in the same parametric family.

Conjugate prior distribution

Advantages:

* mathematically convenient
* easy to interpret
* can provide good approximation to many prior opinions

(especially if we allow mixtures of distributions from the conjugate family)

Disadvantages:

may not be realistic

Binomial model

$X|\theta \sim bin(n, \theta)$
$\theta \sim beta(\alpha, \beta)$.
$g(\theta|x) = beta(x + \alpha, n - x + \beta)$.
posterior mean is $\hat{\theta} = \frac{x+\alpha}{n+\alpha+\beta}$.