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The Galois Theory for Linear Homogeneous Partial Differential Equations of the First Order

By

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Dedicated to Professor Tosihusa Kimura on his 60th birthday

Contents

Introduction

Stimulated by the celebrated Galois theory for algebraic extensions of fields, Lie, Picard and many mathematicians devoted themselves to establishing the Galois theory for differential equations. For linear ordinary differential equations, the theory is completed and is called the Picard-Vessiot theory (Kolchin [5]). This theory is a Galois theory for finitely generated extensions of ordinary differential fields. Kolchin [6] further defined the strongly normal extension, which is also a finitely generated extension and contains the concept of Picard-Vessiot extension, and he applied it to the study of some class of non-linear differential equations. (Recently Umemura [16] and Nishioka [10] have proved the irreducibility of the first transcendental function of Painlevé by using this concept.)

On the other hand Drach [2, 3] initiated a Galois theory for general non-linear ordinary differential equations by considering the first integrals of the equation. Therefore his theory is a Galois theory for a linear homogeneous partial differential equation of the first order, because the first integrals satisfy such an equation. Several definitions appearing in the algebraic Galois theory can be extended to this case in a natural way. However, since a partial differential equation has a solution space of an infinite dimension, some difficulties arise from the infiniteness, which leaves the Drach's theory incomplete. Introducing the concept of automorphic extensions, Vessiot [18] made an effort to complete the Drach's work, but had not succeeded.

The purpose of this paper is to complete the Drach-Vessiot theory by considering the objects in the local analytic category. We define a Galois extension to be some infinitely generated extension and establish a one-way Galois correspondence. In the following we illustrate the Drach-Vessiot theory, point out the defects of the theory and explain the way of our justification.

For an ordinary differential equation of order n

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}), \tag{0.1}$$

the first integrals of (0.1) satisfy the following linear homogeneous partial differential equation of the first order

$$\frac{\partial z}{\partial x} + y'\frac{\partial z}{\partial y} + \cdots + F\frac{\partial z}{\partial y^{(n-1)}} = 0. \tag{0.2}$$

Drach and Vessiot considered in general partial differential equations of the form

$$\frac{\partial z}{\partial x_0} + a_1(x)\frac{\partial z}{\partial x_1} + \cdots + a_n(x)\frac{\partial z}{\partial x_n} = 0, \tag{0.3}$$

where $x = (x_0, x_1, \dots, x_n)$. By a fundamental system of solutions of (0.3) they meant an n-vector $(\zeta_1(x), \dots, \zeta_n(x))$ of solutions of (0.3) with $\dfrac{\partial(\zeta_1, \dots, \zeta_n)}{\partial(x_1, \dots, x_n)} \neq 0$; then any solution of (0.3) becomes a function of $\zeta_1, \dots, \zeta_n$. Thus another fundamental system of solutions of (0.3) is written as $(F_1(\zeta_1, \dots, \zeta_n), \dots, F_n(\zeta_1, \dots, \zeta_n))$, where $\dfrac{\partial(F_1, \dots, F_n)}{\partial(z_1, \dots, z_n)} \neq 0$. Such transformations $(F_1, \dots, F_n)$ form the pseudo-group Γ_n of biholomorphic transformations in $\boldsymbol{C}^n$. The base field K was taken to be a partial differential field with derivations $\dfrac{\partial}{\partial x_0}, \dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}$ which contains all the coefficients $a_1, \dots, a_n$ of (0.3). It seems that Drach and Vessiot defined the Galois extension for (0.3) over K to be the differential field $K\langle\zeta_1, \dots, \zeta_n\rangle$, which is the smallest differential extension field of K containing $\zeta_1, \dots, \zeta_n$. And they defined the Galois group for (0.3) over K to be the set of the transformations of fundamental systems of solutions that leave all the differential algebraic relations over K between $\zeta_1, \dots, \zeta_n$ invariant. Hence the Galois group is a subset of Γ_n.

This is the fundamental definitions in Drach-Vessiot theory, and they afterwards examined the Galois correspondence. However there exist two serious defects in their definitions. The first one is the ambiguity for the domains of definition of the coefficients and solutions. Even in the analytic case the domain of definition of a solution of a partial differential equation may become narrower than that of coefficients of the equation, so that strict conventions are needed for the domains of definition. This ambiguity causes not only the vagueness of their definitions but also an essential difficulty in the development of their theory. The second defect is inconsistency of the Galois group with the group of the differential automorphisms of the Galois extension field over the base field. In fact, when a transformation of a fundamental system of solutions leaves all the relations between $\zeta_1, \dots, \zeta_n$ invariant, it may happen that the image of each $\zeta_1, \dots, \zeta_n$ by the transformation, which in general is an (analytic) function of $\zeta_1, \dots, \zeta_n$, does not belong to $K\langle\zeta_1, \dots, \zeta_n\rangle$. Conversely if one wants to define the Galois group to be the group of the differential automorphisms of $K\langle\zeta_1, \dots, \zeta_n\rangle$ over K, one should restrict the transformations under consideration to, for example, the birational transformations in $\boldsymbol{C}^n$. Owing to this inconsistency, for two fundamental systems of solutions, the corresponding Galois groups may not be isomorphic. Therefore the normality of the Galois extension and hence the Galois correspondence are not established in the Drach-Vessiot theory.

We will resolve the above difficulties by introducing some universal field which plays a role similar to an algebraically closed field in the theory for algebraic extensions. In consideration of solving partial differential equations, we take the field of germs of all meromorphic functions as the universal field; namely we consider everything in the local analytic category. Then the Cauchy-Kowalevsky theorem is at our disposal, and substitution into analytic functions will be justified. The Galois extension for (0.3) is defined by adjunction of all solutions in the universal field, which in general becomes an infinitely generated extension. As this definition is independent of the choice of fundamental systems of solutions, the Galois group is canonically defined as the group of the automorphisms of the Galois extension field over the base field. Thus we have resolved the difficulties in the Drach-Vessiot theory, and moreover a kind of normality holds for Galois extensions.

In §1 of this paper we consider an equation (0.3) in the local analytic category and define fundamental systems of solutions. The set of all fundamental systems of solutions is regarded as a principal homogeneous space of the pseudo-group Γ_n. In §2, the Galois extension and the Galois group for (0.3) are defined as above. Since we treat infinitely generated extensions, we need to introduce several concepts in differential algebra. The first theorem asserts that, for any fundamental system of solutions, there exists a representation of the Galois group into Γ_n with respect to the fundamental system. And we have a similar representation with respect to another fundamental system of solutions, which is stated in Theorem 2. §3 is devoted to the study of the structure of Galois groups. In comparison with the Picard-Vessiot theory in which Picard-Vessiot groups are algebraic over the field of constants, the Galois groups in our theory are found to be defined by some differential equations (Theorem 5). The main theorem, the normality of Galois extensions, is proved in §4. In general there are two definitions for a normal extension; we adopt one of them (Definition 6) and prove this normality of Galois extensions (Theorem 6). The other normality does not hold in general, because Galois extensions are infinitely generated (see Example at the end of §4). Thus we establish a one-way Galois correspondence in Theorem 8. In §5 we return to an ordinary differential equation (0.1) and define the Galois group for (0.1) to be that for the associated equation (0.2). We calculate several Galois groups for reducible ordinary differential equations; namely for linear ordinary equation, linearizable equations and equations reducible to equations of lower order.

The present theory is applicable to the examination of irreducibility of non-linear ordinary differential equations; if we define a reducible equation to be a linearizable or a reducible to lower order ones equation, we see, by Theorem 8, that a given equation is irreducible whenever the Galois group for it cannot be contained in the groups of the form given in §5. For the first equation of Painlevé [11]

$$y'' = 6y^2 + x\,, \tag{0.4}$$

Drach [4] claimed, by using the results of the theory for infinite dimensional Lie groups (Lie [8], Vessiot [17], Cartan [1]), that the Galois group coincides with

$$G = \left\{(g_1(z_1, z_2), g(z_1, z_2)) \in \Gamma_2; \frac{\partial(g_1, g_2)}{\partial(z_1, z_2)} = 1\right\}.$$

Our theory asserts that, if the Galois group for (0.4) is given by

$$G_p = G \cap \Gamma_{2,p}\,,$$

then the equation (0.4) is irreducible. The example above shows that it is necessary to study the theory for infinite dimensional Lie groups, especially the classification of Lie subgroups, to determine Galois groups (see Kuranishi [7]).

We have completed the Drach-Vessiot theory by considering objects in the local analytic category. However the exact Galois correspondence cannot be established. It is caused by the inconsistency of algebraic concepts and analytic concepts. For the Drach-Vessiot theory other completions may be possible, and the exact Galois correspondence may be established by combining plural completions.

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§1. Fundamental systems of solutions

For a fixed integer n, let $x = (x_0, x_1, \dots, x_n)$ denote the coordinates of $\boldsymbol{C}^{n+1}$. Take a point $x^0 = (x_0^0, x_1^0, \dots, x_n^0) \in \boldsymbol{C}^{n+1}$ and fix it. The field ${}_{n+1}\mathscr{M}_{x^0}$ of all germs of meromorphic functions in x at x^0 can be considered as a partial differential field under the usual commutative derivations $\dfrac{\partial}{\partial x_0}, \dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}$; we denote this differential field by Ω and call it the *universal differential field.* The set ${}_{n+1}\mathcal{O}_{x^0}$ of all elements in Ω that are holomorphic at x^0 makes a differential subring of Ω and is denoted by Ω_0. The field of constants of Ω is the set $\boldsymbol{C}$ of all complex numbers. For any vector $(\xi_1, \dots, \xi_m) \in (\Omega_0)^m$, we put

$$p(\xi_1, \dots, \xi_m) = (\xi_1(x^0), \dots, \xi_m(x^0)) \in \boldsymbol{C}^m$$

and call it the *center* of $(\xi_1, \dots, \xi_m)$.

Take a differential subfield K of Ω whose field of constants is $\boldsymbol{C}$ and fix it. Throughout this paper we shall consider a linear homogeneous partial differential equation of the first order

$$X(z) = \frac{\partial z}{\partial x_0} + a_1(x)\frac{\partial z}{\partial x_1} + \cdots + a_n(x)\frac{\partial z}{\partial x_n} = 0 \tag{1}$$

with $a_i(x) \in K \cap \Omega_0$, $i = 1, \dots, n$. By Cauchy-Kowalevsky theorem we have

Proposition 1. *For any $\varphi(x_1, \dots, x_n) \in {}_n\mathcal{O}_{(x_1^0, \dots, x_n^0)}$ there exists one and only one solution of* (1) *that belongs to Ω_0 and satisfies the initial condition*

$$z(x_0^0, x_1, \dots, x_n) = \varphi(x_1, \dots, x_n)\,.$$

A solution of (1) which belongs to Ω_0 is called an Ω_0-solution of (1). For any Ω_0-solution $\zeta(x_0, x_1, \dots, x_n)$ it is clear that $\zeta(x_0^0, x_1, \dots, x_n)$ belongs to ${}_n\mathcal{O}_{(x_1^0, \dots, x_n^0)}$. Hence we can identify the set of all Ω_0-solutions with the set ${}_n\mathcal{O}_{(x_1^0, \dots, x_n^0)}$.

Let L be a differential extension field of K which is not necessarily a differential subfield of Ω. For $\xi_1, \dots, \xi_m \in L$, we say that $\xi_1, \dots, \xi_m$ are *independent* if

$$\operatorname{rank}\left(\frac{\partial \xi_i}{\partial x_j}\right)_{\substack{i=1,\dots,m\\ j=0,1,\dots,n}} = \min\{m, n+1\}\,.$$

When $n+1$ elements $\xi_0, \xi_1, \dots, \xi_n$ of L are solutions of (1), we obtain

$$\det\left(\frac{\partial \xi_i}{\partial x_j}\right)_{i,j=0,\dots,n} = \frac{\partial(\xi_0, \dots, \xi_n)}{\partial(x_0, \dots, x_n)} = 0$$

from $X(\xi_0) = \cdots = X(\xi_n) = 0$. Hence no differential extension field of K contains more than n independent solutions of (1). This fact lead us to the following definition.

Definition 1. An n-vector $(\zeta_1, \dots, \zeta_n)$ with elements in Ω_0 is called a *fundamental system of solutions* of (1), if each ζ_i $(i = 1, \dots, n)$ is a solution of (1) and $\zeta_1, \dots, \zeta_n$ are independent with respect to $(x_1, \dots, x_n)$ at x^0; namely

$$X(\zeta_1) = \cdots = X(\zeta_n) = 0\,,$$

$$\frac{\partial(\zeta_1, \dots, \zeta_n)}{\partial(x_1, \dots, x_n)}(x^0) \neq 0\,.$$

By using Proposition 1, we can show the existence of fundamental systems of solutions. Put

$$\Phi = \left\{(\varphi_1, \dots, \varphi_n) \in ({}_n\mathcal{O}_{(x_1^0, \dots, x_n^0)})^n;\ \frac{\partial(\varphi_1, \dots, \varphi_n)}{\partial(x_1, \dots, x_n)}(x_1^0, \dots, x_n^0) \neq 0\right\}.$$

Proposition 2. *For any* $(\varphi_1, \dots, \varphi_n) \in \Phi$ *there exists one and only one fundamental system of solutions* $(\zeta_1, \dots, \zeta_n)$ *of* (1) *such that*

$$\zeta_i(x_0^0, x_1, \dots, x_n) = \varphi_i(x_1, \dots, x_n) \tag{2}$$

for $i = 1, \dots, n$.

Proof. By Proposition 1 there is one and only one Ω_0-solution $\zeta_i(x_0, x_1, \dots, x_n)$ of (1) that satisfies (2), for $i = 1, \dots, n$. Then $(\zeta_1, \dots, \zeta_n)$ becomes a fundamental system of solutions of (1) since

$$\frac{\partial(\zeta_1, \dots, \zeta_n)}{\partial(x_1, \dots, x_n)}(x^0) = \frac{\partial(\varphi_1, \dots, \varphi_n)}{\partial(x_1, \dots, x_n)}(x_1^0, \dots, x_n^0) \neq 0\,.$$

We denote by Σ the set of all fundamental systems of solutions of (1). Take any $\zeta = (\zeta_1, \dots, \zeta_n)$ in Σ and fix it. Put

$${}_n\mathcal{O}_{p(\zeta)}(\zeta) = \{f(\zeta_1, \dots, \zeta_n);\ f \in {}_n\mathcal{O}_{p(\zeta)}\}\,,$$

then clearly we have ${}_n\mathcal{O}_{p(\zeta)}(\zeta) \subset \Omega_0$.

Proposition 3. ${}_n\mathcal{O}_{p(\zeta)}(\zeta)$ *is the set of all* Ω_0-*solutions of* (1) *for any* $\zeta \in \Sigma$. *And hence, for any* ζ *and* $\eta \in \Sigma$, *we have*

$${}_n\mathcal{O}_{p(\zeta)}(\zeta) = {}_n\mathcal{O}_{p(\eta)}(\eta)\,.$$

Proof. For any $f(\zeta_1, \dots, \zeta_n) \in {}_n\mathcal{O}_{p(\zeta)}(\zeta)$, $X(f(\zeta_1, \dots, \zeta_n)) = 0$ since $X(\zeta_i) = 0$, $i = 1, \dots, n$. Thus ${}_n\mathcal{O}_{p(\zeta)}(\zeta)$ is contained in the set of all Ω_0-solutions of (1). Conversely take any solution $\xi \in \Omega_0$ of (1) and put $\psi(x_1, \dots, x_n) = \xi(x_0^0, x_1, \dots, x_n)$, then $\psi \in {}_n\mathcal{O}_{(x_1^0, \dots, x_n^0)}$. If we put $\varphi_i(x_1, \dots, x_n) = \zeta_i(x_0^0, x_1, \dots, x_n)$ for $i = 1, \dots, n$, $(\varphi_1, \dots, \varphi_n)$ belongs to Φ. Then, by the inverse mapping theorem, there is an $F \in {}_n\mathcal{O}_{p(\zeta)}$ such that $\psi = F(\varphi_1, \dots, \varphi_n)$. Since $F(\zeta_1, \dots, \zeta_n)$ is an Ω_0-solution of (1), the uniqueness of the solution (Proposition 1) proves $\xi = F(\zeta_1, \dots, \zeta_n)$, and hence $\xi \in {}_n\mathcal{O}_{p(\zeta)}(\zeta)$.

Now we shall study the structure of Σ. Proposition 2 shows that Σ can be identified with Φ, so that we first examine the structure of Φ. Consider the set

$$\Gamma_n = \bigcup_{p \in \mathbf{C}^n} \left\{ g = (g_1(z_1, \dots, z_n), \dots, g_n(z_1, \dots, z_n)) \in ({}_n\mathcal{O}_p)^n;\ \frac{\partial(g_1, \dots, g_n)}{\partial(z_1, \dots, z_n)}(p) \neq 0 \right\}.$$

When $g \in \Gamma_n \cap ({}_n\mathcal{O}_p)^n$, p is said to be the *source* of g and is denoted by $s(g)$, and $g(p) \in \mathbf{C}^n$ is said to be the *target* of g and is denoted by $t(g)$. For any $g, h \in \Gamma_n$ such that $t(g) = s(h)$, the composition $h \circ g$ can be defined and belongs to Γ_n, and, by the inverse mapping theorem, the inverse g^{-1} of any $g \in \Gamma_n$ exists and belongs to Γ_n. Thus Γ_n has the structure of pseudo-groups; we call Γ_n the *pseudo-group of germs of bioholomorphic transformations of* $\mathbf{C}^n$. Note that $s(g^{-1}) = t(g)$ and $t(g^{-1}) = s(g)$.

Γ_n acts on the set Φ as follows; take any $\varphi = (\varphi_1, \dots, \varphi_n) \in \Phi$ and any $g = (g_1, \dots, g_n) \in \Gamma_n$ such that $s(g) = p(\varphi)$, and put

$$\varphi^g = (g_1(\varphi_1, \dots, \varphi_n), \dots, g_n(\varphi_1, \dots, \varphi_n)), \tag{3}$$

then it is easy to see that $\varphi^g \in \Phi$. The mapping $(\varphi, g) \mapsto \varphi^g$ defines the action of Γ_n on Φ.

Lemma 1. *The action of Γ_n on Φ is transitive, and Φ is a principal homogeneous space of Γ_n;* i.e. *for any $\varphi, \psi \in \Phi$, there exists one and only one $g \in \Gamma_n$ such that $\psi = \varphi^g$.*

Proof. Φ can be regarded as a subset of Γ_n which consists of elements with the source $(x_1^0, \dots, x_n^0)$. Then, for any $\varphi, \psi \in \Phi$, we see that $\varphi^{-1} \in \Gamma_n$ and $t(\varphi^{-1}) = s(\psi) = (x_1^0, \dots, x_n^0)$, so that the composition $\psi \circ \varphi^{-1}$ belongs to Γ_n. It is clear, by the definition (3), that $g = \psi \circ \varphi^{-1}$ has the asserted property.

The action of Γ_n on Φ can be extended canonically to the action on Σ. For any $\zeta = (\zeta_1, \dots, \zeta_n) \in \Sigma$ and for any $g = (g_1, \dots, g_n) \in \Gamma_n$ such that $s(g) = p(\zeta)$, put

$$\zeta^g = (g_1(\zeta_1, \dots, \zeta_n), \dots, g_n(\zeta_1, \dots, \zeta_n)).$$

From the proof of Proposition 3, we see that $\zeta^g \in \Sigma$, then the mapping $(\zeta, g) \mapsto \zeta^g$ defines the action of Γ_n on Σ. By Lemma 1 and Proposition 2, we have

Proposition 4. *The action of Γ_n on Σ is transitive, and Σ is a principal homogeneous space of Γ_n;* i.e. *for any ζ, $\eta \in \Sigma$, there exists one and only one $g \in \Gamma_n$ such that $\eta = \zeta^g$.*

§2. Galois extensions

In this section we shall define a Galois extension for a differential equation of the form (1) over a differential subfield of Ω, and give a representation of the Galois group into Γ_n. For this purpose we introduce several new concepts in differential algebra in the first half of this section.

As in the preceding section let K be a differential subfield of Ω whose field of constants is $\boldsymbol{C}$. For differential indeterminates $z_1, \ldots, z_m$ with respect to the derivations $\dfrac{\partial}{\partial x_0}, \ldots, \dfrac{\partial}{\partial x_n}$, $K\{z_1, \ldots, z_m\}$ denotes the differential ring of all differential polynomials in $z_1, \ldots, z_m$ with coefficients in K. Take a point $p = (p_1, \ldots, p_m) \in \boldsymbol{C}^m$. By regarding $z_1, \ldots, z_m$ as ordinary indeterminates, we have the ring ${}_m\mathcal{O}_p(z_1, \ldots, z_m)$ of all convergent power series in $z_1 - p_1, \ldots, z_m - p_m$ with coefficients in $\boldsymbol{C}$. For any vector $(f_1, \ldots, f_k) \in ({}_m\mathcal{O}_p(z_1, \ldots, z_m))^k$, we put

$$p(f_1, \ldots, f_k) = (f_1(p), \ldots, f_k(p)) \in \boldsymbol{C}^k$$

and call it also the *center* of $(f_1, \ldots, f_k)$. Note that both $K\{z_1, \ldots, z_m\}$ and ${}_m\mathcal{O}_p(z_1, \ldots, z_m)$ are $\boldsymbol{C}$-algebras; then we define a $\boldsymbol{C}$-algebra R by

$$R = K\{z_1, \ldots, z_m\} \otimes_{\boldsymbol{C}} {}_m\mathcal{O}_p(z_1, \ldots, z_m).$$

We shall extend the derivations $\dfrac{\partial}{\partial x_0}, \dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}$ on $K\{z_1, \ldots, z_m\}$ to those on R; for $i = 0, 1, \ldots, n$, define an additive mapping $\partial_i \colon R \to R$ by

$$\partial_i(\alpha \otimes f) = \frac{\partial \alpha}{\partial x_i} \otimes f + \sum_{j=1}^m \alpha \frac{\partial z_j}{\partial x_i} \otimes \frac{\partial f}{\partial z_j},$$

where $\alpha \in K\{z_1, \ldots, z_m\}$ and $f \in {}_m\mathcal{O}_p(z_1, \ldots, z_m)$. Certainly ∂_i maps R into R since $\dfrac{\partial f}{\partial z_j} \in {}_m\mathcal{O}_p(z_1, \ldots, z_m)$ for any $f \in {}_m\mathcal{O}_p(z_1, \ldots, z_m)$. It is easy to see that ∂_i's are derivations on R, and hence R becomes a partial differential ring with derivations $\partial_0, \partial_1, \ldots, \partial_n$. The ring of constants of R is the field $\boldsymbol{C}$. By the natural injections $K\{z_1, \ldots, z_m\} \hookrightarrow R$ and ${}_m\mathcal{O}_p(z_1, \ldots, z_m) \hookrightarrow R$, we regard

$K\{z_1, \ldots, z_m\}$ and ${}_m\mathcal{O}_p(z_1, \ldots, z_m)$ as subrings of R. As ∂_i is an extension of $\dfrac{\partial}{\partial x_i}$ on $K\{z_1, \ldots, z_m\}$, we often use $\dfrac{\partial}{\partial x_i}$ instead of ∂_i for $i = 0, 1, \ldots, n$. We denote R by $K\{{}_m\mathcal{O}_p(z_1, \ldots, z_m)\}$.

Take any $(\eta_1, \ldots, \eta_m) \in (\Omega_0)^m$ with $p(\eta_1, \ldots, \eta_m) = p \in \boldsymbol{C}^m$. By replacing $(z_1, \ldots, z_m)$ with $(\eta_1, \ldots, \eta_m)$ we similarly define a differential ring $K\{{}_m\mathcal{O}_p(\eta_1, \ldots, \eta_m)\}$. In this case $K\{\eta_1, \ldots, \eta_m\} \subset \Omega$ and ${}_m\mathcal{O}_p(\eta_1, \ldots, \eta_m) \subset \Omega$ since, for any $f \in {}_m\mathcal{O}_p(z_1, \ldots, z_m)$, $f(\eta_1, \ldots, \eta_m) \in \Omega_0$. Then we regard $K\{{}_m\mathcal{O}_p(\eta_1, \ldots, \eta_m)\}$ as a differential subring of the universal differential field Ω. We denote by $K\langle\langle \eta_1, \ldots, \eta_m \rangle\rangle$ the quotient field of $K\{{}_m\mathcal{O}_p(\eta_1, \ldots, \eta_m)\}$ in Ω.

Remark 1. By the definition we have

$$K\langle\langle \eta_1, \ldots, \eta_m \rangle\rangle = K\langle {}_m\mathcal{O}_p(\eta_1, \ldots, \eta_m) \rangle ,$$

where, for a subset M of Ω, $K\langle M \rangle$ denotes the smallest differential subfield of Ω containing both K and M.

Let R_0 be a differential subring of $K\{{}_m\mathcal{O}_p(z_1, \ldots, z_m)\}$, and let R_Ω be $\Omega\{{}_m\mathcal{O}_p(z_1, \ldots, z_m)\}$ or Ω.

Definition 2. A ring homomorphism $\sigma\colon R_0 \to R_\Omega$ is called a *differential-analytic homomorphism over* K if

i) $\sigma | K\{z_1, \ldots, z_m\} \cap R_0$ is a differential homomorphism over $K \cap R_0$, and if

ii) $p(\sigma(z_1), \ldots, \sigma(z_m)) = p$ and, for any $f(z_1, \ldots, z_m) \in {}_m\mathcal{O}_p(z_1, \ldots, z_m) \cap R_0$,

$$\sigma(f(z_1, \ldots, z_m)) = f(\sigma(z_1), \ldots, \sigma(z_m)) .$$

Similarly we have the following definition; let R_0 be a differential subring of $K\{{}_m\mathcal{O}_p(\eta_1, \ldots, \eta_m)\}$ and let R_Ω be the same as above.

Definition 2′. A ring homomorphism $\sigma\colon R_0 \to R_\Omega$ is called a *differential-analytic homomorphism over* K *with respect to* $(\eta_1, \ldots, \eta_m)$ if

i)′ $\sigma | K\{\eta_1, \ldots, \eta_m\} \cap R_0$ is a differential homomorphism over $K \cap R_0$, and if

ii)′ $p(\sigma(\eta_1), \ldots, \sigma(\eta_m)) = p$ and, for any $f(\eta_1, \ldots, \eta_m) \in {}_m\mathcal{O}_p(\eta_1, \ldots, \eta_m) \cap R_0$,

$$\sigma(f(\eta_1, \ldots, \eta_m)) = f(\sigma(\eta_1), \ldots, \sigma(\eta_m)) .$$

It is clear that a differential-analytic homomorphism over K is a differential homomorphism over K. We say that a differential-analytic homomorphism σ over K is a *differential-analytic isomorphism over* K if it is a differential isomorphism over K. And a differential-analytic isomorphism $\sigma\colon R_0 \to R_\Omega$ is called a *differential-analytic automorphism of* R_0 if Image $\sigma = R_0$. A differential-analytic homomorphism (resp. isomorphism) $\sigma: R_0 \to R_\Omega$ over K is uniquely extended to the field of quotients $Q(R_0)$ of R_0 as a differential homomorphism over K, which we call again a differential-analytic homomorphism (resp. isomorphism) over K. A differential-analytic isomorphism σ of $Q(R_0)$ over K is called a *differential-analytic automorphism of* $Q(R_0)$ *over* K if Image $\sigma = Q(R_0)$. The following lemma is frequently used for determining differential-analytic homomorphisms.

Lemma 2.

(i) *For any* $(f_1, \ldots, f_m) \in ({}_m\mathcal{O}_p(z_1, \ldots, z_m))^m$ *with* $p(f_1, \ldots, f_m) = p$ *(resp.* $(\xi_1, \ldots, \xi_m) \in (\Omega_0)^m$ *with* $p(\xi_1, \ldots, \xi_m) = p$*), there exists one and only one differential-analytic homomorphism*

$$\sigma: K\{{}_m\mathcal{O}_p(z_1, \ldots, z_m)\} \to K\{{}_m\mathcal{O}_p(z_1, \ldots, z_m)\}$$

$$(\textit{resp. } \sigma: K\{{}_m\mathcal{O}_p(z_1, \ldots, z_m)\} \to \Omega)$$

over K *such that* $\sigma(z_i) = f_i$ *(resp.* $\sigma(z_i) = \xi_i$*) for* $i = 1, \ldots, m$.

(ii) *A differential-analytic homomorphism*

$$\sigma: K\{{}_m\mathcal{O}_p(\eta_1, \ldots, \eta_m)\} \to \Omega$$

$$(\textit{resp. } \sigma: K\langle\!\langle \eta_1, \ldots, \eta_m \rangle\!\rangle \to \Omega)$$

over K *with respect to* $(\eta_1, \ldots, \eta_m)$ *is uniquely determined by* $(\sigma(\eta_1), \ldots, \sigma(\eta_m)) \in (\Omega_0)^m$. *And we have*

$$\text{Image } \sigma = K\{{}_m\mathcal{O}_p(\sigma(\eta_1), \ldots, \sigma(\eta_m))\}$$

$$(\textit{resp. } \text{Image } \sigma = K\langle\!\langle \sigma(\eta_1), \ldots, \sigma(\eta_m) \rangle\!\rangle) .$$

Proof. (ii) is a direct consequence of the definitions. We prove (i). Note that, for any multi-index $j = (j_0, j_1, \ldots, j_n) \in N^{n+1}$ we have

$$\partial^j f_i = \partial_0^{j_0} \partial_1^{j_1} \ldots \partial_n^{j_n} f_i \in K\{{}_m\mathcal{O}_p(z_1, \ldots, z_m)\} ,$$

since $f_i \in {}_m\mathcal{O}_p(z_1, \ldots, z_m)$, $i = 1, \ldots, m$. Define a ring homomorphism

$$\sigma_1: K\{z_1, \ldots, z_m\} \to K\{{}_m\mathcal{O}_p(z_1, \ldots, z_m)\}$$

over K by

$$\sigma_1(z_i) = f_i , \qquad (i = 1, \ldots, m) ,$$

$$\sigma_1\left(\left(\frac{\partial}{\partial x}\right)^j z_i\right) = \partial^j f_i , \qquad (i = 1, \ldots, m, j \in N^{n+1}) .$$

As there is no relation between the differential indeterminates $z_1, \ldots, z_m$, σ_1 is well-defined and makes a differential homomorphism over K. Next, for any $F \in {}_m\mathcal{O}_p(z_1, \dots, z_m)$ we have

$$F(f_1, \dots, f_m) \in {}_m\mathcal{O}_p(z_1, \dots, z_m),$$

since $p(f_1, \dots, f_m) = p$. Then, if we define

$$\sigma_2: {}_m\mathcal{O}_p(z_1, \dots, z_m) \to K\{{}_m\mathcal{O}_p(z_1, \dots, z_m)\}$$

by $\sigma_2(F) = F(f_1, \dots, f_m)$, $(F \in {}_m\mathcal{O}_p(z_1, \dots, z_m))$, σ_2 is well-defined and makes a ring homomorphism over K since no relation exists between the (analytic) indeterminates $z_1, \dots, z_m$. Now σ_1 and σ_2 coincide on the intersection $K\{z_1, \dots, z_m\} \cap {}_m\mathcal{O}_p(z_1, \dots, z_m) = \boldsymbol{C}[z_1, \dots, z_m]$, and $z_1, \dots, z_m$ have no relation, so that σ_1 and σ_2 are uniquely extended to a common homomorphism

$$\sigma: K\{z_1, \dots, z_m\} \otimes_C {}_m\mathcal{O}_p(z_1, \dots, z_m) \to K\{{}_m\mathcal{O}_p(z_1, \dots, z_m)\}$$

over K. Then σ is clearly a differential-analytic homomorphism over K. The claim for $(\xi_1, \dots, \xi_m) \in (\Omega_0)^m$ can be shown similarly.

Take any $g = (g_1, \dots, g_n) \in \Gamma_n$. Put $s(g) = p$ and $t(g) = q$, then in particular $g_i \in {}_n\mathcal{O}_p(z_1, \dots, z_n)$ for $i = 1, \dots, n$. We define a differential homomorphism

(4) $$\sigma_g: K\{{}_n\mathcal{O}_q(z_1, \dots, z_n)\} \to K\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\}$$

over K by

$$\sigma_g(F) = F(g_1, \dots, g_n), \qquad (F \in {}_n\mathcal{O}_q(z_1, \dots, z_n)).$$

In a way similar to the proof of Lemma 2 (i), we see that σ_g is well-defined. For any $\zeta = (\zeta_1, \dots, \zeta_n) \in (\Omega_0)^n$ with $p(\zeta) = p$, a differential-analytic homomorphism

(5) $$v_\zeta: K\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\} \to \Omega$$

over K is defined by $v_\zeta(z_i) = \zeta_i$, (Lemma 2 (i)). Note that, for the above g and ζ, $\zeta^g = (g_1(\zeta), \dots, g_n(\zeta)) \in (\Omega_0)^n$ with $p(\zeta^g) = q$. Then we have the following commutative diagram;

(6) $$\begin{tikzcd} K\{{}_n\mathcal{O}_q(z_1, \dots, z_n)\} \arrow[rr, "\sigma_g"] \arrow[dr, "v_{\zeta^g}"'] & & K\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\} \arrow[dl, "v_\zeta"] \\ & \Omega & \end{tikzcd}$$

i.e. $v_{\zeta^g} = v_\zeta \circ \sigma_g$.

Now consider a differential equation

(1) $$X(z) = \frac{\partial z}{\partial x_0} + a_1(x)\frac{\partial z}{\partial x_1} + \cdots + a_n(x)\frac{\partial z}{\partial x_n} = 0$$

with $a_i(x) \in K \cap \Omega_0$, $i = 1, \dots, n$. Take any fundamental system of solutions $(\zeta_1, \dots, \zeta_n)$ of (1) and put $p(\zeta_1, \dots, \zeta_n) = p \in \boldsymbol{C}^n$. By Remark 1, the differential subfield $K\langle\langle\zeta_1, \dots, \zeta_n\rangle\rangle$ of Ω coincides with $K\langle {}_n\mathcal{O}_p(\zeta_1, \dots, \zeta_n)\rangle$ and ${}_n\mathcal{O}_p(\zeta_1, \dots, \zeta_n)$ is the set of all Ω_0-solutions of (1) which is independent of the choice of a fundamental system of solutions $(\zeta_1, \dots, \zeta_n)$ (Proposition 3), so that $K\langle\langle\zeta_1, \dots, \zeta_n\rangle\rangle$ is determined only by K and the equation (1). In other words $K\langle\langle\zeta_1, \dots, \zeta_n\rangle\rangle$ is the smallest differential subfield of Ω containing both K and the set of all Ω_0-solutions of (1). Put $L = K\langle\langle\zeta_1, \dots, \zeta_n\rangle\rangle$. Then, from the above consideration, we have

Definition 3. L is called the *Galois extension field of K for* (1).

Lemma 3. *Let L be the Galois extension field of K for* (1). *If $\sigma: L \to L$ is a differential-analytic automorphism of L over K with respect to a fundamental system of solutions $(\zeta_1, \dots, \zeta_n)$ of* (1), *then σ is a differential-analytic automorphism of L over K with respect to any fundamental system of solutions of* (1).

Proof. Take another fundamental system of solutions $(\eta_1, \dots, \eta_n)$ of (1) and put $p(\eta_1, \dots, \eta_n) = q$. By Proposition 4 there exists a unique $g = (g_1, \dots, g_n) \in \Gamma_n$ such that $(\eta_1, \dots, \eta_n) = (\zeta_1, \dots, \zeta_n)^g$, i.e. $\eta_i = g_i(\zeta_1, \dots, \zeta_n)$ for $i = 1, \dots, n$. Note that $s(g) = p$ and $t(g) = q$, where $p = p(\zeta_1, \dots, \zeta_n)$.

As σ is a differential-analytic homomorphism over K with respect to $(\zeta_1, \dots, \zeta_n)$, σ maps η_i to $\sigma(\eta_i) = g_i(\sigma(\zeta_1), \dots, \sigma(\zeta_n))$, $i = 1, \dots, n$. Then

$$\begin{aligned}\sigma\left(\frac{\partial}{\partial x_j}\eta_i\right) &= \sigma\left(\frac{\partial}{\partial x_j}(g_i(\zeta_1, \dots, \zeta_n))\right) \\ &= \frac{\partial}{\partial x_j}(g_i(\sigma(\zeta_1), \dots, \sigma(\zeta_n))) \\ &= \frac{\partial}{\partial x_j}(\sigma(\eta_i))\,,\end{aligned}$$

$i = 1, \dots, n$; $j = 0, 1, \dots, n$. Thus $\sigma | K\{\eta_1, \dots, \eta_n\}$ is a differential homomorphism. For any $F \in {}_n\mathcal{O}_q(z_1, \dots, z_n)$,

$$\begin{aligned}\sigma(F(\eta_1, \dots, \eta_n)) &= \sigma(F \circ g(\zeta_1, \dots, \zeta_n)) \\ &= F \circ g(\sigma(\zeta_1), \dots, \sigma(\zeta_n)) \\ &= F(\sigma(\eta_1), \dots, \sigma(\eta_n))\end{aligned}$$

since $F \circ g \in {}_n\mathcal{O}_p(z_1, \dots, z_n)$. Hence σ is a differential-analytic homomorphism over K with respect to $(\eta_1, \dots, \eta_n)$.

On the other hand, it is easy to see that σ is a differential isomorphism over K. Then $(\sigma(\zeta_1), \ldots, \sigma(\zeta_n))$ is a fundamental system of solutions of (1), so that $(\sigma(\eta_1), \ldots, \sigma(\eta_n)) = (\sigma(\zeta_1), \ldots, \sigma(\zeta_n))^g$ is also a fundamental system of solutions of (1) because of Proposition 4. Then, by Lemma 2, Image $\sigma = K\langle\langle\sigma(\eta_1), \ldots, \sigma(\eta_n)\rangle\rangle = L$. Thus σ is a differential-analytic automorphism of L over K with respect to $(\eta_1, \ldots, \eta_n)$ and this completes the proof.

Then we say that σ is a *differential-analytic automorphism of* L *over* K if it is a differential-analytic automorphism of L over K with respect to one (and hence any) fundamental system of solutions of (1). The set of all differential-analytic automorphisms of L over K makes a group with respect to composition.

Definition 4. Let L be the Galois extension field of K for (1). The group of all differential-analytic automorphisms of L over K is called the *Galois group of* L *over* K or the *Galois group for* (1) *over* K, and is denoted by Gal (L/K).

In the remainder of this section we shall construct the representation of Gal (L/K) into Γ_n. For $p \in \boldsymbol{C}^n$ we put

$$\Gamma_{n,p} = \{g \in \Gamma_n;\ s(g) = t(g) = p\}\ ,$$

then $\Gamma_{n,p}$ is the subgroup of Γ_n which fixes the point p. For any $g \in \Gamma_{n,p}$, σ_g which is introduced in (4) becomes a differential-analytic automorphism of $K\{{}_n\mathcal{O}_p(z_1, \ldots, z_n)\}$ over K, and the inverse of σ_g is $\sigma_{g^{-1}}$.

Let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be a fundamental system of solutions of (1) with $p(\zeta) = p$. For a differential-analytic homomorphism ν_ζ (see (5)), we put

$$\mathfrak{J} = \operatorname{Ker} \nu_\zeta = \{\varphi \in K\{{}_n\mathcal{O}_p(z_1, \ldots, z_n)\};\ \nu_\zeta(\varphi) = 0\}\ .$$

Then $\mathfrak{J}$ is a prime differential ideal of $K\{{}_n\mathcal{O}_p(z_1, \ldots, z_n)\}$, which we call the *prime ideal for* ζ. Thus we have an exact sequence

$$(7) \qquad 0 \longrightarrow \mathfrak{J} \longrightarrow K\{{}_n\mathcal{O}_p(z_1, \ldots, z_n)\} \xrightarrow{\nu_\zeta} K\{{}_n\mathcal{O}_p(\zeta)\} \longrightarrow 0\,,$$

where each arrow of the sequence represents a differential-analytic homomorphism over K.

Take any $\sigma \in \operatorname{Gal}(L/K)$ and fix it. Then $\eta = (\sigma(\zeta_1), \ldots, \sigma(\zeta_n))$ is also a fundamental system of solutions of (1) with $p(\eta) = p(\zeta) = p$, so that, by Proposition 4, there exists a unique $g \in \Gamma_{n,p}$ such that $\eta = \zeta^g$. σ induces a differential-analytic isomorphism

$$(8) \qquad \sigma\colon K\{{}_n\mathcal{O}_p(\zeta)\} \xrightarrow{\sim} K\{{}_n\mathcal{O}_p(\eta)\}$$

over K, which is denoted by the same letter. Let us calculate the prime ideal for η.

$$\begin{aligned}\operatorname{Ker} \nu_\eta &= \{\psi \in K\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\};\ \nu_\eta(\psi) = 0\} \\ &= \{\psi;\ \nu_{\zeta^g}(\psi) = 0\} \\ &= \{\psi;\ \nu_\zeta(\sigma_g(\psi)) = 0\} \\ &= \{\psi;\ \sigma_g(\psi) \in \mathfrak{I}\} \\ &= \{\sigma_{g^{-1}}(\varphi);\ \varphi \in \mathfrak{I}\}\,,\end{aligned}$$

where we used (6). We denote the above ideal by $\mathfrak{I}^{g^{-1}}$. Similarly to (7) we have an exact sequence

$$0 \longrightarrow \mathfrak{I}^{g^{-1}} \longrightarrow K\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\} \xrightarrow{\nu_\eta} K\{{}_n\mathcal{O}_p(\eta)\} \longrightarrow 0\,. \tag{9}$$

Combining (7), (8) and (9), we obtain the following commutative and exact diagram;

$$\begin{tikzcd} & 0 \arrow[d, dotted] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathfrak{I} \arrow[r] \arrow[d, dotted] & K\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\} \arrow[r] \arrow[d, "\text{id.}"] & K\{{}_n\mathcal{O}_p(\zeta)\} \arrow[r] \arrow[d, "\sigma"] & 0 \\ 0 \arrow[r] & \mathfrak{I}^{g^{-1}} \arrow[r] \arrow[d, dotted] & K\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\} \arrow[r] \arrow[d] & K\{{}_n\mathcal{O}_p(\eta)\} \arrow[r] \arrow[d] & 0\,. \\ & 0 & 0 & 0 & \end{tikzcd} \tag{10}$$

The chase of (10) shows that $\mathfrak{I} = \mathfrak{I}^{g^{-1}}$.

Conversely, for two fundamental systems of solutions ζ, η with $p(\zeta) = p(\eta) = p$, the prime ideal for η is $\mathfrak{I}^{g^{-1}}$, where $\mathfrak{I}$ is the prime ideal for ζ and $g \in \Gamma_{n,p}$ is determined by $\eta = \zeta^g$. If we have $\mathfrak{I} = \mathfrak{I}^{g^{-1}}$, the chase of (10) shows the existence of a differential-analytic isomorphism (8) over K. It is naturally extended to a differential-analytic automorphism $\sigma: L \to L$ over K, and hence belongs to Gal(L/K).

Thus we have

Theorem 1. *Let $\zeta = (\zeta_1, \dots, \zeta_n)$ be a fundamental system of solutions of* (1) *with a center p, and put $L = K\langle\!\langle \zeta_1, \dots, \zeta_n \rangle\!\rangle$, the Galois extension field of K for* (1). *Then we have an injective homomorphism of groups*

$$0 \longrightarrow \operatorname{Gal}(L/K) \xrightarrow{\rho_\zeta} \Gamma_{n,p} \qquad \text{(exact)}$$

defined by $(\sigma(\zeta_1), \ldots, \sigma(\zeta_n)) = \zeta^{\rho_\zeta(\sigma)}$ *for any* $\sigma \in \mathrm{Gal}\,(L/K)$. *Moreover we have*

$$\text{Image}\ \rho_\zeta = \{g \in \Gamma_{n,p};\ \mathfrak{I} = \mathfrak{I}^{g^{-1}}\},$$

where $\mathfrak{I}$ *is the prime ideal for* ζ.

We call ρ_ζ the *representation of* $\mathrm{Gal}(L/K)$ *into* $\Gamma_{n,p}$ *with respect to* ζ. Now we study how ρ_ζ depends on ζ. Take another fundamental system of solutions $\xi = (\xi_1, \ldots, \xi_n)$ of (1). From Proposition 4 there exists a unique $h \in \Gamma_n$ such that $\xi = \zeta^h$. For any $\sigma \in \mathrm{Gal}\,(L/K)$, we have

$$(\sigma(\zeta_1), \ldots, \sigma(\zeta_n)) = \zeta^{\rho_\zeta(\sigma)},$$

$$(\sigma(\xi_1), \ldots, \sigma(\xi_n)) = \xi^{\rho_\xi(\sigma)}.$$

Simple calculation shows that the left hand member of the second equation coincides with $(\sigma(\zeta_1), \ldots, \sigma(\zeta_n))^h$. Thus we have

$$\zeta^{\rho_\zeta(\sigma)} = \zeta^{h\rho_\xi(\sigma)h^{-1}}.$$

Theorem 2. *The notation being as above, two representations* ρ_ζ *and* ρ_ξ *are similar;*

$$\rho_\zeta(\sigma) = h\rho_\xi(\sigma)h^{-1} \qquad (\sigma \in \mathrm{Gal}\,(L/K)).$$

Theorem 1 and 2 assure that, in order to obtain the Galois group for (1), it is enough to calculate the set $\{g \in \Gamma_{n,p(\zeta)};\ \mathfrak{I} = \mathfrak{I}^{g^{-1}}\}$ for the prime ideal $\mathfrak{I}$ for *any one* fundamental system of solutions ζ of (1). Moreover we have

Proposition 5. *Let* ζ *and* $\mathfrak{I}$ *be the same as in Theorem* 1. *If the set* $H = \{g \in \Gamma_{n,p};\ \mathfrak{I} \subset \mathfrak{I}^{g^{-1}}\}$ *is a group, then*

$$\text{Image}\ \rho_\zeta = H.$$

It is not known whether the assumption of Proposition 5 is always satisfied, while, in the Galois theory for algebraic equations and in the Picard-Vessiot theory for linear ordinary differential equations, the corresponding conditions always hold (Kolchin [5]). We will not go into this problem in this paper.

§3. The structure of Galois groups

In the Galois theory for algebraic equations, Galois groups are isomorphic to subgroups of symmetric groups of finite degree. And in the Picard-Vessiot theory, Picard-Vessiot groups are isomorphic to algebraic groups over fields of constants. We shall show that the Galois groups introduced in the preceding section are isomorphic to solution spaces of some kind of systems of differential equations, when some conditions are satisfied.

We quote several terminology and results from differential algebra which will be used in this section. For the proofs and the details, see Kolchin [6].

Let R be a differential ring. A differential ideal $\mathfrak{A}$ of R is said to be *perfect* if whenever the m-th power x^m of an element x in R belongs to $\mathfrak{A}$ for some $m \in N$, then x belongs to $\mathfrak{A}$. For any subset M of R there is the smallest perfect differential ideal of R containing M, which is denoted by $(M)_P$. A subset M of a perfect differential ideal $\mathfrak{A}$ of R is called a *basis* of $\mathfrak{A}$ if $\mathfrak{A} = (M)_P$, and is called a *finite basis* of $\mathfrak{A}$ if moreover it is a finite set.

Definition 5. A differential ring R is said to be *differential Noetherian* if any perfect differential ideal of R has a finite basis.

The following is an analogue of the basis theorem of Hilbert and is said to be the basis theorem of Ritt-Kolchin.

Theorem 3 (Kolchin [6], Chapter III, section 4, Theorem 1). *Let R_0 be a differential subring of a differential ring R over which R is finitely generated* (*as a differential ring*). *Suppose that the ring of constants of R_0 contains the field $\boldsymbol{Q}$. If R_0 is differential Noetherian, then R is also differential Noetherian.*

Now we consider the differential ring $\boldsymbol{C}\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\}$. From the definition $R = \boldsymbol{C}\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\} = \boldsymbol{C}\{z_1, \dots, z_n\} \otimes_{\boldsymbol{C}} {}_n\mathcal{O}_p(z_1, \dots, z_n)$ and we know that $R_0 = {}_n\mathcal{O}_p(z_1, \dots, z_n)$ is a Noetherian ring, but Theorem 3 cannot be applied to these rings because R_0 is not a differential ring. However we have

Proposition 6. *$\boldsymbol{C}\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\}$ is differential Noetherian.*

To prove this proposition, we define a ranking of elements of $\boldsymbol{C}\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\}$ as follows;

a ranking of $\boldsymbol{C}\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\}$ is a mapping

$$r\colon \boldsymbol{C}\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\} \to \boldsymbol{N}$$

which satisfies that

i) $r(f) = 0$ for any $f \in {}_n\mathcal{O}_p(z_1, \dots, z_n)$,

ii) for $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_0, \beta_1, \dots, \beta_n) \in \boldsymbol{N}^{n+1}$, $r\left(\left(\frac{\partial}{\partial x}\right)^{\alpha} z_i\right) > r\left(\left(\frac{\partial}{\partial x}\right)^{\beta} z_j\right)$ if and only if $(|\alpha|, i, \alpha_0, \alpha_1, \dots, \alpha_n)$ exceeds $(|\beta|, j, \beta_0, \beta_1, \dots, \beta_n)$ in lexicographic order,

iii) for a general element in $\boldsymbol{C}\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\}$, r can be defined so that the convention in Kolchin [6], Chapter I, section 8 holds.

Once a ranking r is defined, using the fact that ${}_n\mathcal{O}_p(z_1, \ldots, z_n)$ is Noetherian, we can prove this proposition in exact the same way as the proof of Theorem 3, and hence we omit the proof.

The following lemma can be shown in the same manner as the case of ordinary rings (see Nagata [9]).

Lemma 4. *Let R be a differential Noetherian ring and let S be a multiplicatively closed subset of R which does not contain* 0. *Then the quotient ring R_S with respect to S is also differential Noetherian.*

Theorem 4. *Suppose that a differential field K is finitely generated as a differential field over its field of constants $\boldsymbol{C}$. Then the differential ring $K\{{}_n\mathcal{O}_p(z_1, \ldots, z_n)\}$ is differential Noetherian.*

Proof. From the assumption there are $u_1, \ldots, u_m \in K$ such that $K = \boldsymbol{C}\langle u_1, \ldots, u_m\rangle$. If we put $S = \boldsymbol{C}\{u_1, \ldots, u_m\} \backslash \{0\}$, S is multiplicatively closed and $K = \boldsymbol{C}\{u_1, \ldots, u_m\}_S$. Put $R = \boldsymbol{C}\{{}_n\mathcal{O}_p(z_1, \ldots, z_n)\}$, then it is differential Noetherian by Proposition 6. Now we have

$$\begin{aligned} K\{{}_n\mathcal{O}_p(z_1, \ldots, z_n)\} &\cong K \otimes_C R \\ &= \boldsymbol{C}\{u_1, \ldots, u_m\}_S \otimes_C R \\ &\cong R\{u_1, \ldots, u_m\}_S \,. \end{aligned}$$

From Theorem 3, $R\{u_1, \ldots, u_m\}$ is differential Noetherian, and then, from Lemma 4, $R\{u_1, \ldots, u_m\}_S$ is also differential Noetherian, which establishes the proof.

Consider the equation

$$X(z) = \frac{\partial z}{\partial x_0} + a_1(x)\frac{\partial z}{\partial x_1} + \cdots + a_n(x)\frac{\partial z}{\partial x_n} = 0 \tag{1}$$

over K (i.e. $a_i(x) \in K \cap \Omega_0$ for $i = 1, \ldots, n$) and take a fundamental system of solutions $\zeta = (\zeta_1, \ldots, \zeta_n)$ of (1) with a center p. Let L be the Galois extension field of K for (1). Then, from Theorem 1, the Galois group $\mathrm{Gal}\,(L/K)$ is isomorphic to $G = \{g \in \Gamma_{n,p};\, \mathfrak{I}^g = \mathfrak{I}\}$ (which is apparently the same set as $\{g \in \Gamma_{n,p};\, \mathfrak{I} = \mathfrak{I}^{g^{-1}}\}$), where $\mathfrak{I}$ is the prime ideal for ζ. We are interested in the structure of the group G.

Before stating the theorem we must prepare several notation. For indeterminates $z_1, \ldots, z_n$, the field ${}_n\mathcal{M}_p(z_1, \ldots, z_n)$ of all germs of meromorphic functions in $(z_1, \ldots, z_n)$ at p makes a differential field with derivations $\dfrac{\partial}{\partial z_1}, \ldots, \dfrac{\partial}{\partial z_n}$. Let $y_1, \ldots, y_n$ be differential indeterminates with respect to these derivations, and let R_0 be a differential subring of ${}_n\mathscr{M}_p(z_1, \dots, z_n)$ with the ring of constants $\boldsymbol{C}$. We denote by $R_0\{y_1, \dots, y_n\}^*$ the ring of all differential polynomials in $y_1, \dots, y_n$ with coefficients in R_0, where $*$ is used to distinguish it from a differential polynomial ring with respect to the derivations $\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. In the same manner as in §2, we obtain a differential ring

$$R = R_0\{y_1, \dots, y_n\}^* \otimes_{\boldsymbol{C}} {}_n\mathcal{O}_p(y_1, \dots, y_n)$$

with derivations $\delta_1, \dots, \delta_n$ defined by

$$\delta_j(\xi \otimes F) = \frac{\partial \xi}{\partial z_j} \otimes F + \sum_{i=1}^n \xi \frac{\partial y_i}{\partial z_j} \otimes \frac{\partial F}{\partial y_i}, \qquad i = 1, \dots, n,$$

for $\xi \in R_0\{y_1, \dots, y_n\}^*$ and for $F \in {}_n\mathcal{O}_p(y_1, \dots, y_n)$. R is denoted by $R_0\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^*$ and we often use $\frac{\partial}{\partial z_j}$ instead of δ_j for $j = 1, \dots, n$. Note that $R_0\{y_1, \dots, y_n\}^*$ and ${}_n\mathcal{O}_p(y_1, \dots, y_n)$ are canonically regarded as subrings of $R_0\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^*$. The concept of differential-analytic homomorphisms can be defined similarly. Take any $g = (g_1, \dots, g_n) \in \Gamma_{n,p}$. We define a differential-analytic homomorphism

$$\alpha_g\colon R_0\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^* \to {}_n\mathscr{M}_p(z_1, \dots, z_n) \tag{11}$$

by $\alpha_g(y_i) = g_i(z_1, \dots, z_n)$ (see Lemma 2 (i)). For a subset $\mathscr{D}$ of $R_0\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^*$, g is said to be a *zero* (or a *solution*) of $\mathscr{D}$ if $\mathscr{D}$ is contained in the kernel of α_g.

Now we state the main theorem of this section. While we are interested in the structure of the Galois group, the theorem is concerned with a distinct set; the relation to the Galois group shall be mentioned in Remark 2 after the proof of the theorem.

Theorem 5. *Let ζ be a fundamental system of solutions of the differential equation* (1) *over K and let H be the set*

$$\{g \in \Gamma_{n,p}; \mathfrak{I}^g \subset \mathfrak{I}\},$$

where $\mathfrak{I}$ is the prime ideal for ζ and p is the center of ζ. Suppose that K is finitely generated over $\boldsymbol{C}$ as a differential field. Then there exists a finite subset $\mathscr{D}$ of ${}_n\mathscr{M}_p(z_1, \dots, z_n)\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^*$ such that $g \in \Gamma_{n,p}$ belongs to H if and only if g is a solution of $\mathscr{D}$.

Proof. Put $\zeta = (\zeta_1, \dots, \zeta_n)$. Take $g = (g_1, \dots, g_n) \in H$ and fix it. Then, for any $\varphi \in \mathfrak{I}$, we have $\sigma_g(\varphi) \in \mathfrak{I}$, so that $\nu_\zeta \circ \sigma_g(\varphi) = 0$ (σ_g and ν_ζ are defined in (4) and (5), respectively, in §2). We shall extract the equations satisfied by g from the last equation.

Consider a ring $R = K\{z_1, \dots, z_n\} \otimes_C C\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^*$ and define additive mappings $\partial_i^*\colon R \to R$, $i = 0, 1, \dots, n$, by

$$\partial_i^*(\xi \otimes \Xi) = \partial_i \xi \otimes \Xi + \sum_{j=1}^n \xi \frac{\partial z_j}{\partial x_i} \otimes \delta_j \Xi$$

for $\xi \in K\{z_1, \dots, z_n\}$ and for $\Xi \in C\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^*$. We see that, for $i = 0, 1, \dots, n$, ∂_i^* is a derivation of R and can be regarded as an extension of $\dfrac{\partial}{\partial x_i}$, so that R becomes a differential overring of $K\{z_1, \dots, z_n\}$. Now we decompose σ_g into two differential-analytic homomorphisms over K. First we define a differential-analytic homomorphism over K

$$\sigma_y\colon K\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\} \to K\{z_1, \dots, z_n\} \otimes_C C\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^*$$

by $\sigma_y(z_i) = 1 \otimes y_i$ for $i = 1, \dots, n$. Second we have a differential-analytic homomorphism over K

$$\alpha_g\colon K\{z_1, \dots, z_n\} \otimes_C C\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^* \to K\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\}$$

defined by $\alpha_g(1 \otimes y_i) = g_i$ and $\alpha_g(z_i \otimes 1) = z_i$ for $i = 1, \dots, n$. For a differential subring R_0 of ${}_n\mathcal{M}_p(z_1, \dots, z_n)$, α_g canonically induces a differential-analytic homomorphism

$$R_0\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^* \to {}_n\mathcal{M}_p(z_1, \dots, z_n)\,,$$

which is nothing but the differential-analytic homomorphism introduced in (11) (hence we use the same notation). In consequence we have the decomposition

$$\sigma_g = \alpha_g \circ \sigma_y\,. \tag{12}$$

Next we define two other differential-analytic homomorphisms β and γ_g over K as follows;

$$\beta\colon K\{{}_n\mathcal{O}_p(z_1, \dots, z_n)\} \to \Omega\,, \qquad \text{or}$$

$$\beta\colon K\{z_1, \dots, z_n\} \otimes_C C\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^*$$
$$\to K\{\zeta_1, \dots, \zeta_n\} \otimes_C C\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^*$$

is defined by $\beta(z_i) = \zeta_i$ or $\beta(z_i \otimes f) = \zeta_i \otimes f$, respectively, for $i = 1, \dots, n$ and for any $f \in C\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^*$ (hence β is nothing but ν_ζ), and

$$\gamma_g\colon K\{\zeta_1, \dots, \zeta_n\} \otimes_C C\{{}_n\mathcal{O}_p(y_1, \dots, y_n)\}^* \to \Omega$$

is so defined that the following diagram commutes:

$$
(13)\qquad
\begin{tikzcd}
K\{z_1,\dots,z_n\}\otimes_{\boldsymbol{C}}\boldsymbol{C}\{{}_n\mathcal{O}_p(y_1,\dots,y_n)\}^* \arrow[r, "\alpha_g"] \arrow[d, "\beta"] & K\{{}_n\mathcal{O}_p(z_1,\dots,z_n)\} \arrow[d, "\beta"] \\
K\{\zeta_1,\dots,\zeta_n\}\otimes_{\boldsymbol{C}}\boldsymbol{C}\{{}_n\mathcal{O}_p(y_1,\dots,y_n)\}^* \arrow[r, "\gamma_g"] & \Omega
\end{tikzcd}
$$

(γ_g is well defined by the definitions of α_g and β). The homomorphisms α_g, β and γ_g canonically induce the following commutative diagram:

$$
(14)\qquad
\begin{tikzcd}
{}_n\mathcal{M}_p(z_1,\dots,z_n)\otimes_{\boldsymbol{C}}\boldsymbol{C}\{{}_n\mathcal{O}_p(y_1,\dots,y_n)\}^* \arrow[r, "\alpha_g"] \arrow[d, "\beta"] & {}_n\mathcal{M}_p(z_1,\dots,z_n) \arrow[d, "\beta"] \\
{}_n\mathcal{M}_p(\zeta_1,\dots,\zeta_n)\otimes_{\boldsymbol{C}}\boldsymbol{C}\{{}_n\mathcal{O}_p(y_1,\dots,y_n)\}^* \arrow[r, "\gamma_g"] & {}_n\mathcal{M}_p(\zeta_1,\dots,\zeta_n),
\end{tikzcd}
$$

where α_g, β and γ_g denote the induced homomorphisms from the former ones respectively. In this diagram β in each column is an isomorphism, since $\zeta_1,\dots,\zeta_n$ are independent with respect to $(x_1,\dots,x_n)$, while β: $K\{{}_n\mathcal{O}_p(z_1,\dots,z_n)\}\to\Omega$ is not necessarily an isomorphism (see (7) in §2).

As we have seen above, for any $\varphi\in\mathfrak{I}$, $\beta\circ\sigma_g(\varphi)=0$ when g belongs to H. Then, by using (12) and (13), we obtain

$$
(15)\qquad \beta\circ\alpha_g\circ\sigma_y(\varphi)=\gamma_g\circ\beta\circ\sigma_y(\varphi)=0\,.
$$

By regarding $\beta\circ\sigma_y(\varphi)$ as a polynomial in elements of $\boldsymbol{C}\{{}_n\mathcal{O}_p(y_1,\dots,y_n)\}^*$ with coefficients in $K\{\zeta_1,\dots,\zeta_n\}$, we take the maximal subset $\{\xi_1^*,\dots,\xi_m^*\}$ of all coefficients appeared in $\beta\circ\sigma_y(\varphi)$ that is linearly independent over the field ${}_n\mathcal{M}_p(\zeta_1,\dots,\zeta_n)$. Then we obtain

$$
(16)\qquad \beta\circ\sigma_y(\varphi)=\theta_1^*\xi_1^*+\cdots+\theta_m^*\xi_m^*\,,
$$

where $\theta_1^*,\dots,\theta_m^*\in{}_n\mathcal{M}_p(\zeta_1,\dots,\zeta_n)\otimes_{\boldsymbol{C}}\boldsymbol{C}\{{}_n\mathcal{O}_p(y_1,\dots,y_n)\}^*$. Applying γ_g to each member of (16) and using (15), we have

$$
0=\gamma_g\circ\beta\circ\sigma_y(\varphi)=\gamma_g(\theta_1^*)\xi_1^*+\cdots+\gamma_g(\theta_1^*)\xi_1^*\,.
$$

Note that γ_g maps $\boldsymbol{C}\{{}_n\mathcal{O}_p(y_1,\dots,y_n)\}^*$ into ${}_n\mathcal{M}_p(\zeta_1,\dots,\zeta_n)\subset\Omega$. Thus $\gamma_g(\theta_1^*)$, $\dots$, $\gamma_g(\theta_m^*)$ belongs to the field ${}_n\mathcal{M}_p(\zeta_1,\dots,\zeta_n)$ over which $\xi_1^*,\dots,\xi_m^*$ are linearly independent. Hence $\gamma_g(\theta_1^*)=\cdots=\gamma_g(\theta_m^*)=0$. Put $\theta_i=\beta^{-1}(\theta_i^*)$ for $i=1,\dots,m$. Then $\gamma_g(\theta_i^*)=\gamma_g\circ\beta(\theta_i)$, therefore, by using (14), we have $\beta\circ\alpha_g(\theta_i)=0$ for $i=1,\dots,m$. Thus $\alpha_g(\theta_i)=0$ for $i=1,\dots,m$. Namely g is a zero of $\theta_1,\dots,\theta_m$. Recall that θ's are obtained independently of g. Hence, with each $\varphi\in\mathfrak{I}$, a finite subset $\{\theta_1,\dots,\theta_m\}$ of ${}_n\mathcal{M}_p(z_1,\dots,z_n)\otimes_{\boldsymbol{C}}\boldsymbol{C}\{{}_n\mathcal{O}_p(y_1,\dots,y_n)\}^*$ is associated such that $\beta\circ\alpha_g\circ\sigma_y(\varphi)=0$ if and only if $\alpha_g(\theta_1)=\cdots=\alpha_g(\theta_m)=0$.

By the assumption on K and Theorem 4, $K\{{}_n\mathcal{O}_p(z_1, \ldots, z_n)\}$ is differential Noetherian, so that $\mathfrak{I}$ has a finite basis $\{\varphi_1, \ldots, \varphi_N\}$. Let $\{\theta_1^i, \ldots, \theta_{m_i}^i\}$ be the subset of ${}_n\mathcal{M}_p(z_1, \ldots, z_n)\{{}_n\mathcal{O}_p(y_1, \ldots, y_n)\}^*$ associated with φ_i for $i = 1, \ldots, N$. Then the set $\mathscr{D} = \{\theta_j^i; i = 1, \ldots, N, j = 1, \ldots, m_i\}$ has the asserted property. In fact, if $g \in H$, then clearly $\alpha_g(\theta_j^i) = 0$ for every (i, j). Conversely suppose that $g \in \Gamma_{n,p}$ is a zero of $\mathscr{D}$. Then $\beta \circ \alpha_g \circ \sigma_y(\varphi_i) = 0$ for $i = 1, \ldots, N$ and so $\beta \circ \alpha_g \circ \sigma_y(\varphi) = 0$ for any $\varphi \in \mathfrak{I}$ because $\{\varphi_1, \ldots, \varphi_N\}$ is a basis of $\mathfrak{I}$. Therefore $\mathfrak{I}^g \subset \mathfrak{I}$, hence $g \in H$. This completes the proof.

Remark 2. Theorem 5 is concerned with the set

$$H = \{g \in \Gamma_{n,p}; \mathfrak{I}^g \subset \mathfrak{I}\}.$$

By Proposition 5, when H is a group, it coincides with the set

$$G = \{g \in \Gamma_{n,p}; \mathfrak{I}^g = \mathfrak{I}\},$$

which is isomorphic to the Galois group for (1) over K. And in this case Theorem 5 gives the characterization of the structure of the Galois group.

Several examples of Galois groups will be given in §5.

§4. Normality of Galois extensions

It is known in the theory of commutative algebra that, for an algebraic extension L/K of fields with char $K = 0$, the following two conditions are equivalent:

(N1) L contains with every element x also all conjugates of x over K,

(N2) the elements of L which are left invariant under all the automorphisms of L over K belongs to K.

L is called a *normal* extension of K when the two equivalent conditions are satisfied. It will be shown that, for the Galois extension defined in §2, a condition corresponding to (N1) holds. While, in general, a condition corresponding to (N2) does not hold; we shall explain by an example the cause at the end of this section.

Let K be a differential subfield of Ω whose field of constants is $\boldsymbol{C}$. Two elements α, β of Ω are said to be *conjugate over* K if there is a differential isomorphism

$$K\langle\alpha\rangle \to K\langle\beta\rangle$$

over K which sends α into β. We also say that β is a conjugate of α over K if α and β are conjugate over K. For an element α of Ω, define a differential homomorphism

$$\nu_\alpha\colon K\{z\} \to K\{\alpha\}$$

over K by $v_\alpha(z)=\alpha$ (see (5) in §2). Let $\mathfrak{I}_\alpha$ be the kernel of v_α. Then we see that α and β are conjugate over K if and only if $\mathfrak{I}_\alpha=\mathfrak{I}_\beta$. Indeed this is shown by the chase of the following exact and commutative diagram;

$$\begin{tikzcd}
 & 0 \arrow[d, dotted] & 0 \arrow[d] & 0 \arrow[d, dotted] & \\
0 \arrow[r] & \mathfrak{I}_\alpha \arrow[r] \arrow[d, dotted, "\mathrm{d}"] & K\{z\} \arrow[r] \arrow[d, "\mathrm{id}"] & K\{\alpha\} \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \mathfrak{I}_\beta \arrow[r] \arrow[d, dotted] & K\{z\} \arrow[r] \arrow[d] & K\{\beta\} \arrow[r] \arrow[d, dotted] & 0\,. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Definition 6. Let K be a differential subfield of Ω with the field of constants C. A differential extension field L of K in Ω is called a *normal extension of* K if any conjugate of every element of L over K belongs to L.

Theorem 6. *The Galois extensions are normal.*

For proving the theorem we need a result due to Riquier in differential algebra which claims the existence of analytic solutions of some kind of partial differential equations. We illustrate it below.

Let $(x_1,\dots,x_n)$ be coordinates of $\boldsymbol{C}^n$, and let $y_1,\dots,y_m$ be differential indeterminates with respect to derivations $\dfrac{\partial}{\partial x_1},\dots,\dfrac{\partial}{\partial x_n}$. With each x_i (resp. y_j), s-tuple of non-negative integers $(u_{i1},\dots,u_{is})$ (resp. $(v_{j1},\dots,v_{js})$) is associated and is called the *mark* of x_i (resp. y_j), $i=1,\dots,n$; $j=1,\dots,m$. With each derivative

$$\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{k_n} y_i\,,$$

of y_i, we associate the mark $(w_{i1},\dots,w_{is})$ with

$$w_{ij}=v_{ij}+k_1u_{1j}+\cdots+k_nu_{nj}\,,\qquad j=1,\dots,s\,.$$

We put the lexicographic order to the set of all marks, and consequently we have an ordering of derivatives of y's. Suppose that a difference in order exists between any two distinct derivatives of y's (it is always possible by increasing the number s if necessary). We are now in position to introduce the system of partial differential equations which is treated in Riquier's work; it is a finite system of partial differential equations

$$\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{k_n} y_i = G_{k_1\ldots k_n, i} \tag{17}$$

where

i) in each equation, G is a function of $x_1, \ldots, x_n$ and of a certain number of derivatives of y's, every derivative in G being lower in order than the left hand member of the equation,

ii) the left hand members of any two equations are distinct,

iii) if w is a left hand member of some equation, no derivative of w appears in the right hand member of any equation,

iv) the functions G are all analytic at some point in the space of all the arguments involved in them.

Such a system is called *orthonomic.*

The derivatives of y's which are derivatives of left hand members in the orthonomic system are called *principal derivatives.* All other derivatives are called *parametric derivatives.* For simplicity we consider the system (17) at $x_1 = \cdots = x_n = 0$. For each y_i, let numerical values be assigned to the parametric derivatives of y_i, at the origin, such that the right hand member in (17) are analytic for the values given to the derivatives in them and that the series

$$I_i = \sum \frac{a_{j_1\ldots j_n}}{j_1!\ldots j_n!} x_1{}^{j_1}\ldots x_n{}^{j_n},$$

where a's are the values of the parametric derivatives, the subscripts indicating the type of differentiation, converges in a neighbourhood of the origin. We call the series I_i the *initial determination* of y_i. Let an initial determination I_i be given for each y_i. Then, determining principal derivatives by using (17) with I_i, we obtain the unique vector $(y_1, \ldots, y_m)$ of functions, which will turn out analytic at the origin; we denote this vector by $(y_1{}^I, \ldots, y_m{}^I)$, where we put $I = (I_1, \ldots, I_m)$.

For every orthonomic system (17), there exists the smallest differential extension field F of $\boldsymbol{C}\langle y_1, \ldots, y_m\rangle$ that contains every right hand member of the equations (17). Then a finite subset $\{\mu_1, \ldots, \mu_M\}$ of F, which consists of functions of parametric derivatives, is associated with the system (17) in some definite way; we call it the *compatibility condition* of (17).

Theorem 7 (Riquier, see Ritt [15], Chapter VIII). *Let $\{\mu_1, \ldots, \mu_M\}$ be the compatibility condition of an orthonomic system* (17). *If* (17) *has a solution analytic at the point in the domain of definition, then every μ_i vanishes when the parametric derivatives are replaced by those of the solution. Conversely, for a* *vector* $I = (I_1, \dots, I_m)$ *of initial determination of* $(y_1, \dots, y_m)$, *if every* μ_i *vanishes when the parametric derivatives are replaced by those of* $(y_1{}^I, \dots, y_m{}^I)$, *then* $(y_1{}^I, \dots, y_m{}^I)$ *makes an analytic solution of* (17).

Proof of Theorem 6. Let K be as in Definition 6 and let L be the Galois extension field of K for a linear homogeneous partial differential equation of the first order

$$X(z) = 0 \tag{1}$$

over K. Take a fundamental system of solutions $(\zeta_1, \dots, \zeta_n)$ of (1) with a center p. We have to prove that $\beta \in \Omega$ belongs to L whenever β is a conjugate of some element α of L over K. Let

$$\sigma\colon K\langle\sigma\rangle \to K\langle\beta\rangle$$

be a differential isomorphism over K which sends α into β. Put $K_1 = K\langle\alpha\rangle$ and $K_2 = K\langle\beta\rangle$. We may assume that α is in $K\{{}_n\mathcal{O}_p(\zeta_1, \dots, \zeta_n)\}$. Then we have an expression

$$\alpha = \textstyle\sum_{k=1}^{m} A_k \otimes \xi_k ,$$

where $A_k \in K\{\zeta_1, \dots, \zeta_n\}$ and $\xi_k = f_k(\zeta_1, \dots, \zeta_n)$ for some $f_k \in {}_n\mathcal{O}_p(z_1, \dots, z_n)$, $k = 1, \dots, m$. We introduce new differential indeterminates $y_1, \dots, y_m$ with respect to derivations $\dfrac{\partial}{\partial x_0}, \dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}$ and consider a differential homomorphism

$$K_1\{z_1, \dots, z_n, y_1, \dots, y_m\} \to K\{\zeta_1, \dots, \zeta_n, \xi_1, \dots, \xi_m\}$$

over K which sends z_i and y_j into ζ_i and ξ_j respectively, $i = 1, \dots, n$; $j = 1, \dots, m$. The kernel of this homomorphism has a finite basis because $K_1\{z_1, \dots, z_n, y_1, \dots, y_m\}$ is differential Noetherian by Theorem 3 in §3. Let $\{\varphi_1, \dots, \varphi_N\}$ be the basis of the kernel. Then $(\zeta_1, \dots, \zeta_n, \xi_1, \dots, \xi_m)$ satisfies the following system;

$$\varphi_i(z_1, \dots, z_n, y_1, \dots, y_m) = 0 , \qquad i = 1, \dots, N . \tag{18}$$

By introducing other indeterminates if necessary, we may assume that (18) is an orthonomic system. Let $\{\mu_1, \dots, \mu_M\}$ denote the compatibility condition of (18). Since (18) has an analytic solution $(\zeta_1, \dots, \zeta_n, \xi_1, \dots, \xi_m)$, we see, by Theorem 7, that every μ_i vanishes when the parametric derivatives are replaced by those of the solution.

The differential isomorphism $\sigma\colon K_1 \to K_2$ induces the differential isomorphism

$$\sigma \otimes \mathrm{id}\colon K_1\{z_1, \dots, z_n, y_1, \dots, y_m\} \to K_2\{z_1, \dots, z_n, y_1, \dots, y_m\}$$

over K (where $K_i\{z_1, \ldots, z_n, y_1, \ldots, y_m\}$ is identified with $K_i \otimes_C C\{z_1, \ldots, y_n, y_1, \ldots, y_m\}$). Then the orthonomic system (18) is transformed by it to the system

$$(19) \qquad (\sigma \otimes \mathrm{id})(\varphi_i)(z_1, \ldots, z_n, y_1, \ldots, y_m) = 0\,, \qquad i = 1, \ldots, N\,,$$

which is again an orthonomic system. Note that the parametric derivatives of (19) are the same as those of (18). Let $I = (I_1, \ldots, I_n, I_{n+1}, \ldots, I_{n+m})$ be the initial determination of $(\zeta_1, \ldots, \zeta_n, \xi_1, \ldots, \xi_m)$, and let $(\eta_1, \ldots, \eta_n, \theta_1, \ldots, \theta_m)$ be the vector $(y_1{}^I, \ldots, y_n{}^I, z_1{}^I, \ldots, z_m{}^I)$ of analytic functions obtained from the same initial determination I by using (19). Note that the compatibility condition $\{\tilde{\mu}_1, \ldots, \tilde{\mu}_M\}$ of (19) is just the image of that of (18) by the differential isomorphism $\sigma \otimes \mathrm{id}$. Now we show that every $\tilde{\mu}_j$ vanishes when the parametric derivatives are replaced by those of $(\eta_1, \ldots, \eta_n, \theta_1, \ldots, \theta_m)$.

For simplicity we use (ζ, ξ) and (η, θ) instead of $(\zeta_1, \ldots, \zeta_n, \xi_1, \ldots, \xi_m)$ and $(\eta_1, \ldots, \eta_n, \theta_1, \ldots, \theta_m)$, respectively. As we have seen above, for each i,

$$\mu_i(\zeta, \xi) = 0\,,$$

only the parametric derivatives of (ζ, ξ) appearing in the left hand member. Then for any $j = (j_0, j_1, \ldots, j_n) \in \boldsymbol{N}^{n+1}$ we have

$$\left(\frac{\partial}{\partial x}\right)^j \mu_i(\zeta, \xi) = 0\,.$$

By using the equations (18), the left hand member of this equation can be written as follows:

$$\sum_{j'} \left(\left(\frac{\partial}{\partial x}\right)^{j'} \mu_i\right)(\zeta, \xi) G_{j'}(\zeta, \xi) = 0\,, \qquad \text{(finite sum)}\,,$$

where G's belong to $K_1\{z_1, \ldots, z_n, y_1, \ldots, y_m\}$ and only the parametric derivatives appear in the left hand member. Thus in particular we have

$$\sum_{j'} \left(\left(\frac{\partial}{\partial x}\right)^{j'} \mu_i\right)((a)) G_{j'}((a)) = 0\,,$$

where (a) denotes the vector of the numerical values assigned to the parametric derivatives of (ζ, ξ). Operating σ to each member of this equation, we obtain

$$\sum_{j'} \left(\left(\frac{\partial}{\partial x}\right)^{j'} \tilde{\mu}_i\right)((a)) \tilde{G}_{j'}((a)) = 0\,.$$

In this equation we can regard $\tilde{G}$'s as elements in $K_2\{z_1, \ldots, z_n, y_1, \ldots, y_m\}$ which are used for expressing the principal derivatives of (η, θ) by the parametric derivatives of it by the help of equations (19). Since (η, θ) has the same initial determination as that of (ζ, ξ), it follows that

$$\left(\frac{\partial}{\partial x}\right)^j \tilde{\mu}_i(\eta(x^0), \theta(x^0)) = 0\,,$$

for any $j \in \mathcal{N}^{n+1}$. Regarding $\tilde{\mu}_i(\eta, \theta)$ as an analytic function at x^0, we conclude that $\tilde{\mu}_i(\eta, \theta) = 0$ from the theorem of identity.

Hence the compatibility condition of (19) is satisfied by (η, θ), so that (η, θ) makes an analytic solution of (19) by Theorem 7. With $\zeta_1, \dots, \zeta_n$ also $\xi_1, \dots, \xi_m$ being solutions of (1) by Proposition 3 in §1, $\mathfrak{J}$ contains the elements

$$X(z_1), \dots, X(z_n), X(y_1), \dots, X(y_m)\,,$$

which are invariant under the differential isomorphism $\sigma \otimes \mathrm{id}$ over K. Thus $\eta_1, \dots, \eta_n$ and $\theta_1, \dots, \theta_m$ are solutions of (1), and hence belong to L. Now we see that the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \mathfrak{J} \arrow[r] \arrow[d, "\sigma\otimes\mathrm{id}"] & K_1\{z_1, \dots, z_n, y_1, \dots, y_m\} \arrow[r] \arrow[d, "\sigma\otimes\mathrm{id}"] & K_1\{\zeta_1, \dots, \zeta_n, \xi_1, \dots, \xi_m\} \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \mathfrak{K} \arrow[r] & K_2\{z_1, \dots, z_n, y_1, \dots, y_m\} \arrow[r] \arrow[d] & K_2\{\eta_1, \dots, \eta_n, \theta_1, \dots, \theta_m\} \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

is exact and commutative, where $\mathfrak{K}$ is the kernel of the differential homomorphism in the second row of this diagram; in fact, $\sigma \otimes \mathrm{id}$ maps $\mathfrak{J}$ into $\mathfrak{K}$ because $(\eta_1, \dots, \eta_n, \theta_1, \dots, \theta_m)$ is a solution of (19) which is the image of the basis of $\mathfrak{J}$. Then we have a differential homomorphism

$$\tilde{\sigma}\colon K_1\{\zeta_1, \dots, \zeta_n, \xi_1, \dots, \xi_m\} \to K_2\{\eta_1, \dots, \eta_n, \theta_1, \dots, \theta_m\}$$

over K which is compatible with σ. Thus we have

$$\begin{aligned}\beta = \sigma(\alpha) &= \sigma(\textstyle\sum_{k=1}^m A_k \otimes \xi_k)\\ &= \textstyle\sum_{k=1}^m \tilde{\sigma}(A_k)\tilde{\sigma}(\xi_k)\\ &= \textstyle\sum_{k=1}^m \tilde{\sigma}(A_k)\theta_k\,,\end{aligned}$$

where $\tilde{\sigma}(A_k) \in K\{\eta_1, \dots, \eta_n\} \subset L$ and $\theta_k \in L$ for $k = 1, \dots, m$, and hence $\beta \in L$. This proves our assertion.

Theorem 8. *Let K be a differential subfield of Ω with a field of constants $\boldsymbol{C}$ and let M be a Galois extension of K. Then, for every intermediate differential field L between K and M, M is a Galois extension of L and* $\mathrm{Gal}(M/L)$ *is a subgroup of* $\mathrm{Gal}(M/K)$. *Moreover, if L is a normal extension of K, then* $\mathrm{Gal}(M/L)$ *is a normal subgroup of* $\mathrm{Gal}(M/K)$.

Proof. Let M be a Galois extension of K for a differential equation

$$X(z) = 0 \tag{1}$$

over K. Then we have $M = K\langle Z\rangle$, where Z is the set of all Ω_0-solutions of (1). As $L \supset K$, (1) can be regarded as an equation over L, then $L\langle Z\rangle$ is a Galois equation of L for (1). And as $M \supset L$, we have $M = L\langle Z\rangle$, which proves the first half of the assertion. Clearly $\mathrm{Gal}(M/L)$ is a subgroup of $\mathrm{Gal}(M/K)$.

Note that, for any $\sigma \in \mathrm{Gal}(M/K)$ and for any $\alpha \in M$, α and $\sigma(\alpha)$ are conjugate over K. Then, if L is a normal extension of K, $\sigma(\alpha)$ belongs to L for every $\alpha \in L$. Take any $\tau \in \mathrm{Gal}(M/L)$ and any $\alpha \in L$. Since $\sigma(\alpha)$ belongs to L for any $\sigma \in \mathrm{Gal}(M/K)$ and since τ is a homomorphism over L, we have

$$\sigma^{-1}\tau\sigma(\alpha) = \sigma^{-1}\sigma(\alpha) = \alpha\,,$$

thus $\sigma^{-1}\tau\sigma$ leaves every element of L invariant. Hence $\sigma^{-1}\tau\sigma$ belongs to $\mathrm{Gal}(M/L)$, which proves that $\mathrm{Gal}(M/L)$ is a normal subgroup of $\mathrm{Gal}(M/K)$.

Thus, for a Galois extension M/K, we have a one-way correspondence

$$L \mapsto \mathrm{Gal}(M/L)$$

of differential subfields of M which contain K with subgroups of $\mathrm{Gal}(M/K)$. In particular M corresponds with $\{e\}$ (e is the unit element), and K corresponds with the whole $\mathrm{Gal}(M/K)$.

However the Galois correspondence in the opposite direction does not exist. Now we give a typical example.

Example. Let K be the field $\boldsymbol{C}(x_0, x_1, \ldots, x_n)$ of all rational functions in $(x_0, x_1, \ldots, x_n)$. Considering the germs at x^0, we can regard K as a differential subfield of Ω. As a differential equation under consideration, we take

$$X(z) = \frac{\partial z}{\partial x_0} = 0 \tag{20}$$

over K. Then we have a fundamental system of solutions $(x_1, \ldots, x_n)$ of (20). Thus the Galois extension field of K for (20) is

$$L = K\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle\,.$$

Since each x_i belongs to K, the Galois group $\mathrm{Gal}(L/K)$ is $\{e\}$. However L

cannot coincide with K because any analytic function in $(x_1, \dots, x_n)$ which is not rational belongs to $L \backslash K$; for example $\exp(x_1) \in L \backslash K$.

This example implies that the lack of the Galois correspondence in the opposite direction results from the definition in which Galois extensions are defined as infinitely generated in general, on the other hand it is necessary to define Galois extensions as above in order to assure the normality of the extensions.

§5. Galois groups for reducible ordinary differential equations

In this section x denotes a complex variable. Take $x^0 \in \boldsymbol{C}$ and fix it. Let ω be the field ${}_1\mathscr{M}_{x^0}$ of all germs of meromorphic functions in x at x^0, which makes an ordinary differential field with a derivation $\frac{d}{dx}$, whose field of constants is $\boldsymbol{C}$. We denote by ω_0 the set of all elements in ω that are holomorphic at x^0. Let y be a differential indeterminates with respect to $\frac{d}{dx}$. We often use y', y'', $\dots$, $y^{(n)}$, $\dots$ instead of $\frac{d}{dx}y$, $\left(\frac{d}{dx}\right)^2 y$, $\dots$, $\left(\frac{d}{dx}\right)^n y$, $\dots$, respectively. A positive integer n being fixed, we take a point $y^0 = (y_0^0, y_1^0, \dots, y_{n-1}^0) \in \boldsymbol{C}^n$ and put $\Omega = {}_{n+1}\mathscr{M}_{(x^0, y^0)}$, which is understood to be the universal differential field in preceding sections, where we employ $(x, y, y', \dots, y^{(n-1)})$ as coordinates of $\boldsymbol{C}^{n+1}$ instead of $(x_0, x_1, \dots, x_n)$.

We consider an ordinary differential equation of order n

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}), \tag{21}$$

where F belongs to Ω. It is well known that first integrals of (21) are solutions of

$$\frac{\partial z}{\partial x} + y' \frac{\partial z}{\partial y} + \cdots + y^{(n-1)} \frac{\partial z}{\partial y^{(n-2)}} + F \frac{\partial z}{\partial y^{(n-1)}} = 0, \tag{22}$$

which is of the form (1) in §1. We call (22) the associated partial differential equation with (21). If we have n solutions of (22) which are independent with respect to $(y, y', \dots, y^{(n-1)})$ at (x^0, y^0), a general solution of (21) which contains n constants can be expressed implicitly by them. In this point of view we call a fundamental system of solutions of (22) a *fundamental system of solutions of* (21). Let K be a differential subfield of Ω with the field of constants $\boldsymbol{C}$ which contains y', $\dots$, $y^{(n-1)}$ and F. Then the Galois extension of K for (22) and the Galois group for (22) over K are defined as in §2. They shall be called the *Galois extension of* K *for* (21) and the *Galois group for* (21) *over* K, respectively.

It may happen that a general solution of (21) is expressed by solutions of a linear ordinary differential equation, or by solutions of ordinary differential equations of orders less than n. On that occasion we say that a differential equation (21) is *reducible*. We shall study the characterization of Galois groups for reducible ordinary differential equations.

5.1. Linear ordinary differential equations

Let k be a differential subfield of ω with the field of constants $\boldsymbol{C}$. Consider a linear ordinary differential equation

$$\ell(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0\,, \tag{23}$$

where $a_i \in k \cap \omega_0$ for $i = 1, \ldots, n$. For simplicity we put $y^0 = (0, \ldots, 0) \in \boldsymbol{C}^n$. The associated partial differential equation with (23) is

$$\frac{\partial z}{\partial x} + y' \frac{\partial z}{\partial y} + \cdots + (-a_1 y^{(n-1)} - \cdots - a_n y) \frac{\partial z}{\partial y^{(n-1)}} = 0\,, \tag{24}$$

hence we consider this equation over $K = k(y, y', \ldots, y^{(n-1)}) \subset \Omega$. Let u be another differential indeterminate with respect to $\dfrac{d}{dx}$. Then the adjoint differential equation of (23) is

$$\ell^*(u) = \textstyle\sum_{i=0}^{n} (-1)^{n-i} (a_i u)^{(n-i)} = 0\,, \tag{25}$$

where we put $a_0 = 1$. Define the bilinear concomitant by

$$\Phi(y, u) = \textstyle\sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (-1)^{j-i} (a_i u)^{(j-i)} y^{(n-1-j)}\,, \tag{26}$$

then we have Lagrange's identity

$$\frac{d}{dx} \Phi(y, u) = u\ell(y) - y\ell^*(u)\,. \tag{27}$$

We obtain a fundamental system of solutions of (24) by using (23) and (26) as follows. Take any n solutions $u_1, \ldots, u_n$ of (25) which are linearly independent over $\boldsymbol{C}$, and put

$$\zeta_i(x, y, \ldots, y^{(n-1)}) = \Phi(y, u_i)$$

for $i = 1, \ldots, n$. Then $\zeta_1, \ldots, \zeta_n$ are first integrals of (23) because of Lagrange's identity (27). Moreover we have

$$\frac{\partial(\zeta_1, \ldots, \zeta_n)}{\partial(y, y', \ldots, y^{(n-1)})} = W(u_1, \ldots, u_n)\,,$$

where $W(u_1, \dots, u_n)$ denotes the Wronskian of $u_1, \dots, u_n$, which is different from 0 since $u_1, \dots, u_n$ are linearly independent over the field of constants. Clearly ζ_i belongs to Ω_0 for $i = 1, \dots, n$, thus $(\zeta_1, \dots, \zeta_n)$ makes a fundamental system of solutions of (24) (and hence of (23)) with a center 0.

Let L be the Galois extension field of K for (24). By Theorems 1 and 2, the Galois group $\mathrm{Gal}\,(L/K)$ is entirely determined up to isomorphism by the image of $(\zeta_1, \dots, \zeta_n)$ by every element of $\mathrm{Gal}\,(L/K)$. Take any $\sigma \in \mathrm{Gal}\,(L/K)$. Note that

$$u_i = \frac{\partial \zeta_i}{\partial y^{(n-1)}}$$

by (26), and hence u_i belongs to L for $i = 1, \dots, n$. Since ζ_i is linear in $(y, y', \dots, y^{(n-1)})$ and since $y, y', \dots, y^{(n-1)}$ belong to K, we obtain

$$\begin{aligned} \sigma(\zeta_i) &= \sigma(\Phi(y, u_i)) \\ &= \Phi(y, \sigma(u_i)) \end{aligned} \tag{28}$$

for $i = 1, \dots, n$.

Here we recall the Picard-Vessiot theory for linear ordinary differential equations. The equation (25) being considered over k, the Picard-Vessiot extension of k for (25) is $k\langle u_1, \dots, u_n\rangle$, where, for a subset U of ω, $k\langle U\rangle$ denotes the smallest differential subfield of ω containing k and U. The Picard-Vessiot group for (25) over k is the set of all differential automorphisms of $k\langle u_1, \dots, u_n\rangle$ over k, which is known to be isomorphic to an algebraic subgroup $\mathfrak{G}$ of $\mathrm{GL}\,(n; \boldsymbol{C})$. The isomorphism of groups Picard-Vessiot group $\ni \sigma \mapsto \gamma \in \mathfrak{G}$ is defined by

$$\begin{bmatrix} \sigma(u_1) \\ \vdots \\ \sigma(u_n) \end{bmatrix} = \gamma \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}. \tag{29}$$

Now let us return to the study of the Galois group for (23). As $k \subset K$ and $k\langle u_1, \dots, u_n\rangle \subset L$, every $\sigma \in \mathrm{Gal}\,(L/K)$ induces a differential automorphism of $k\langle u_1, \dots, u_n\rangle$ over k, which we know is an element of the Picard-Vessiot group for (25) over k. Then there exists a matrix $\gamma = (c_{ij}) \in \mathfrak{G}$ such that (29) holds. Combining (28) and (29), we have

$$\begin{aligned} \sigma(\zeta_i) &= \Phi(y, \textstyle\sum_{j=1}^n c_{ij} u_j) \\ &= \textstyle\sum_{j=1}^n c_{ij} \zeta_j\,, \end{aligned}$$

for $i = 1, \dots, n$, because $\Phi(y, u)$ is linear in $(u, u', \dots, u^{(n-1)})$. Moreover $(\sigma(\zeta_1), \dots, \sigma(\zeta_n))$ has the same center 0 as $(\zeta_1, \dots, \zeta_n)$. Thus we have

Theorem 9. *Let k be a differential subfield of ω with the field of constants $\boldsymbol{C}$ and put $K = k(y, y', \dots, y^{(n-1)})$. Then, for a linear ordinary differential equation*

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \tag{23}$$

over k, the Galois group for (23) *over K is isomorphic to the Picard-Vessiot group for the adjoint equation of* (23) *over k, in particular to an algebraic group over $\boldsymbol{C}$.*

5.2. Linearizable differential equations

As in the preceding paragraph, k is a differential subfield of ω with the field of constants $\boldsymbol{C}$. Before defining linearizable differential equations, we give two examples.

A non-linear differential equation of the form

$$y' = ay^2 + by + c\,, \tag{30}$$

where a, b, $c \in k \cap \omega_0$ with $a(x^0) \neq 0$, is called a *Riccati equation* over k. It is known that, if we put

$$y = -\frac{1}{a}\frac{u'}{u}\,, \tag{31}$$

then u satisfies the linear ordinary differential equation of the second order

$$u'' - \left(\frac{a'}{a} + b\right)u' + acu = 0 \tag{32}$$

whenever y is a solution of (30). The coefficients of the equation (32) belong to $k \cap \omega_0$. Conversely, for any non zero ω_0-solution u of (32), the function y defined by (31) satisfies the Riccati equation (30). In other words, the Riccati equation (30) is obtained from the linear equation (32) by the rational transformation (31). Note that the right hand member of (31) belongs to $k\langle u\rangle$.

We proceed to the second example. Consider a linear ordinary differential equation of the second order

$$u'' + au' + bu = 0 \tag{33}$$

over k. By a *projective solution* of (33) we mean a ratio u_2/u_1 of two solutions of u_1, u_2 of (33) that are linearly independent over $\boldsymbol{C}$. Then projective solutions of (33) satisfy a non-linear differential equation of the third order

$$\{y; x\} = 2b - a' - \frac{1}{2}a^2\,, \tag{34}$$

where the left hand member of this equation is called the Schwarzian derivative and is defined by

$$\{y; x\} = \left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2 .$$

Conversely a differential equation

$$(34)' \qquad \{y; x\} = q$$

with $q \in k \cap \omega_0$ is satisfied by

$$(35) \qquad y = \frac{u_2}{u_1},$$

where u_1, u_2 are linearly independent solutions over $\boldsymbol{C}$ of a linear equation

$$(33)' \qquad u'' + \frac{1}{2} qu = 0$$

over k. We call (34) or (34)′ a *Schwarzian equation* over k. Thus the Schwarzian equation (34)′ (resp. (34)) is obtained from the linear equation (33)′ (resp. (33)) by the rational transformation (35). In this case, u_1 and u_2 being differential indeterminates, the right hand member of the transformation (35) belongs to $k\langle u_1, u_2\rangle$.

By the above examples we are led to the definition of linearizable differential equations. A differential equation

$$(21) \qquad y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$$

is called *rational over* k if $F \in K = k(y, y', \ldots, y^{(n-1)})$, and is called *algebraic over* k if $F \in \bar{K}$, where $\bar{K}$ denotes the algebraic closure of K in Ω. Let $u, u_1, u_2, \ldots, u_m, \ldots$ be differential indeterminates with respect to the derivation $\frac{d}{dx}$.

Definition 7. A rational or algebraic differential equation (21) over k is said to be *linearizable over* k, if there exists a triple (ℓ, Δ, P) consisting of

i) ℓ, a linear differential operator of some order m over k:

$$\ell(u) = u^{(m)} + a_1 u^{(m-1)} + \cdots + a_m u ,$$

where $a_i \in k$ for $i = 1, \ldots, m$,

ii) $\Delta \subset k\{u_1, \ldots, u_m\}$, a finite subset, and

iii) $P \in k\langle u_1, \ldots, u_m\rangle$,

such that, for any $(y, u_1, \ldots, u_m)$ which satisfies

$$\ell(u_1) = \cdots = \ell(u_m) = 0\,,$$

$$W(u_1, \ldots, u_m) \neq 0\,,$$

$$\delta(u_1, \ldots, u_m) = 0 \qquad \text{for any} \quad \delta \in \Delta\,, \qquad \text{and}$$

$$y = P(u_1, \ldots, u_m)\,, \tag{36}$$

the conditions

$$\text{tr. deg. } k\langle y, u_1, \ldots, u_m\rangle/k = \text{tr. deg. } k\langle y\rangle/k = n$$

hold, and that the algebraic dependence of $(y, y', \ldots, y^{(n)})$ yields the equation (21).

We call (ℓ, Δ, P) a *linearization of* (21) *over* k. Now we explain what role Δ plays in a linearization. Once a linear operator ℓ is given, there exist, a priori, relations over k which any fixed solutions linearly independent over C satisfy. Hence the value tr. deg $k\langle u_1, \ldots, u_m\rangle/k$, which coincides with tr. deg. $k\langle y, u_1, \ldots, u_m\rangle/k$ when y satisfies (36), is determined only by ℓ. However, without the relations Δ, (21) is not necessarily obtained by virtue of (36). Now we give an example. Consider the Riccati equation (30) over k, which has a linearization $\left((32),\ \varnothing,\ -\dfrac{u_1'}{au_1}\right)$; in this case we do not need Δ. On the other hand we have another linearization of (30) as follows. By Lemmas 5 and 7 below, we see that there exists a linear operator $\tilde{\ell}(v)$ of which u_1, u_2, u_1' and u_2' are solutions, where u_1 and u_2 are any solutions of (32) linearly independent over C. The order of $\tilde{\ell}$ is 4 or 3, according to the linear independence of u_1, u_2, u_1' and u_2'. We assume that the order of $\tilde{\ell}$ is 4; namely u_1, u_2, u_1' and u_2' are linearly independent over C. Put $v_1 = u_1$, $v_2 = u_2$, $v_3 = u_1'$ and $v_4 = u_2'$, then $v_1, \ldots, v_4$ are solutions of $\tilde{\ell}(v) = 0$ linearly independent over C and satisfy the relations

$$v_1' = v_3\,, \qquad v_2' = v_4\,.$$

We see that (31) is now written as

$$y = -\frac{1}{a}\frac{v_3}{v_1}\,. \tag{31$'$}$$

However, for any solutions $v_1, \ldots, v_4$ of $\tilde{\ell}$, $\tilde{\ell}(v)$ and (31)$'$ do not necessarily yield the Riccati equation (30); $\tilde{\ell}(v)$ and (31)$'$ yield (30) only when the relations $v_1' = v_3$ and $\ell(v_1) = 0$ are satisfied. Thus we must take Δ as $\{v_1' - v_3, \ell(v_1)\}$. Hence Δ plays a role of indicating some class of solutions of $\tilde{\ell}$.

As we have shown just above, for a linearizable equation, its linearization is not unique; however we show that there is the canonical one.

Proposition 7. *If an equation* (21) *is linearizable over* k, *there exists the linearization* (ℓ, Δ, P) *over* k *with* $P = \dfrac{u_2}{u_1}$.

In proving this proposition, we use three lemmas. For simplicity we say that "linearly independent" instead of "linearly independent over $\boldsymbol{C}$".

Lemma 5. *Let* $u_1, \dots, u_m$ *be linearly independent solutions of a linear differential equation of order* m *over* k. *Then, for any* $a \in k$ *and for any* $i \in \boldsymbol{N}$, *there exists a linear differential equation over* k *of which* $a u_1^{(i)}, \dots, a u_m^{(i)}$ *are solutions.*

Lemma 6. *The assumption being the same as in Lemma* 5, *let* $\{q_1, \dots, q_M\}$ *be the set of all monomials in* $u_1, \dots u_m$ *of degree* s. *Then there exists a linear differential equation over* k *of which* $q_1, \dots, q_M$ *are solutions.*

Lemma 7. *The assumption being the same as in Lemma* 5, *let* $v_1, \dots, v_p$ *be linearly independent solutions of another linear differential equation of order* p *over* k. *Then there exists a linear differential equation over* k *of which* $u_1, \dots, u_m$ *and* $v_1, \dots, v_p$ *are solutions.*

Proof of Proposition 7. Take one linearization (ℓ, Δ, P) of (21) over k, then P can be written as

$$P = \frac{q_{21} + \cdots + q_{2t}}{q_{11} + \cdots + q_{1s}},$$

where each q_{ij} is a monomial in u's and their derivatives with a coefficient in k. By Lemmas 5, 6 and 7, there exists a linear differential equation $\tilde{\ell}(v) = 0$ over k of which $q_{11}, \dots, q_{1s}, q_{21}, \dots, q_{2t}, u_1, \dots, u_m$ are solutions. Let the order of $\tilde{\ell}$ be M and let $v_1, \dots, v_M$ denote linearly independent solutions of $\tilde{\ell}(v) = 0$. We can take $v_1, \dots, v_M$ as follows:

$$u_i = v_i\,, \qquad i = 1, \dots, m\,,$$

$$q_{1e} = v_{i_e}\,, \qquad q_{2f} = v_{j_f}\,, \qquad e = 1, \dots, s\,, \quad f = 1, \dots, t\,.$$

Then this convention yields the relations $\tilde{\Delta}$ between $v_1, \dots, v_M$ in which $\delta(v_1, \dots, v_m)$ is contained for every $\delta \in \Delta$. Put

$$\tilde{P} = \frac{v_{j_1} + \cdots + v_{j_t}}{v_{i_1} + \cdots + v_{i_s}}.$$

Then clearly $(\tilde{\ell}, \tilde{\Delta}, \tilde{P})$ becomes another linearization of (21) over k. Moreover if we put $w_1 = v_{i_1} + \cdots + v_{i_s}$, $w_2 = v_{j_1} + \cdots + v_{j_t}$ and introduce $w_3, \dots, w_M$ so that $(v_1, \dots, v_M)$ and $(w_1, \dots, w_M)$ are obtained by linear transformations from each other, there corresponds $\hat{\Delta} \subset k\{w_1, \dots, w_M\}$ to $\tilde{\Delta}$ canonically and $\left(\tilde{\ell}, \hat{\Delta}, \frac{w_2}{w_1}\right)$ is also a linearization of (21). This completes the proof.

The proofs of Lemmas 5, 6 and 7 are omitted here (see Picard [12]). Now we have the main result of this paragraph.

Theorem 10. *Let k be a differential subfield of ω with the field of constants $\boldsymbol{C}$. Suppose that a rational or algebraic differential equation over k*

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}) \tag{21}$$

is linearizable over k. Then the Galois group for (21) *over $\overline{k(y, y', \dots, y^{(n-1)})}$ is isomorphic to an algebraic group over $\boldsymbol{C}$.*

When (21) has such a linearization (ℓ, Δ, P) that $P \in k\{u_1, \dots, u_m\}$, we see, from the proof of Proposition 7, that $y = P(u_1, \dots, u_m)$ satisfies a linear differential equation over k. Hence Theorem 10 is a direct consequence of Theorem 9 in this case. Previous to the proof of Theorem 10 for a general case, we calculate the Galois groups for Riccati equations and Schwarzian equations, which will illustrate the idea of the proof.

As we have seen above, the Riccati equation (30) has a linearization $\left((32), \varnothing, -\frac{1}{a}\frac{u_1'}{u_1}\right)$ over k. The adjoint equation of (32) is

$$v'' + \left(\frac{a'}{a} + b\right)v' + \left(\left(\frac{a'}{a} + b\right)' + ac\right)v = 0\,, \tag{37}$$

and we take linearly independent solutions v_1, v_2 of (37). Put

$$\eta_i(x, u, u') = v_i u' - \left(v_i' + \left(\frac{a'}{a} + b\right)v_i\right)u\,, \qquad i = 1, 2\,,$$

then (η_1, η_2) makes a fundamental system of solutions of (32) (see §5.1). By virtue of the relation (31), we can write

$$\frac{\eta_2(x, u, u')}{\eta_1(x, u, u')} = \frac{v_2 u' - \left(v_2' + \left(\frac{a'}{a} + b\right)v_2\right)u}{v_1 u' - \left(v_1' + \left(\frac{a'}{a} + b\right)v_1\right)u}$$
$$= \frac{a v_2 y + \left(v_2' + \left(\frac{a'}{a} + b\right)v_2\right)}{a v_1 y + \left(v_1' + \left(\frac{a'}{a} + b\right)v_1\right)}.$$

Let $\zeta(x, y)$ denote the last member of the above. For any solution y of (30), every u which is determined by (31) is a solution of (32), so that η_1, η_2 and hence ζ become constants. Thus ζ is a first integral of (30), and moreover becomes a fundamental system of solutions of (30) by virtue of the linear independence of v_1 and v_2.

Let L be the Galois extension field of $K = k(y)$ for (30), then, from the expression of ζ, we see that $\frac{v_2}{v_1}$, $\frac{v_1'}{v_1}$ and $\frac{v_2'}{v_1}$ belong to L. Now we examine how these elements are transformed by the elements of $\mathrm{Gal}(L/K)$. Let X be a neighborhood of x^0 in x-space. Consider a point

$$\xi = (v_1(x) : v_2(x) : v_1'(x) : v_2'(x)) \in \boldsymbol{P}^3 \times X .$$

The ideal $\mathfrak{I}_{\mathrm{homog}} \subset k[V_1, V_2, V_1', V_2']$ is defined as consisting of all homogeneous algebraic relations over k between v_1, v_2, v_1' and v_2'. Any $x \in X$ being fixed, let Ξ_x be the projective algebraic subvariety of $\boldsymbol{P}^3 \times \{x\}$ defined by $\mathfrak{I}_{\mathrm{homog}}$. For any $F \in \mathfrak{I}_{\mathrm{homog}}$, we have

$$F(v_1, v_2, v_1', v_2') = 0 ,$$

so that

$$F\left(1, \frac{v_2}{v_1}, \frac{v_1'}{v_1}, \frac{v_2'}{v_1}\right) = 0 .$$

Take any $\sigma \in \mathrm{Gal}(L/K)$ and operate it to each member of the above equation, then we obtain

$$F\left(1, \sigma\left(\frac{v_2}{v_1}\right), \sigma\left(\frac{v_1'}{v_1}\right), \sigma\left(\frac{v_2'}{v_1}\right)\right) = 0 ;$$

hence $\left(1 : \sigma\left(\frac{v_2}{v_1}\right) : \sigma\left(\frac{v_1'}{v_1}\right) : \sigma\left(\frac{v_2'}{v_1}\right)\right) \in \Xi_x$. Namely σ maps the point $\xi \in \Xi_x$ into Ξ_x itself. Put $\Xi = \prod_{x \in X} \Xi_x$, then, by the help of (37), any point of Ξ is expressed as

$$(c_{11}v_1(x) + c_{12}v_2(x) : c_{21}v_1(x) + c_{22}v_2(x) : c_{11}v_1'(x) + c_{12}v_2'(x) \\ : c_{21}v_1'(x) + c_{22}v_2'(x))$$

for some $\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \in \mathfrak{G}$, where $\mathfrak{G}$ is a projective algebraic subgroup of $\mathrm{PGL}(2; \boldsymbol{C})$, which we call the projective Picard-Vessiot group relative to (v_1, v_2). Thus we have

$$\sigma\left(\frac{v_2}{v_1}\right) = \frac{c_{21}v_1 + c_{22}v_2}{c_{11}v_1 + c_{12}v_2}, \qquad \sigma\left(\frac{v_1'}{v_1}\right) = \frac{c_{11}v_1' + c_{12}v_2'}{c_{11}v_1 + c_{12}v_2},$$

$$\sigma\left(\frac{v_2'}{v_1}\right) = \frac{c_{21}v_1' + c_{22}v_2'}{c_{11}v_1 + c_{12}v_2},$$

for some $(c_{ij}) \in \mathfrak{G}$. Then we have

$$\begin{aligned}\sigma(\zeta) &= \frac{(ay + A)\sigma\left(\dfrac{v_2}{v_1}\right) + \sigma\left(\dfrac{v_2'}{v_1}\right)}{(ay + A) + \sigma\left(\dfrac{v_1'}{v_1}\right)} \\ &= \frac{c_{21}((ay + A)v_1 + v_1') + c_{22}((ay + A)v_2 + v_2')}{c_{11}((ay + A)v_1 + v_1') + c_{12}((ay + A)v_2 + v_2')} \\ &= \frac{c_{21} + c_{22}\zeta}{c_{11} + c_{12}\zeta},\end{aligned}$$

where $\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \in \mathfrak{G}$ and we put $A = \dfrac{a'}{a} + b$. Thus we obtain

Proposition 8. *The Galois group for a Riccati equation*

$$y' = ay^2 + by + c \tag{30}$$

over k is isomorphic to the group

$$\left\{g(z) = \frac{c_{21} + c_{22}z}{c_{11} + c_{12}z}; \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \in \mathfrak{G}, g(z^0) = z^0\right\},$$

where $\mathfrak{G}$ is a projective algebraic subgroup of PGL $(2; \boldsymbol{C})$ *which is called the projective Picard-Vessiot group for a linear differential equation*

$$v'' + \left(\frac{a'}{a} + b\right)v' + \left(\left(\frac{a'}{a} + b\right)' + ac\right)v = 0 \tag{37}$$

over k.

The Schwarzian equation $(34)'$ has a linearization $\left((33)', \varnothing, \dfrac{u_2}{u_1}\right)$ over k. We take linearly independent solutions v_1 and v_2 of the adjoint equation

$$v'' + \frac{1}{2}qv = 0$$

of $(33)'$, and put

$$\eta_i(x, u, u') = v_i u' - v_i' u$$

for $i = 1, 2$, then (η_1, η_2) makes a fundamental system of solutions of (33)'. On the other hand (35) gives the following expression:

$$\frac{u_1'}{u_1} = -\frac{y''}{2y'}, \qquad \frac{u_2'}{u_1} = y' - \frac{y''y}{2y'}. \tag{38}$$

In the same manner as in the case of Riccati equations, we can show by using the expression (38) that

$$\frac{\eta_j(x, u_t, u_t')}{\eta_i(x, u_s, u_s')}$$

becomes a function of x, y, y' and y'' for $i, j, s, t = 1, 2$. Then they are first integrals of (34)' and we can obtain a fundamental system of solutions of (34)' by using them. In fact define ζ_1, ζ_2 and ζ_3 by the following:

$$\zeta_1 = \frac{\eta_2(x, u_1, u_1')}{\eta_1(x, u_1, u_1')} = \frac{v_2 u_1' - v_2' u_1}{v_1 u_1' - v_1' u_1} = \frac{v_2 y'' + 2v_2' y'}{v_1 y'' + 2v_1' y'},$$

$$\begin{aligned}\zeta_2 &= \frac{\eta_2(x, u_2, u_2')}{\eta_1(x, u_2, u_2')} = \frac{v_2 u_2' - v^{2\prime} u_2}{v_1 u_2' - v_1' u_2} \\ &= \frac{v_2(y''y - 2y'^2) + 2v_2' y' y}{v_1(y''y - 2y'^2) + 2v_1' y' y},\end{aligned}$$

$$\begin{aligned}\zeta_3 &= \frac{\eta_1(x, u_2, u_2')}{\eta_1(x, u_1, u_1')} = \frac{v_1 u_2' - v_1' u_2}{v_1 u_1' - v_1' u_1} \\ &= \frac{v_1(y''y - 2y'^2) + 2v_1' y' y}{v_1 y'' + 2v_1' y'},\end{aligned}$$

then we see that $(\zeta_1, \zeta_2, \zeta_3)$ makes a fundamental system of solutions of (34)' by virtue of the linear independence of v_1 and v_2. Let L be the Galois extension field of $K = k(y, y', y'')$ for (34)'. From the expression of ζ's we see that $\frac{v_2}{v_1}$, $\frac{v_1'}{v_1}$ and $\frac{v_2'}{v_1}$ belongs to L. Thus, in the same manner as in the case of Riccati equations, we obtain

Proposition 9. *The Galois group for a Schwarzian equation*

$$\{y; x\} = q \tag{34'}$$

over k is isomorphic to the group

$$\left\{ (g_1(z), g_2(z), g_3(z)) \in \Gamma_{3,p}; \right.$$

$$g_1(z) = \frac{c_{21} + c_{22}z_1}{c_{11} + c_{12}z_1}, \; g_2(z) = \frac{c_{21} + c_{22}z_2}{c_{11} + c_{12}z_2},$$

$$\left. g_3(z) = \frac{z_3(c_{11} + c_{12}z_2)}{c_{11} + c_{12}z_1}, \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \in \mathfrak{I} \right\},$$

where $\mathfrak{G}$ *is the projective Picard-Vessiot group for a linear differential equation*

$$v'' + \frac{1}{2}qv = 0$$

over k.

Proof of Theorem 10. By Proposition 7, for the linearizable equation

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}), \tag{21}$$

we have the canonical linearization $\left(\ell, \Delta, \frac{u_2}{u_1}\right)$ over k. Let m be the order of ℓ. Take linearly independent solutions $v_1, \dots, v_m$ of the adjoint equation ℓ^* of ℓ and put

$$\eta_j(u) = \Phi(u, v_i)$$

for $i = 1, \dots, m$, where Φ is the bilinear concomitant for ℓ (see (26)). Then $(\eta_1, \dots, \eta_m)$ makes a fundamental system of solutions of ℓ. On the other hand, by the expression $y = \frac{u_2}{u_1}$ and Definition 7, we have

$$\frac{u_i^{(j)}}{u_1} \in \bar{K} = \overline{k(y, y', \dots, y^{(n-1)})} \tag{39}$$

for $i = 1, 2$ and for $j = 0, 1, \dots, m-1$.

Consider any first integral ζ of (21). Putting the expression $y = \frac{u_2}{u_1}$ into ζ, we see that ζ is a function of x, u_1, u_2 and their derivatives and becomes a constant for solutions u_1 and u_2 of ℓ. Then, as $(\eta_1, \dots, \eta_m)$ is a fundamental system of solutions of ℓ, ζ can be written as a function of $\eta_1(u_1), \dots, \eta_m(u_1)$, $\eta_1(u_2), \dots, \eta_m(u_2)$. On the other hand ζ originally a function of x, y, y', $\dots$, $y^{(n-1)}$. Hence ζ must be a function of $\frac{\eta_j(u_t)}{\eta_i(u_s)}$, $i, j = 1, \dots, m$; $s, t = 1, 2$.

Therefore we can take a fundamental system of solutions of (21) which consists of $\zeta_{i,j;s,t}=\dfrac{\eta_j(u_t)}{\eta_i(u_s)}$'s. Let $\zeta_e=\zeta_{i_e,j_e;s_e,t_e}$ be a component of a fundamental system of solutions of (21) for $e=1,\dots,n$, then each $\zeta_{i,j;s,t}$ becomes a function of $\zeta_1,\dots,\zeta_n$:

$$\zeta_{i,j;s,t}=F_{i,j;s,t}(\zeta_1,\dots,\zeta_n)\,.$$

Let L be the Galois extension field $\bar{K}\langle\langle\zeta_1,\dots,\zeta_n\rangle\rangle$ of $\bar{K}$ for (21). Then, as we have seen in the above examples, every $\sigma\in\mathrm{Gal}(L/\bar{K})$ corresponds to an element of projective Picard-Vessiot group for ℓ^* over k. Thus we obtain

$$\begin{aligned}\sigma(\zeta_e)&=\sigma(\zeta_{i_e,j_e;s_e,t_e})\\&=\sigma\left(\frac{\eta_{j_e}(u_{t_e})}{\eta_{i_e}(u_{s_e})}\right)\\&=\frac{\sum_{j=1}^m c_{j_ej}\eta_j(u_{t_e})}{\sum_{j=1}^m c_{i_ej}\eta_j(u_{s_e})}\\&=\frac{\sum_{j=1}^m c_{j_ej}\zeta_{i_e,j;s_e,t_e}}{\sum_{j=1}^m c_{i_ej}\zeta_{i_e,j;s_e,s_e}}\\&=\frac{\sum_{j=1}^m c_{j_ej}F_{i_e,j;s_e,t_e}(\zeta_1,\dots,\zeta_n)}{\sum_{j=1}^m c_{i_ej}F_{i_e,j;s_e,s_e}(\zeta_1,\dots,\zeta_n)},\end{aligned}$$

for $e=1,\dots,n$, where (c_{ij}) represents an element of projective Picard-Vessiot group relative to $(v_1,\dots,v_m)$. Hence $\mathrm{Gal}(L/\bar{K})$ is isomorphic to an algebraic group over $\boldsymbol{C}$.

5.3. Equations reducible to lower order ones

Consider a rational equation

$$y^{(n)}=F(x,y,y',\dots,y^{(n-1)}) \tag{21}$$

over k of order n; namely F belongs to the field $K=k(y,y',\dots,y^{(n-1)})$. Take $n_1,n_2\in\boldsymbol{N}$ such that $n_1+n_2=n$. We shall define (n_1,n_2)-reducible equations.

Let there be two differential equations

$$u^{(n_1)}=A(x,u,u',\dots,u^{(n_1-1)})\,, \tag{40}$$

$$y^{(n_2)}=B(x,u,u',\dots,u^{(n_1-1)},y,y',\dots,y^{(n_2-1)}) \tag{41}$$

with

$$A \in k(u, u', \dots, u^{(n_1-1)}),$$

$$B \in k(u, u', \dots, u^{(n_1-1)}, y, y', \dots, y^{(n_2-1)}),$$

and suppose that, for any solution y of (21), there exists a solution u of (40) such that y is a solution of (41) with the u. Then any solution of (21) is obtained by solving the equations (40) and (41) that are of lower order than (21). It seems natural that we define (n_1, n_2)-reducible equations by the above, but there is some ambiguity, especially in the term "any solution". Therefore we have to define (n_1, n_2)-reducible equations in an exact and algebraic way.

By operating $\frac{d}{dx}$ successively to each member of (41) and eliminating higher derivatives of u and y in the right hand members by the help of (40) and (41), we obtain the following expressions:

$$(42)\qquad \begin{cases} y^{(n_2)} = B(x, u, \dots, u^{(n_1-1)}, y, \dots, y^{(n_2-1)}), \\ y^{(n_2+1)} = B_1(x, u, \dots, u^{(n_1-1)}, y, \dots, y^{(n_2-1)}), \\ \quad \cdots\cdots\cdots\cdots \\ y^{(n-1)} = B_{n_1-1}(x, u, \dots, u^{(n_1-1)}, y, \dots, y^{(n_2-1)}), \end{cases}$$

where each $B_i \in k(u, \dots, u^{(n_1-1)}, y, \dots, y^{(n_2-1)})$. Regarding (42) as a system of algebraic equations in $(u, u', \dots, u^{(n_1-1)})$, we obtain the solutions

$$(43)\qquad (C_t^{(0)}, C_t^{(1)}, \dots, C_t^{(n_1-1)}), \qquad t = 1, \dots, T,$$

where $C_t^{(s)} = C_t^{(s)}(x, y, y', \dots, y^{(n-1)}) \in \bar{K}$ for $t = 1, \dots, T$ and for $s = 0, 1, \dots, n-1$ ($\bar{K}$ denotes the algebraic closure of K). We call (43) a *parametric root* for the pair ((40), (41)). Note that

$$(44)\qquad \det\left(\frac{\partial C_t^{(s)}}{\partial y^{(j)}}\right)_{\substack{s=0,\dots,n_1-1 \\ j=n_2,\dots,n-1}} \neq 0\,.$$

This can be shown by differentiating each member of the equations in (42) with respect to $y^{(n_2)}, \dots, y^{(n-1)}$.

Now operate $\frac{d}{dx}$ to each member of

$$u^{(n_1-1)} = C_t^{(n_1-1)}(x, y, y', \dots, y^{(n-1)})\,.$$

Then we have, by using the equation (40),

$$A(x, C_t^{(0)}, \dots, C_t^{(n_1-1)}) = \frac{\partial C_t^{(n_1-1)}}{\partial x} + y' \frac{\partial C_t^{(n_1-1)}}{\partial y} + \cdots + y^{(n)} \frac{\partial C_t^{(n_1-1)}}{\partial y^{(n-1)}}\,.$$

If (21) is (n_1, n_2)-reducible in the above sense, $y^{(n)}$ in this equation must coincide with F. Hence we are led to the following definition.

Definition 8. A rational differential equation

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}) \tag{21}$$

over k is said to be (n_1, n_2)-*reducible over* k if $n_1 + n_2 = n$ and if the following holds;

there exist two differential equations

$$u^{(n_1)} = A(x, u, u', \dots, u^{(n_1-1)}), \tag{40}$$

$$y^{(n_2)} = B(x, u, u', \dots, u^{(n_1-1)}, y, y', \dots, y^{(n_2-1)}) \tag{41}$$

with $A \in k(u, u', \dots, u^{(n_1-1)})$, $B \in k(u, \dots, u^{(n_1-1)}, y, \dots, y^{(n_2-1)})$, and, for any parametric root $(C_t^{(0)}, \dots, C_t^{(n_1-1)})$ for the pair ((40), (41)), the expression

$$\left(\frac{\partial C_t^{(n_1-1)}}{\partial y^{(n-1)}}\right)^{-1}\left[A(x, C_t^{(0)}, \dots, C_t^{(n_1-1)}) - \frac{\partial C_t^{(n_1-1)}}{\partial x}\right.$$
$$\left. - y'\frac{\partial C_t^{(n_1-1)}}{\partial y} - \cdots - y^{(n-1)}\frac{\partial C_t^{(n_1-1)}}{\partial y^{(n-2)}}\right]$$

coincides with F.

The equation (40) is called the *parametric equation* for (21), and the equation (41) is called the *reduced equation* for (21).

Example. The rational equation

$$y'' = \frac{3y'^2}{y}$$

is (1, 1)-reducible over C with the parametric equation

$$u' = u^3$$

and with the reduced equation

$$y' = u^2 y\,.$$

For any partition of n, $n = n_1 + \cdots + n_m$, we define $(n_1, \dots, n_m)$-reducible equations inductively as follows; if the equation (21) is $(n_1, \dots, n_{m-2}, n')$-reducible and if the reduced equation for (21), which is of order n', is (n_{m-1}, n_m)-reducible (where $n' = n_{m-1} + n_m$), then (21) is said to be $(n_1, \dots, n_m)$-reducible and the new reduced equation for (21) is of order n_m.

We are now ready to state the characterization of Galois groups for equations reducible to lower order ones.

Theorem 11. *Let k be a differential subfield of ω with the field of constants C. Suppose that a rational equation*

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}) \tag{21}$$

over k of order n is (n_1, n_2)-reducible over k. Let L be the Galois extension field of $K = k(y, y', \dots, y^{(n-1)})$ for (21). *Then there exists a normal subgroup G of* $\mathrm{Gal}\,(L/K)$ *which is isomorphic to a subgroup of the group*

$$G_{n_1,n_2} = \left\{(g_1(z), \dots, g_n(z)) \in \Gamma_{n,p};\ \frac{\partial g_i}{\partial z_j} = 0 \text{ for } i = 1, \dots, n_1, j = n_1 + 1, \dots, n\right\},$$

such that the quotient group $\mathrm{Gal}\,(L/K)/G$ *is isomorphic to a subgroup of the symmetric group of a finite degree.*

For a subgroup G of $\Gamma_{n,p}$, the set of Jacobian matrices

$$J(G) = \left\{\left(\frac{\partial g_i}{\partial z_j}\right)_{i,j=1,\dots,n};\ (g_1, \dots, g_n) \in G\right\}$$

makes a group, which we call the *Jacobian matrix group* of G. The group G_{n_1,n_2} appeared in the above theorem is nothing but the group of which Jacobian matrix group is of the (n_1, n_2)-block triangular form. Thus Theorem 11 asserts that *the Galois group for an (n_1, n_2)-reducible equation over K is (n_1, n_2)-block triangularizable.*

Proof of Theorem 11. Since the equation (21) is (n_1, n_2)-reducible over k, we use the notation in Definition 8. Take a fundamental system of solutions $(\eta_1, \dots, \eta_{n_1})$ of the parametric equation

$$u^{(n_1)} = A\,, \tag{40}$$

take a $t \in \{1, \dots, T\}$ and fix them. For $i = 1, \dots, n_1$, define a function ζ_i by

$$\zeta_i(x, y, \dots, y^{(n-1)}) = \eta_i(x, C_t^{(0)}, \dots, C_t^{(n_1-1)})\,,$$

then we see that it becomes a first integral of (21) by virtue of Definition 8. We denote the right hand member by $\hat{\eta}_i$. Then we have

$$\frac{\partial \zeta_i}{\partial y^{(j)}} = \sum_{s=0}^{n_1-1} \frac{\partial \hat{\eta}_i}{\partial u^{(s)}} \frac{\partial C_t^{(s)}}{\partial y^{(j)}}$$

for $j=0, \ldots, n-1$ and for $i=1, \ldots, n_1$. Let us write down this relations in the form of matrices:

$$
(45)\qquad
\begin{bmatrix}
\dfrac{\partial \zeta_1}{\partial y} & \cdots & \dfrac{\partial \zeta_{n_1}}{\partial y} \\
 & \cdots\cdots\cdots & \\
\dfrac{\partial \zeta_1}{\partial y^{(n_2-1)}} & \cdots & \dfrac{\partial \zeta_{n_1}}{\partial y^{(n_2-1)}} \\
\dfrac{\partial \zeta_1}{\partial y^{(n_2)}} & \cdots & \dfrac{\partial \zeta_{n_1}}{\partial y^{(n_2)}} \\
 & \cdots\cdots\cdots & \\
\dfrac{\partial \zeta_1}{\partial y^{(n-1)}} & \cdots & \dfrac{\partial \zeta_{n_1}}{\partial y^{(n-1)}}
\end{bmatrix}
=
\begin{bmatrix}
\dfrac{\partial C_t^{(0)}}{\partial y} & \cdots & \dfrac{\partial C_t^{(n_1-1)}}{\partial y} \\
 & \cdots\cdots\cdots & \\
\dfrac{\partial C_t^{(0)}}{\partial y^{(n_2-1)}} & \cdots & \dfrac{\partial C_t^{(n_1-1)}}{\partial y^{(n_2-1)}} \\
\dfrac{\partial C_t^{(0)}}{\partial y^{(n_2)}} & \cdots & \dfrac{\partial C_t^{(n_1-1)}}{\partial y^{(n_2)}} \\
 & \cdots\cdots\cdots & \\
\dfrac{\partial C_t^{(0)}}{\partial y^{(n-1)}} & \cdots & \dfrac{\partial C_t^{(n_1-1)}}{\partial y^{(n-1)}}
\end{bmatrix}
\begin{bmatrix}
\dfrac{\partial \hat{\eta}_1}{\partial u} & \cdots & \dfrac{\partial \hat{\eta}_{n_1}}{\partial u} \\
 & \cdots\cdots\cdots & \\
\dfrac{\partial \hat{\eta}}{\partial u^{(n_1-1)}} & \cdots & \dfrac{\partial \hat{\eta}_{n_1}}{\partial u^{(n_1-1)}}
\end{bmatrix}.
$$

Put

$$
M=\begin{bmatrix}
\dfrac{\partial C_t^{(0)}}{\partial y^{(n_2)}} & \cdots & \dfrac{\partial C_t^{(n_1-1)}}{\partial y^{(n_2)}} \\
 & \cdots\cdots\cdots & \\
\dfrac{\partial C_t^{(0)}}{\partial y^{(n-1)}} & \cdots & \dfrac{\partial C_t^{(n_1-1)}}{\partial y^{(n-1)}}
\end{bmatrix},
$$

then we see that $\det M \neq 0$ by (44). Therefore we have

$$
\begin{bmatrix} \dfrac{\partial \hat{\eta}_1}{\partial u} & \cdots & \dfrac{\partial \hat{\eta}_{n_1}}{\partial u} \\ & \cdots\cdots & \\ \dfrac{\partial \hat{\eta}_1}{\partial u^{(n_1-1)}} & \cdots & \dfrac{\partial \hat{\eta}_{n_1}}{\partial u^{(n_1-1)}} \end{bmatrix} = M^{-1} \begin{bmatrix} \dfrac{\partial \zeta_1}{\partial y^{(n_2)}} & \cdots & \dfrac{\partial \zeta_{n_1}}{\partial y^{(n_2)}} \\ & \cdots\cdots & \\ \dfrac{\partial \zeta_1}{\partial y^{(n-1)}} & \cdots & \dfrac{\partial \zeta_{n_1}}{\partial y^{(n-1)}} \end{bmatrix} \tag{46}
$$

from (45). Since $(\eta_1, \ldots, \eta_{n_1})$ is a fundamental system of solutions of (40), the determinant of the left hand member of (46) differs from 0, so that we have

$$
\frac{\partial(\zeta_1, \ldots, \zeta_{n_1})}{\partial(y^{(n_2)}, \ldots, y^{(n-1)})} \neq 0 .
$$

Hence we can take $\zeta_{n_1+1}, \ldots, \zeta_n$ so that $(\zeta_1, \ldots, \zeta_n)$ makes a fundamental system of solutions of (21). Now we put

$$
M_0 = \begin{bmatrix} \dfrac{\partial C_t^{(0)}}{\partial y} & \cdots & \dfrac{\partial C_t^{(n_1-1)}}{\partial y} \\ & \cdots\cdots & \\ \dfrac{\partial C_t^{(0)}}{\partial y^{(n_2-1)}} & \cdots & \dfrac{\partial C_t^{(n_1-1)}}{\partial y^{(n_2-1)}} \end{bmatrix}.
$$

Then, by using (45) and (46), we obtain the important formula:

$$
\begin{bmatrix} \dfrac{\partial \zeta_1}{\partial y} & \cdots & \dfrac{\partial \zeta_{n_1}}{\partial y} \\ & \cdots\cdots & \\ \dfrac{\partial \zeta_1}{\partial y^{(n_2-1)}} & \cdots & \dfrac{\partial \zeta_{n_1}}{\partial y^{(n_2-1)}} \end{bmatrix} = M_0 M^{-1} \begin{bmatrix} \dfrac{\partial \zeta_1}{\partial y^{(n_2)}} & \cdots & \dfrac{\partial \zeta_{n_1}}{\partial y^{(n_2)}} \\ & \cdots\cdots & \\ \dfrac{\partial \zeta_1}{\partial y^{(n-1)}} & \cdots & \dfrac{\partial \zeta_{n_1}}{\partial y^{(n-1)}} \end{bmatrix}. \tag{47}
$$

Take a differential extension field K_1 of K which contains every element of $M_0 M^{-1}$ and let $L_1 = K_1\langle\langle \zeta_1, \ldots, \zeta_n \rangle\rangle$ be the Galois extension of K_1 for (21). Take any $\sigma \in \mathrm{Gal}\,(L_1/K_1)$ and define $(g_1(z), \ldots, g_n(z)) \in \Gamma_{n,p}$ by $\sigma(\zeta_i) = g_i(\zeta_1, \ldots, \zeta_n)$ for $i = 1, \ldots, n$, where p denotes the center of $(\zeta_1, \ldots, \zeta_n)$. As σ leaves every element of $M_0 M^{-1}$ invariant, applying σ to each member of (47), we have

$$
M_0 M^{-1}\begin{bmatrix}
\dfrac{\partial\zeta_1}{\partial y^{(n_2)}} & \cdots & \dfrac{\partial\zeta_{n_1}}{\partial y^{(n_2)}} & \dfrac{\partial\zeta_{n_1+1}}{\partial y^{(n_2)}} & \cdots & \dfrac{\partial\zeta_n}{\partial y^{(n_2)}} \\
 & \cdots\cdots\cdots & & & \cdots\cdots\cdots & \\
\dfrac{\partial\zeta_1}{\partial y^{(n-1)}} & \cdots & \dfrac{\partial\zeta_{n_1}}{\partial y^{(n-1)}} & \dfrac{\partial\zeta_{n_1+1}}{\partial y^{(n-1)}} & \cdots & \dfrac{\partial\zeta_n}{\partial y^{(n-1)}}
\end{bmatrix}
$$

$$
\times\begin{bmatrix}
\dfrac{\partial g_1}{\partial z_1} & \cdots & \dfrac{\partial g_{n_1}}{\partial z_1} \\
 & \cdots\cdots\cdots & \\
\dfrac{\partial g_1}{\partial z_{n_1}} & \cdots & \dfrac{\partial g_{n_1}}{\partial z_{n_1}} \\
\dfrac{\partial g_1}{\partial z_{n_1+1}} & \cdots & \dfrac{\partial g_{n_1}}{\partial z_{n_1+1}} \\
 & \cdots\cdots\cdots & \\
\dfrac{\partial g_1}{\partial z_n} & \cdots & \dfrac{\partial g_{n_1}}{\partial z_n}
\end{bmatrix}
$$

$$
=\begin{bmatrix}
\dfrac{\partial\zeta_1}{\partial y} & \cdots & \dfrac{\partial\zeta_{n_1}}{\partial y} & \dfrac{\partial\zeta_{n_1+1}}{\partial y} & \cdots & \dfrac{\partial\zeta_n}{\partial y} \\
 & \cdots\cdots\cdots & & & \cdots\cdots\cdots & \\
\dfrac{\partial\zeta_1}{\partial y^{(n_2-1)}} & \cdots & \dfrac{\partial\zeta_{n_1}}{\partial y^{(n_2-1)}} & \dfrac{\partial\zeta_{n_1+1}}{\partial y^{(n_2-1)}} & \cdots & \dfrac{\partial\zeta_n}{\partial y^{(n_2-1)}}
\end{bmatrix}
$$

$$
\times\begin{bmatrix}
\dfrac{\partial g_1}{\partial z_1} & \cdots & \dfrac{\partial g_{n_1}}{\partial z_1} \\
 & \cdots\cdots\cdots & \\
\dfrac{\partial g_1}{\partial z_{n_1}} & \cdots & \dfrac{\partial g_{n_1}}{\partial z_{n_1}} \\
\dfrac{\partial g_1}{\partial z_{n_1+1}} & \cdots & \dfrac{\partial g_{n_1}}{\partial z_{n_1+1}} \\
 & \cdots\cdots\cdots & \\
\dfrac{\partial g_1}{\partial z_n} & \cdots & \dfrac{\partial g_{n_1+1}}{\partial z_n}
\end{bmatrix}
$$

Then, using (47) once more, we obtain

$$\left[M_0 M^{-1} \begin{bmatrix} \dfrac{\partial \zeta_{n_1+1}}{\partial y^{(n_2)}} & \cdots & \dfrac{\partial \zeta_n}{\partial y^{(n_2)}} \\ & \cdots\cdots & \\ \dfrac{\partial \zeta_{n_1+1}}{\partial y^{(n-1)}} & \cdots & \dfrac{\partial \zeta_n}{\partial y^{(n-1)}} \end{bmatrix} - \begin{bmatrix} \dfrac{\partial \zeta_{n_1+1}}{\partial y} & \cdots & \dfrac{\partial \zeta_n}{\partial y} \\ & \cdots\cdots & \\ \dfrac{\partial \zeta_{n_1+1}}{\partial y^{(n_2-1)}} & \cdots & \dfrac{\partial \zeta_n}{\partial y^{(n_2-1)}} \end{bmatrix} \right]$$

$$\times \begin{bmatrix} \dfrac{\partial g_1}{\partial z_{n_1+1}} & \cdots & \dfrac{\partial g_{n_1}}{\partial z_{n_1+1}} \\ & \cdots\cdots & \\ \dfrac{\partial g_1}{\partial z_n} & \cdots & \dfrac{\partial g_{n_1}}{\partial z_n} \end{bmatrix} = 0 .$$

If the determinant of the first factor of the left hand member of this equation were 0, we should obtain, by using (47),

$$\frac{\partial(\zeta_1, \ldots, \zeta_n)}{\partial(y, \ldots, y^{(n-1)})} = 0 ,$$

which contradicts the definition of fundamental systems of solutions. Hence we have

$$\left(\frac{\partial g_i}{\partial z_j}\right)_{\substack{i=1,\ldots,n_1 \\ j=n_1+1,\ldots,n}} = 0 .$$

Therefore Gal (L_1/K_1) is isomorphic to a subgroup of G_{n_1,n_2}.

We have been interested in the Galois extension $L = K\langle\langle\zeta_1, \ldots, \zeta_n\rangle\rangle$ of K for (21) and the Galois group Gal (L/K). By virtue of (47) we see that every element of $M_0 M^{-1}$ belongs to L, and moreover, for any parametric root $(C_{t'}^{(0)}, \ldots, C_{t'}^{(n_1-1)})$, every element of the corresponding matrix $M'_0 M'^{-1}$ also belongs to L. Let K_1 be the differential extension field of K which is obtained by adjunction of every element of the matrices $M_0 M^{-1}$ for every parametric root for ((40), (41)). Then we have

$$K \subset K_1 \subset L .$$

By the definition parametric roots are solutions of the system of algebraic equations (42) over K, so that every differential isomorphism over K sends a parametric root to another parametric root. Therefore K_1 is a normal extension of K (see Definition 6 in §4), and hence, by Theorem 8 in §4, Gal (L/K_1) is a normal subgroup of Gal (L/K). On the other hand the group of all differential automorphisms of K_1 over K is isomorphic to a subgroup H of the symmetric group of degree T. Thus the quotient group

Gal (L/K)/Gal (L/K_1) is isomorphic to H. As we have seen that Gal (L/K_1) is isomorphic to a subgroup of G_{n_1,n_2}, this completes the proof.

We can show the following by induction.

Corollary. *The Galois group for an* $(n_1, \ldots, n_m)$*-reducible equation over* K *is* $(n_1, \ldots, n_m)$*-block triangularizable.*

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