π_1 and H_1

Let X be a path connected space with $p \in X$. Then $\pi_1(X,p)$ is the fundamental group of X based at p. It is homotopy equivalence classes of loops based at p with the group operation induced by concatination of paths. $H_1(X)$ is the first singular homology group of X. We will construct a homomorphism from $\pi_1(X,p)$ onto $H_1(X)$, called the Hurewicz homomorphism, and then show that it is equivalent to abelianization. That is,

$$H_1(X) \approx \frac{\pi_1(X,p)}{[\pi_1(X,p), \pi_1(X,p)]}.$$

Our presentation is based on John Lee's book Intro. to Topological Manifolds, pages 351-355.

Let $f:[0,1] \to X$ be cont. with $f(0)=f(1)=p$. Then f is both a loop based at p and a singular 1-simplex. Since $\partial f = 0$, f is a singular 1-cycle and so represents a homology class in $H_1(X)$ as well as a homotopy class in $\pi_1(X,p)$.

Let $[f]_\pi$ denote the homotopy class of f in $\pi_1(X,p)$ and $[f]_H$ denote the homology class of f in $H_1(X)$.

Def The Hurewicz homomorphism, $\gamma: \pi_1(X,p) \to H_1(X)$, is given by $\gamma([f]_\pi) = [f]_H$.

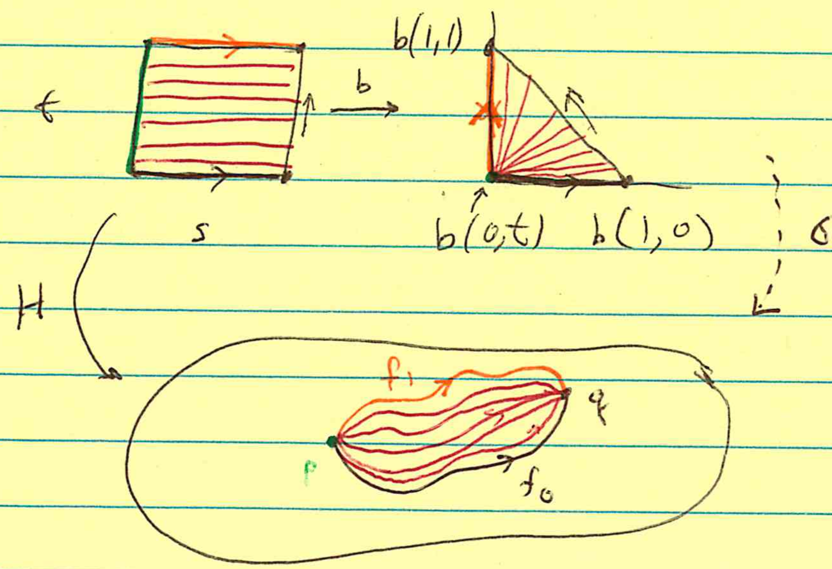
We will show that γ is well defined, that it is an onto homomorphism, and finally that it is equivalent to abelianization.

Well Defined Suppose f_0 and f_1 are homotopic paths in X from p to q. (For now we are not assuming these are loops.) Then f_0 and f_1 considered as 1-chains are homologous.

pf We claim $f_0 - f_1$ is a boundary and thus f_0 is homologous to f_1. Let $H : I \times I \to X$ be a homotopy taking f_0 to f_1 rel. end points.

$$H(s,0) = f_0(s)$$
$$H(s,1) = f_1(s)$$
$$H(0,t) = p$$
$$H(1,t) = q$$

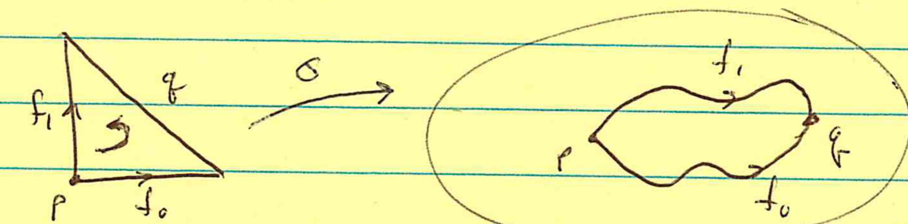
Let $b : I \times I \to \Delta_2$ be $b(s,t) = (s - st, st)$.



You should check that b is a quotient map (see Lemma 4.50 (Closed Map Lemma) on page 100 of Lee's textbook or Thm 22.1 on page 140 of Munkres "Topology" second ed.)

Notice if two points in $I \times I$ are identified by b, then they are also identified by H. Thus, $\exists$ cont. $\sigma : \Delta_2 \to X$ s.t. $\sigma \circ b = H$. (see Thm 3.73 (pg 72) in Lee's textbook or Thm 22.2 (pg 142) in Munkres' textbook

Now $\partial\sigma$ can be computed.



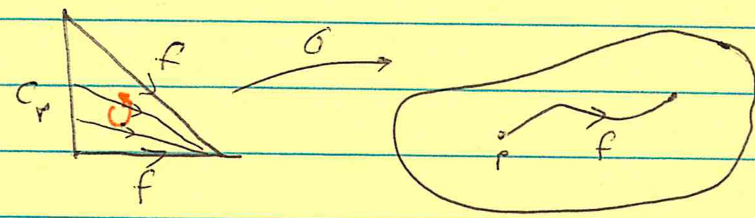
$$\partial\sigma = f_0 - \underbrace{c_q}_{\text{constant map to } q} + (-f_1)$$

Notice, $c_q \sim 0$, i.e. c_q is nulhomologous.
$(\partial c_q = q - q = 0)$ Thus $\partial\sigma \sim f_0 - f_1$ as required. ∎

Fact Let $f: I \to X$ be a path and denote the path given by $f(1-t)$ by $\bar{f}$. Then

$$[\bar{f}]_H = -[f]_H$$

Pf Define a 2-simplex $\sigma: \Delta_2 \to X$ by $\sigma(x,y) = f(x)$.



Let $p = f(0)$. The y-axis in Δ_2 is mapped to p. Each nonvertical segment shown is mapped to the image of f.

Thus $\partial\sigma = f + \bar{f} + \bar{c}_p$. Thus $[f + \bar{f}]_H = 0$. The result follows. ∎

Fact Let f and $g : I \to X$ be paths with $p = f(0)$ and $g(0) = f(1)$. Recall

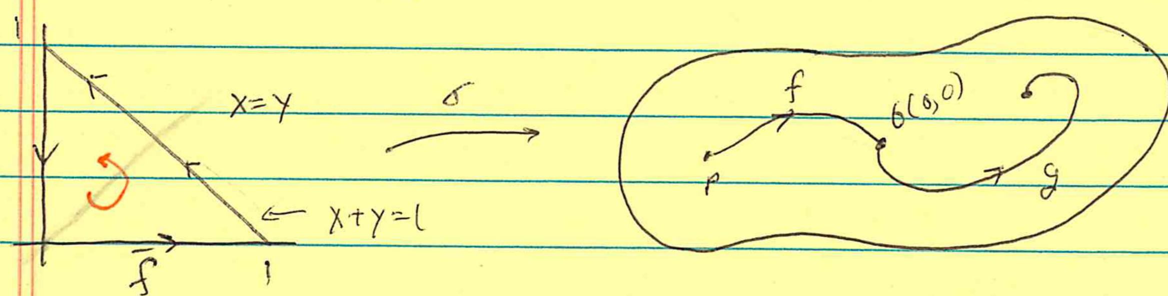
$$(f*g)(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

We know that $[f*g]_\pi = [f]_\pi [g]_\pi$. We claim that

$$[f*g]_H = [f]_H + [g]_H.$$

Pf We will defined a 2-simplex $\sigma : \Delta_2 \to X$ whose boundary is $f*g - (f+g)$.

Let $$\sigma(x,y) = \begin{cases} f(y-x+1) & y \le x \\ g(y-x) & y \ge x \end{cases}$$
for $(x,y) \in \Delta_2$. This is well defined since when $x=y$ we get $f(1)=g(0)$. And σ is cont. by the Pasting Lemma.



When $y=0$, $y \le x$ and $f(0-x+1) = \bar{f}(x)$
When $x=0$, $y \ge x$ and $g(y-0) = g(y)$
When $x+y=1$, we have the following

$$\sigma(1-y, y) = \begin{cases} f(2y) & y \leq 1-y \text{ or } y \leq \frac{1}{2} \\ g(2y-1) & y \geq 1-y \text{ or } y \geq \frac{1}{2}. \end{cases}$$

$$= (f * g)(y).$$

Thus $\partial \sigma = \bar{f} + f * g + \bar{g}$. Thus $f * g$ is homologous to $-(f+g)$. The result follows. $\boxtimes$

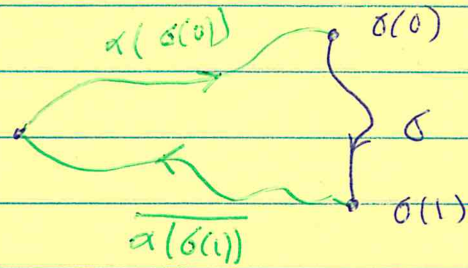
<u>γ is onto</u>

Recall $p \in X$ is our base point. $\forall x \in X$ let $\alpha(x)$ be a specified path from p to x, with $\alpha(p) = c_p$, the constant path at p. Then α extends uniquely to a group homomorphism

$$\alpha : S_0(X) \to S_1(X).$$

Let $\sigma : I \to X$ be any sing. 1-simplex. Define

$$\sigma_\alpha = \alpha(\sigma(0)) * \sigma * \overline{\alpha(\sigma(1))}.$$



Now, $\gamma([\sigma_\alpha]_\pi) = [\sigma_\alpha]_H = [\alpha(\sigma(0))]_H + [\sigma]_H - [\alpha(\sigma(1))]_H$

$$= [\sigma]_H - [\alpha(\sigma(1) - \sigma(0))]_H = [\sigma]_H - [\alpha(\partial\sigma)]_H.$$

Finally we let c be my 1-cycle; thus $[c]_H \in H_1(X)$

Write out c in terms of 1-simplexes.

$$c = \sum_{i=1}^{m} n_i \sigma_i.$$

Let $[f]_\pi \in \pi_1(X, p)$ be given by

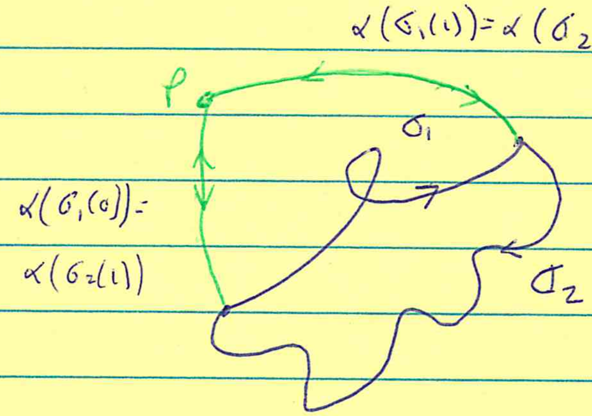
$$f = (\sigma_1)_\alpha^{n_1} * (\sigma_2)_\alpha^{n_2} * \cdots * (\sigma_m)_\alpha^{n_m}.$$

Then, $\gamma([f]_\pi) = \sum_{i=1}^{m} n_i \left([\sigma_i]_H - [\alpha(\partial \sigma_i)]_H \right)$

$$= [c]_H - [\alpha(\partial c)]_H = [c_H] \in H_1(X).$$

Thus γ is onto.

Example



$c = \sigma_1 + \sigma_2.$

Now that we have shown that $\gamma : \pi_1(X,p) \to H_1(X)$ is an onto homomorphism we are ready to prove that γ is abelianization. We review some algebra first.

Def Let G be a group. For $a, b \in G$, let $[a,b] = a^{-1}b^{-1}ab$. Let $[G,G]$ be the subgroup generated by

$$\{ [a,b] \mid a, b \in G \}.$$

Let

$$Ab(G) = G / [G,G].$$

This is called the abelianization of G.

It is easy to show that $Ab(G)$ is in fact abelian. It is a standard exercise to show that if $h : G \to G'$ is a homomorphism onto an abelian group G' then $[G,G] \subset \ker h$ and h has a unique factorization through $Ab(G)$; that is $\exists !\, h' : Ab(G) \to G'$ s.t.

$$\begin{tikzcd} G \arrow[rr, "h"] \arrow[dr, "Ab"'] & & G' \\ & Ab(G) \arrow[ur, "h'"'] & \end{tikzcd}$$

commutes.

Therefore $[\pi_1(X,p), \pi_1(X,p)] \subset \ker \gamma$. We need to show that they are equal.

Let $A = Ab(\pi_1(X,p))$. For a loop f in X based at p let $[f]_A$ be the eq. class of $[f]_\pi$ in A.

For any singular 1-simplex $\sigma : I \to X$, let $\beta(\sigma) = [\sigma_\alpha]_A \in A$. We can extend β to a group homomorphism

Remind yourself what σ_α is.

$$\beta : S_1(X) \to A$$

Since A is abelian and $S_1(X)$ is free abelian. We will show that β takes 1-boundaries to the identity $Id_A \in A$, That is $B_1(X) \subset \ker \beta$.

Assume this has been done. Let $[f]_\pi \in \ker \gamma$. Thus, $[f]_H = 0$, the identity in $H_1(X)$. This means f is a 1-cycle that bounds a 2-chain, $f \in B_1(X)$. Thus $\beta(f) = Id_A$. But also

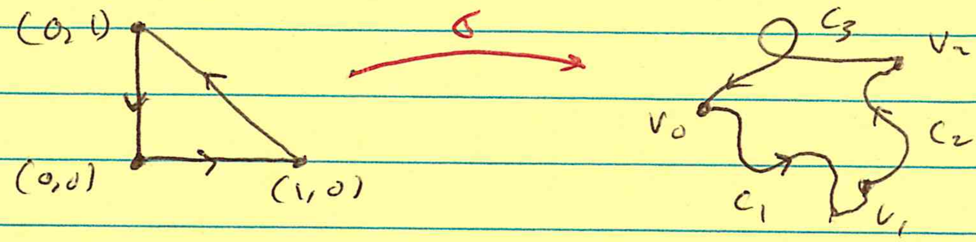
$$\beta(f) = [c_p * f * \bar{c}_p]_A = [f]_A.$$

Thus, $[f]_A = Id_A$ and so $[f]_\pi \in \ker Ab$. Therefore $\ker Ab = \ker \gamma$.

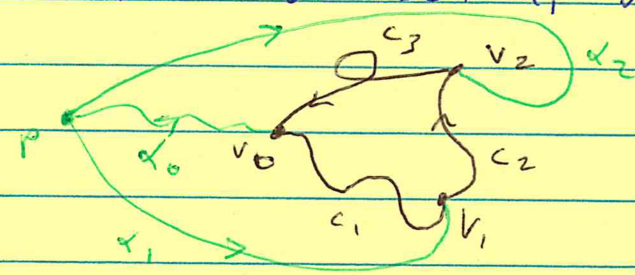
Proof that $B_1(X) \subset \ker \beta$

Let $\sigma : \Delta_2 \to X$ be a sing. 2-simplex. Define $v_0 = \sigma(0,0)$, $v_1 = \sigma(1,0)$, $V_2 = \sigma(0,1)$ and let

$$c_1 : [0,1] \to X \text{ be } c_1(t) = \sigma(t,0),$$
$$c_2 : [0,1] \to X \text{ be } c_2(t) = \sigma(1-t,t), \text{ and}$$
$$c_3 : [0,1] \to X \text{ be } c_3(t) = \sigma(0,1-t).$$



Now $\partial\sigma = c_1 + c_2 + c_3$. Let $\alpha_i = \alpha(v_i)$, $i = 0, 1, 2$.



Thus, $\beta(\partial\sigma) = \beta(c_1) \cdot \beta(c_2) \cdot \beta(c_3)$ [$\cdot$ = the group operation in A.]

$$= [\alpha_0 * c_1 * \bar{\alpha}_1 * \alpha_1 * c_2 * \bar{\alpha}_2 * \alpha_2 * c_3 * \bar{\alpha}_0]_A$$

$$= [\alpha_0 * c_1 * c_2 * c_3 * \bar{\alpha}_0]_A$$

Since $c_1 * c_2 * c_3$ is a loop based at v_0 and the domain of Δ_2 can be continuously retracted to $(0,0)$, we have that $c_1 * c_2 * c_3 \sim c_{v_0}$. Thus, $\beta(\partial\sigma) = [\alpha_0 * c_{v_0} * \bar{\alpha}_0]_A = [c_p]_A = Id_A$.

Since any 2-chain is a sum of 2-simplexes we have $B_1(X) \subset \ker\beta$. $\blacksquare$