The Game of Twenty Questions with noisy answers. Applications to Fast face detection, micro-surgical tool tracking and electron microscopy

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The game of 20 questions

$\mathcal{X}$ is the parameter space ($\{1, \ldots, 1,000,000\}$,$\{A, B\}$,unit interval, unit square, object pose).
$X_1 \in \mathcal{X}$, $X_2 \in \mathcal{X}$, ..., $X_K \in \mathcal{X}$ are the parameters, or targets, or object poses
Ask N questions, $0 \leq n \leq N-1$: select $A_n \subset \mathcal{X}$.
The answer is $Z_n = 1_{X_1 \in A_n} + \ldots + 1_{X_K \in A_n}$ is the number of targets within A_n.
Z_n is not observed
Instead, we observe Y_{n+1}, (a noisy version of Z_n)

$$Y_{n+1} = \begin{cases} \sim f_1 & \text{if } Z_n = 1 \\ \sim f_0 & \text{if } Z_n = 0 \end{cases}$$

where f_0 anf f_1 are probability distributions.
The goal is to choose N questions such that $(X_1, \ldots, X_K)$ can be estimated as accurately as possible after observing the answers.

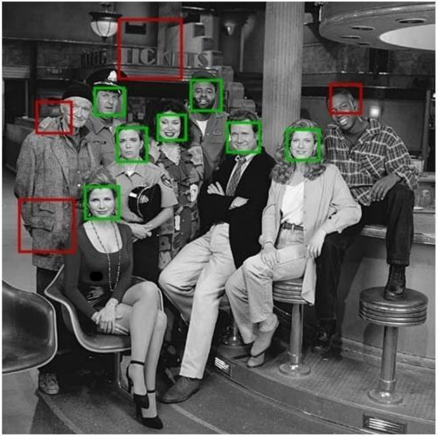
Past and current research projects (applications)

- Road tracking [1996] (Donald Geman, BJ);
- Face detection [2010], surgical tool tracking [2012], electron microscopy [2013] (BJ, Raphael Sznitman);
- table setting analysis [current] (Donald Geman, Yoruk Erdem, Ehsan Jahangiri, BJ, Laurent Younes);
- Root finding using noisy measurements [current] (Peter Frazier, Shane Henderson, BJ)
- Stochastic simulations: screening [current] (Peter Frazier, BJ)
- Experiments in visual perception [current] (Jonathan Flombaum, Heeyeon Im, BJ)

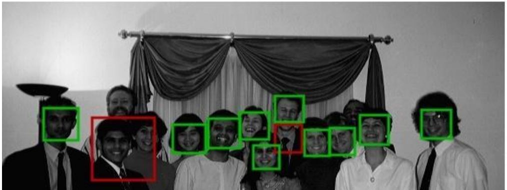
Road Tracking from Remote Sensing Images

Face detection

7150 / 577729 = 0.012



16006 / 1000000 = 0.016



Tool tracking, electron microscopy

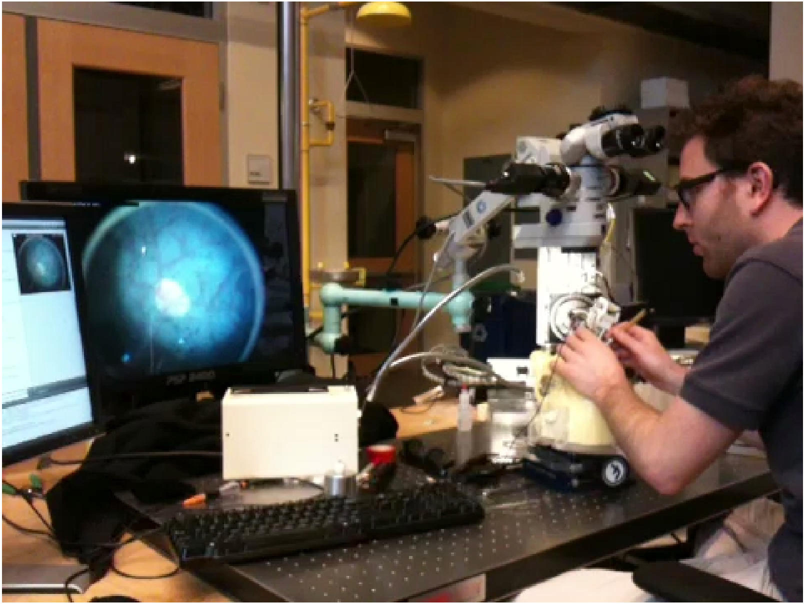
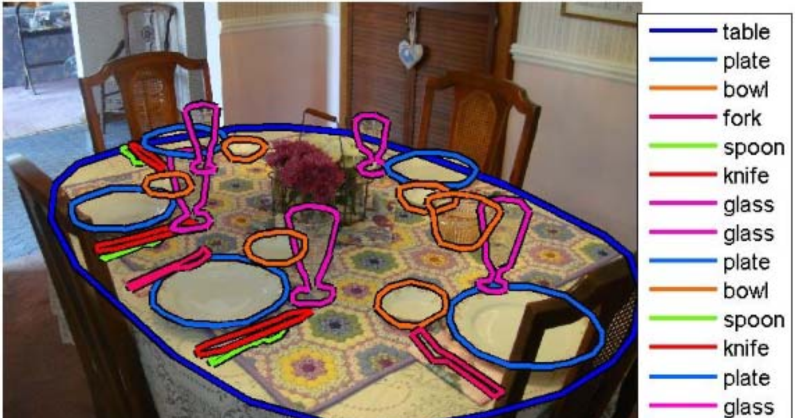


Table setting



Past and current research projects (theory)

- 20 questions as a model for perception [1995] (Donald, Geman, BJ)
- Bayesian optimal policies. Entropy loss. Single target. Noisy answers [2012] (Peter Frazier, Raphael Sznitman, BJ)
- Metric loss functions [current] (Peter Frazier, Shane Henderson, Rolf Waeber, Avner Bar-Hen, BJ)
- Multiple objects. Entropy Loss. [current] (Peter Frazier, Weidong Han, BJ)

Today's talk:

- Specify the mathematical game
- Review a Frequentist result
- Bayesian pont of view with entropy loss
- Play the game
- Probabilistic bisection policy
- Dyadic policy in 1 dimension
- Detecting multiple objects simultaneously
- Application to face detection
- application to tool tracking

References

1. D. Geman and B. Jedynak, “An active testing model for tracking roads from satellite images,” *IEEE Transactions on Pattern Analysis and Machine Intelligence* , 18(1), pp. 1-14, 1996.
2. Raphael Sznitman and Bruno Jedynak, ”Active Testing for Face Detection and Localization”,*IEEE PAMI* 32(10), 2010.
3. Bruno Jedynak, Peter L. Frazier and Raphael Sznitman, ”Twenty Questions with Noise: Bayes Optimal Policies for Entropy Loss”, Journal of Applied Probability, 49(1), March 2012.
4. Sznitman, Raphael; Richa, Rogerio; Taylor, Russell; jedynak, bruno; Hager, Gregory D. ”Unified detection and tracking of instruments during retinal microsurgery”, *IEEE PAMI, 2012 Oct 1*
5. R. Sznitman, A. Lucchi, P. I. Frazier, B. Jedynak and P. Fua, ”An Optimal Policy for Target Localization with Application to Electron Microscopy”, *International Conference on Machine Learning, 2013*

Unidimensional Binary Symmetric Noise

$X \in [0; 1]$ is the parameter of interest. (1D case)
Ask questions: At time $n, 0 \leq n \leq N - 1$, select x_n, $0 \leq x_n \leq 1$
$Z_n = 1_{X \leq x_n}$ is the "true" answer
Z_n is not observed
Instead, we observe Y_{n+1}, a noisy version of Z_n

$$Y_{n+1} = \begin{cases} Z_n & \text{with probability } 1 - \epsilon \\ 1 - Z_n & \text{with probability } \epsilon \end{cases}$$

with $0 \leq \epsilon \leq 1$
Y_{n+1} is Binary and the noise is symmetrical

Frequentist analysis

Choose N questions $x_0, \ldots, x_{N-1}$ and propose an estimator $\hat{X}_N$ of X in order to minimize

$$sup_{X \in [0;1]} E|\hat{X}_N - X|$$

Valid (or adapted) policy: For each time i, the sample locations x_i depends only on the available information at time i.
Non-Adaptive policy: Valid policies for which the sample locations x_i can be chosen simultaneously.

Frequentist analysis

noseless case $\epsilon = 0$

1. restricted to *non-adaptive* policies,

$$sup_{X \in [0;1]} E|\hat{X}_n - X| \leq \frac{1}{2(n+1)}$$

   achieved by choosing $x_0, \ldots, x_{N-1}$ regularly spaced over [0; 1]

$$\{\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}\}$$

2. however,

$$sup_{X \in [0;1]} E|\hat{X}_n - X| \leq \frac{1}{2^{n+1}}$$

   achieved by choosing the **dichotomy policy** which is adaptive.

Frequentist analysis

noisy case $0 < \epsilon < 1$

$$sup_{X \in [0;1]} E|\hat{X}_n - X| \leq 2(\frac{1}{2} + \sqrt{\epsilon(1-\epsilon)})^{n/2}$$

using the **probabilistic bisection policy**. [Please wait a few slides :-)]
Exponential error decay behavior
Is this policy optimal for this criterium ?

Ref: Active Learning and Sampling, Chapter 8, Rui Castro and robert Nowak.

Bayesian analysis

X is random, with density p_0 over $[0;1]$

Consider more general questions of the form $X \in A$? where A is a Lebesgue measurable subset of $[0;1]$

More general answers. f_0 and f_1 are point mass functions or densities

$$Y_{n+1} \sim \begin{cases} f_1 & \text{if } X \in A_n \\ f_0 & \text{if } X \notin A_n \end{cases}$$

Previously $A_n = [0; x_n]$, f_1 is *Bernoulli*$(1-\epsilon)$ and f_0 is *Bernoulli*(ϵ)

Bayes rule

If at time n, the history of answers is $B_n = \{Y_1 = y_1, \ldots, Y_n = y_n\}$, the posterior is p_n, the question is "$X \in A_n$?" and the answer is $Y_{n+1} = y_{n+1}$ then p_{n+1} is the conditional distribution of X at time $n+1$. Using Bayes rule,

$$
\begin{aligned}
p_{n+1}(x) &= p(x|Y_{n+1} = y_{n+1}, B_n) \\
&\alpha\ P(Y_{n+1} = y_{n+1}|X = x, B_n)p(x|B_n) \\
&\alpha\ P(Y_{n+1} = y_{n+1}|X = x)p_n(x) \\
&\alpha\ p_n(x)\begin{cases} f_1(y_{n+1}) & \text{if } x \in A_n \\ f_0(y_{n+1}) & \text{if } x \notin A_n \end{cases}
\end{aligned}
$$

Controlled Markov chain

As questions are asked and answered, the density of X "evolves" becoming p_1, p_2, ... successive posterior densities.
At time n,
The state is the density p_n.
The control is A_n. It affects the transition probability from p_n to p_{n+1}.
The Markov property comes from the fact that the noise is memoryless.
The functional of interest for today is the Shannon Differential Entropy

$$H(p_n) = -\int_0^1 p_n(x) \log_2 p_n(x) dx$$

If $p_n(x) = \frac{1}{b-a}, a \leq x \leq b$, then $2^{H(p_n)} = b - a$

Let’s play the game ...

Define the **value function** as

$$V(p,n) = inf_{\pi} E^{\pi}[H(p_N)|p_n = p], n = 0, \ldots, N$$

where the infimum is taken over all valid policies π.
Let’s play

Bellman optimality principle

Define the **value function** as

$$V(p,n) = inf_{\pi} E^{\pi}[H(p_N)|p_n = p], n = 0,\dots,N$$

where the infimum is taken over all valid policies π.
Principe of optimality: ”An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.” Richard Bellman.
Bellman recursion

$$V(p,n) = inf_{A_n} E[V(p_{n+1}, n+1)|A_n, p_n = p], n = 0,\dots,N$$

and a policy which chooses an x_n attaining the minimum above is optimal.

Greedy policy

Consider minimizing the value function at time N-1.

$$
\begin{aligned}
V(p, N-1) &= inf_{A_{N-1}} E[V(p_N, N)|A_{N-1}, p_{N-1} = p] \\
&= inf_{A_{N-1}} E[H(p_N))|A_{N-1}, p_{N-1} = p] \\
&= inf_{A_{N-1}} (H(p_{N-1}) - I(X, Y_N|A_{N-1}, p_{N-1}) \\
&= H(p_{N-1}) - sup_{A_{N-1}} I(X, Y_N|A_{N-1}, p_{N-1})
\end{aligned}
$$

where $I(X, Y_N|A_{N-1}, p_{N-1})$ denotes the **Mutual Information** between X and Y_N when the density of X is p_{N-1} and the control is A_{N-1}

Computation of the Mutual Information

notate $p(A) = \int_A p(x)dx$ and $u = p_{N-1}(A_{N-1})$

$$I(X, Y_N|A_{N-1}, p_{N-1}) = H(Y_N|A_{N-1}, p_{N-1}) - H(Y_N|X, A_{N-1}, p_{N-1})$$

$$= H(uf_1 + (1-u)f_0) - uH(f_1) - (1-u)H(f_0)$$

$$= \phi(u)$$

ϕ is concave , $\phi(0) = \phi(1) = 0$ and
The **channel capacity** is

$$C = C(f_0, f_1) = max_u \phi(u)$$

Optimal policy

Theorem: The policy for which A_n is such that

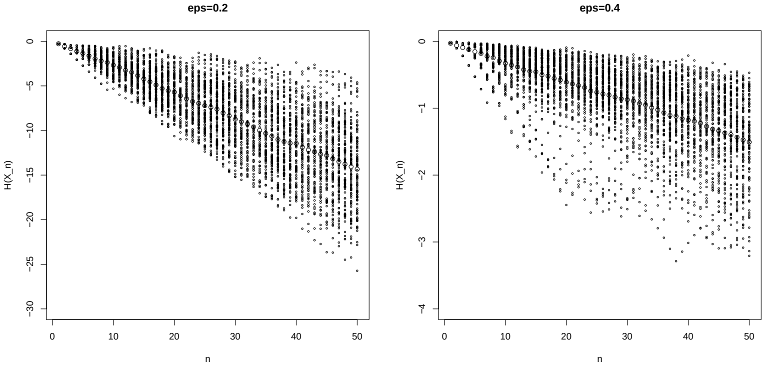
$$p_n(A_n) = u^* = \arg\max_u \phi(u)$$

is optimal and the value function is

$$V(p_n, n) = H(p_n) - (N - n)C$$

Proof: check that this policy verifies Bellman's recursion. The main point is that the **expected** gain in Entropy C realized at each step is independent of the state p_n
When $\phi(u) = \phi(1 - u)$ then $u^* = 1/2$
When moreover $A_n = [0; x_n]$, x_n is the **median** of p_n. This policy is the **probabilistic bisection policy** used in the frequentist analysis

Simulation of the probabilistic bisection policy



The process $H(p_n)$ for the binary symmetric channel. p_0 is $\mathrm{Uniform}([0;1])$. **Left:** $\epsilon = 0.2$ $C = 0.28$ **Right:** $\epsilon = 0.4$ $C = 0.03$

The Diadic policy

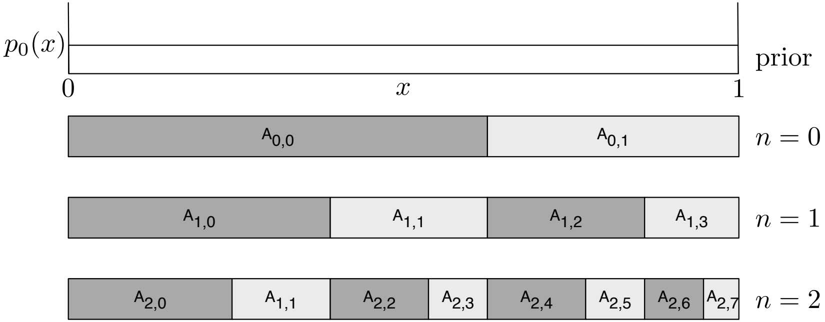
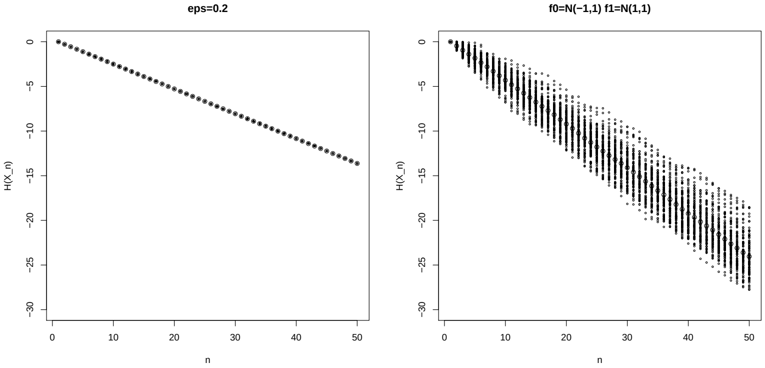


Illustration of the dyadic policy when p_0 is uniform on $[0, 1]$ and $u^* = 5/8$. The prior is displayed on top. Below, the sets $A_{n,k}$ are illustrated for $n = 0, 1, 2$. Each question A_n is the union of the dark grey subsets $A_{n,k}$ for that value of n.

Simulation of the dyadic policy



The process $H(p_n)$ p_0 is $\mathrm{Uniform}([0;1])$. **Left:** Binary symmetric channel $\epsilon = 0.2$ $C = 0.28$ **Right:** Normal channel $C = 0.47$

Properties of the dyadic policy

$$H(p_{n+1}) - H(p_n) = -D\left(B\left(\frac{u^* f_1(Y_{n+1})}{u^* f_1(Y_{n+1}) + (1-u^*) f_0(Y_{n+1})}\right), B(u^*)\right) \tag{1}$$

where Y_n is a sequence of i.i.d. random variables with pmf or density the mixture $u^* f_1 + (1-u^*) f_0$

$$\frac{H(p_n)}{n} \to -C \text{ a.s.} \tag{2}$$

and

$$\frac{H(p_n) + nC}{\sqrt{n}} \to N(0, \sigma^2) \text{ in distribution,} \tag{3}$$

Playing in 2 dimensions

$X^* = (X_1^*, X_2^*)$. Pitfall with minimzing $E[H(p_N)]$
Ex: $X^* \sim U([0; s_1] \times [0; s_2])$, then $H(X^*) = \log(s_1) + \log(s_2)$ which can be arbitrarily small with, say, $s_1 = 1$.
Instead , notate $H_1(p_N) = H(\int p_N(., u_2)du_2)$, similarly for $H_2(p_N)$

$$inf_\pi E^\pi[max(H_1(p_N), H_2(p_N))|p_0 = p]$$

Optimal policy seems out of reach. Instead,

$$V(p) = inf_\pi \liminf_{N \to +\infty} \frac{1}{N} E^\pi[max(H_1(p_N), H_2(p_N))|p_0 = p]$$

Playing in 2 dimensions with questions on the marginals

For further simplification, we assume that questions concern only one coordinate. That is, the sets queried are either of type 1, $A_n = B \times \mathbb{R}$ where B is a finite union of intervals of $\mathbb{R}$, or alternatively of type 2, $A_n = \mathbb{R} \times B$. In each case, we assume that the response passes through a memoryless noisy channel with densities $f_0^{(1)}$ and $f_1^{(1)}$ for questions of type 1, and $f_0^{(2)}$ and $f_1^{(2)}$ for questions of type 2. Let C_1 and C_2 be the channel capacities for questions of type 1 and 2 respectively. We also assume that p_0 is a product of its marginals. This guarantees that p_n for all $n > 0$ remains a product of its marginals and that only one marginal distribution is modified at each point in time.

Playing in 2 dimensions with questions on the marginals

The following policy:
At step n, choose the type of question at random, choosing type 1 with probability $\frac{C_2}{C_1+C_2}$ and type 2 with probability $\frac{C_1}{C_1+C_2}$.
Then, in the dimension chosen, choose the subset to be queried according to the 1-dimensional dyadic policy.
Is optimal. Moreover, it verifies a CLT:

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left[\max(H_1(p_n), H_2(p_n)) + \frac{C_1 C_2}{C_1 + C_2} n\right] =^D \frac{\max\left(\sigma_1\sqrt{C_2}Z_1, \sigma_2\sqrt{C_1}Z_2\right)}{\sqrt{C_1 + C_2}}.$$

Here, Z_1 and Z_2 are independent standard normal random variables, and σ_i^2 is the variance of the increment of $H_i(p_{n+1}) - H_i(p_n)$ when measuring type i

Character localization

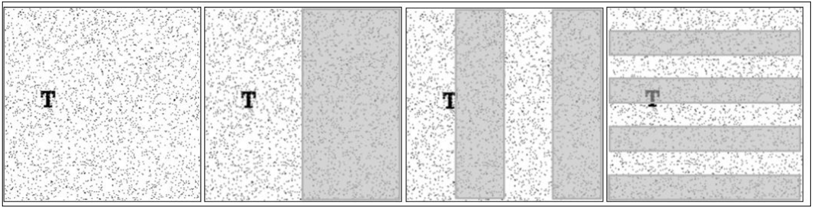


Figure : From left to right: Example of an image containing the character "T". Examples of subset-based questions. In each image, we show the queried region by the gray area.

Character localization

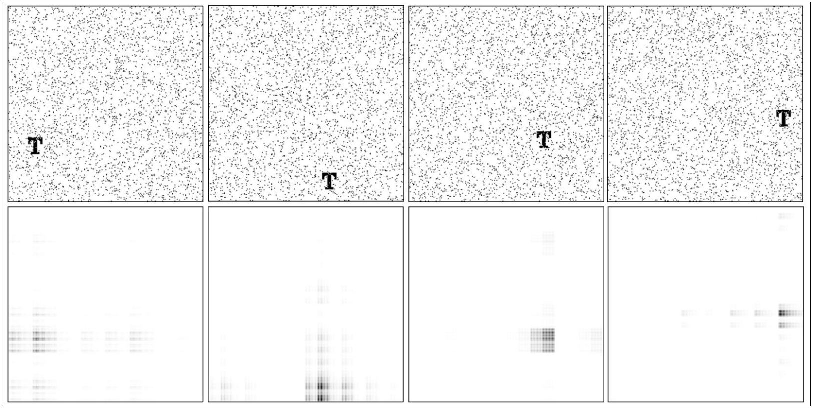


Figure : Pixel Reordering: (*top*) Example images from the test set. (*bottom*) Corresponding ℓ-images. Dark regions indicate pixels more likely to contain the character, while light regions are less likely.

Character localization

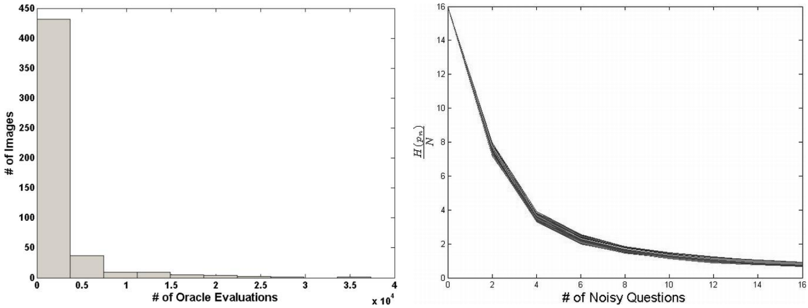


Figure : Noise-free evaluations and convergence in entropy. (a) The distribution of number of noise-free evaluations needed to locate the target character. (b) Plot of $H(p_n)/n$ as a function of n. Each line corresponds to one image, with $H(p_n)/n$ plotted over $n = 1, \ldots, 16$.

Character localization

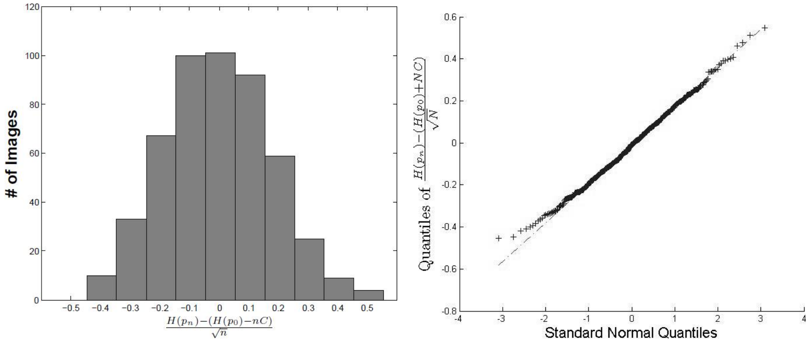


Figure : Central Limit Theorem: (a) Distribution of $\frac{H(p_n)-(H(p_0)-nC)}{\sqrt{n}}$, with mean -0.01. The distribution is close to Gaussian as the Q-Q plot (b) shows.

Detecting two objects

Assume two targets $X = \{X_1, X_2\}$, are independent. We also assume that they have the same prior distribution, notated p_0. For specificity, let's assume that both X_1 and X_2 belong to the interval $[0;1]$. A series of N questions are asked to locate X_1, X_2. The first question is indexed by the set A_0. The first answer is

$$Y_1 = 1_{A_0}(X_1) + 1_{A_0}(X_2)$$

The n^{th} question is indexed by A_{n-1} and has answer

$$Y_n = 1_{A_{n-1}}(X_1) + 1_{A_{n-1}}(X_2)$$

Policies

We define the value function in the usual way

$$V(p, n) = \inf_{\pi} E^{\pi}[H(p_N)|p_n = p], n = 0, \ldots, N \tag{4}$$

where p_n is the posterior distribution over $\{X_1, X_2\}$ after observing the answers to the first n questions. We also define the greedy policy π_G. It is a sequential policy which we define iteratively.

$$A_0 = \arg\min_{A} E[H(p_1)|p_0, A_0 = A] \tag{5}$$

If the first n questions are indexed by $A_0, \ldots, A_{n-1}$ and the answers are $Y_1, \ldots, Y_n$, then A_n is chosen such that

$$A_n = \arg\min_{A} E[H(p_{n+1})|p_n, A_n = A] \tag{6}$$

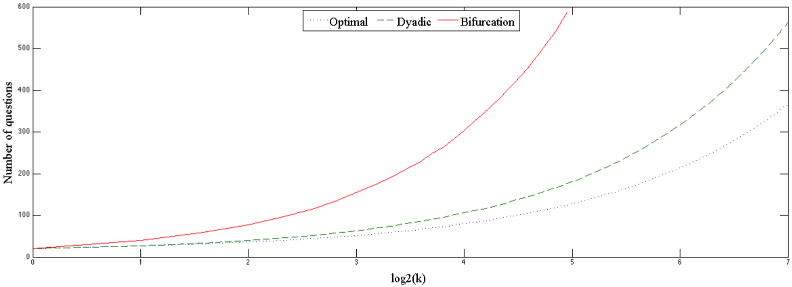
Finally, We note π_D the diadic policy.

Result for the noiseless case

$$
\begin{aligned}
H(p) - \log_2(3)N &< V(p,0) = \inf_{\pi} E^{\pi}[H(p_N)|p_0 = p] \\
V(p,0) &\leq E^{\pi_G}[H(p_N)|p_0 = p] \\
E^{\pi_G}[H(p_N)|p_0 = p] &\leq E^{\pi_D}[H(p_N)|p_0 = p] \\
E^{\pi_D}[H(p_N)|p_0 = p] &= H(p) - 1.5N
\end{aligned}
$$

Interpretation: Learning 20 bits from each object requires only about $\frac{40}{1.5} \sim 27$ questions, compared to 40 if we were to learn first about X_1 and then about X_2.

Multiple objects. No noise.



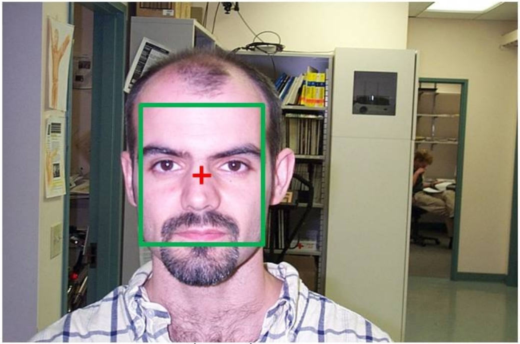
From Idealistic Setting to Computer Vision

- Finite number of identifiable poses (i.e finite number of images)
- Finite set of questions
- Very strong Classifiers available (*i.e.* Oracles)
- Only available when evaluating very small set of poses (*e.g.* single pixel).
- Can create weak classifiers (*i.e.* providing noisy answers for a collection of poses)

Face Localization and Detection

Let us find the center of a face in an image [1]:

- Let $X^* = (X_1^*, X_2^*)$ be a discrete random variable that defines the face center.
- Let $p_0 \sim U([0, M] \times [0, N])$



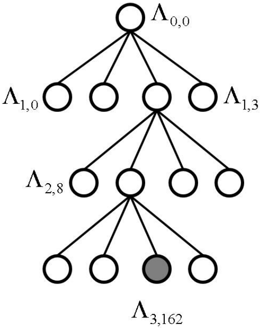
(X_1^*, X_2^*)

[1] Sznitman, Jedynak. **Active Testing for face localization and detection.** *PAMI*, 2010.

Search Space Decomposition

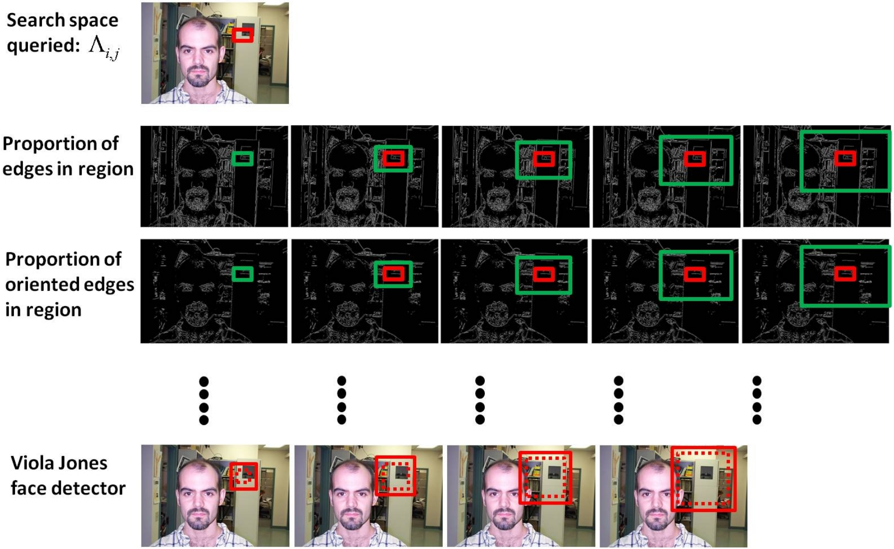
- Let Λ be a regular decomposition of the pose space into quadrants, such that

$$\Lambda = \{\Lambda_{i,j}, i = 0, ..., d, j = 0, ..., 4^{i-1} - 1\}$$



Face Questions

- $\mathcal{K} = 29$ questions available at each node



Active Testing Algorithm

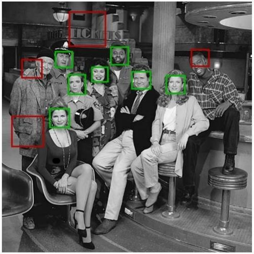
1. Initialize node to query: $i = 0, j = 0$
2. Initialize question type: $k = 0$
3. **Repeat**
   3.1 Test: $y = X_{i,j}^{k}$
   3.2 Update $p_{t+1}(\cdot)$ from y and $p_t(\cdot)$
   3.3 Choose next Question:

$$\{i, j, k\} = \arg\max_{i,j,k} \mathrm{MI}(i, j, k)$$

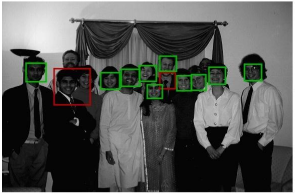
4. **Until** $H(p_{t+1}) < \epsilon$ or a fixed number of iterations.

MIT+CMU Face Dataset:

7150 / 577729 = 0.012

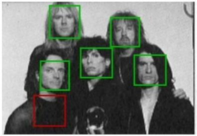


16006 / 1000000 = 0.016

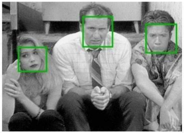


32324 / 881600 = 0.036

13391 / 307200 = 0.043

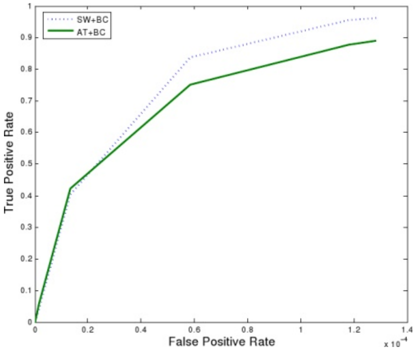


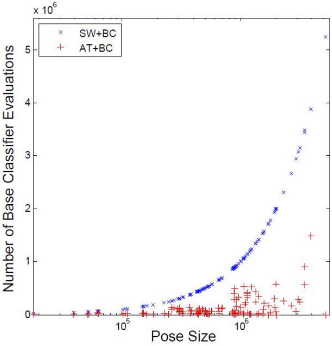
5905 / 426429 =0.013



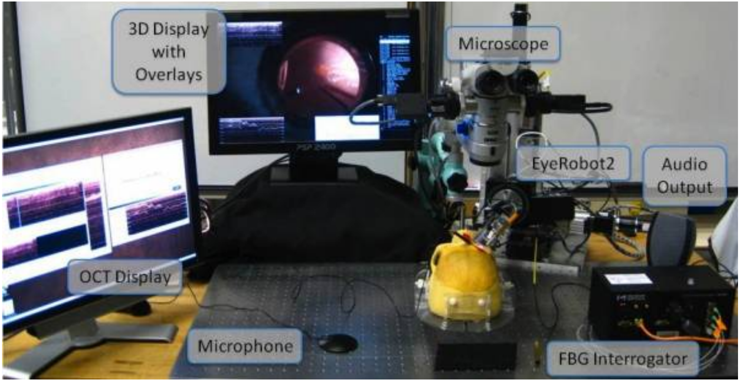
18167 / 372960 =0.048

MIT+CMU: Performance and Iterations

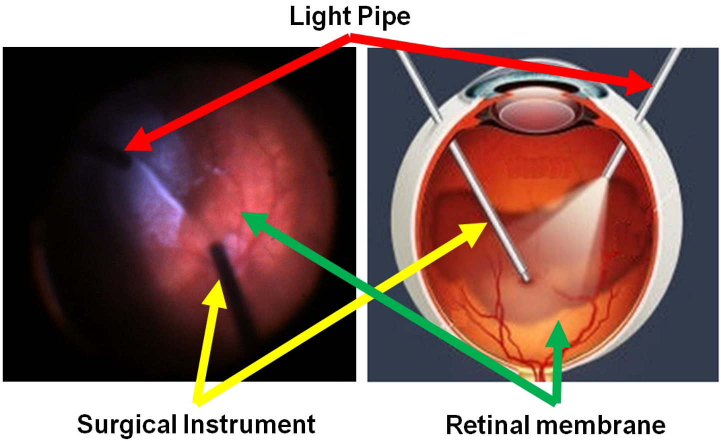




Application:
Instrument detection and Tracking in Retinal Microsurgery



Application:
Instrument detection and Tracking in Retinal Microsurgery



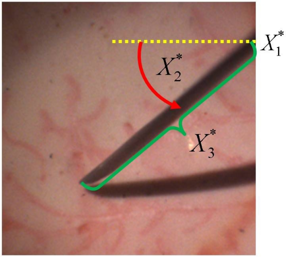
Active Testing for Retinal Tool Detection

Let us find the tool pose in an image [1]:

- Let $X^* = (X_1^*, X_2^*, X_3^*)$ be a discrete random variable that defines the tool pose.
- Let the space of possible tool locations be:

$$\mathcal{S} = [0, P] \times [-\pi/2, \pi/2] \times [\delta, L]$$

- Let $p_0 \sim U(\mathcal{S} \cup \{\square\})$



[1] Sznitman et al. **Unified Detection and Tracking in Retinal Microsurgery.** *MICCAI*, 2011.

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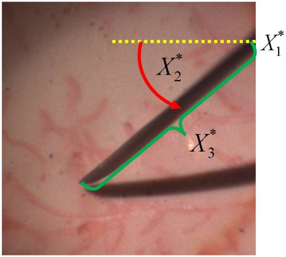


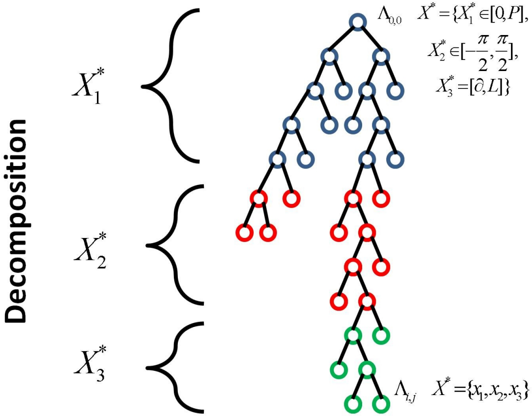
- More complicated density: need a way to organize the search space for efficiency.

[1] Sznitman et al. **Unified Detection and Tracking in Retinal Microsurgery.** *MICCAI*, 2011.

Search Space Decomposition

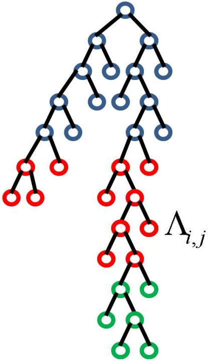
- Let Λ be a regular decomposition of the pose space, such that

$$\Lambda = \{\Lambda_{i,j}, i = 0, ..., d, j = 0, ..., 2^{i-1} - 1\}$$



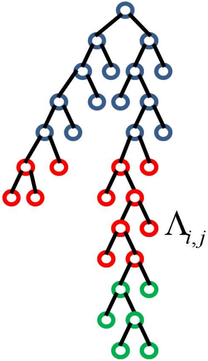
- Will represent p_n via Λ.

Tool Questions



- At each node $\Lambda_{i,j}$, can evaluate a question type: $k = 1, \ldots, \mathcal{K}$

Tool Questions



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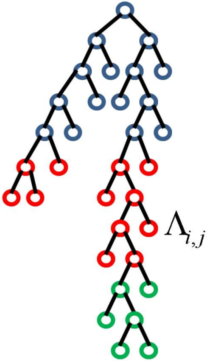
- A question $X_{i,j}^k$ asks: "is $X^* \in \Lambda_{i,j}$" by computing a function k of the image:

$$X_{i,j}^k : I_{\Lambda_{i,j}} \mapsto R$$

Tool Questions



- At each node $\Lambda_{i,j}$, can evaluate a question type: $k = 1, \ldots, \mathcal{K}$
- A question $X^k_{i,j}$ asks: "is $X^* \in \Lambda_{i,j}$" by computing a function k of the image:

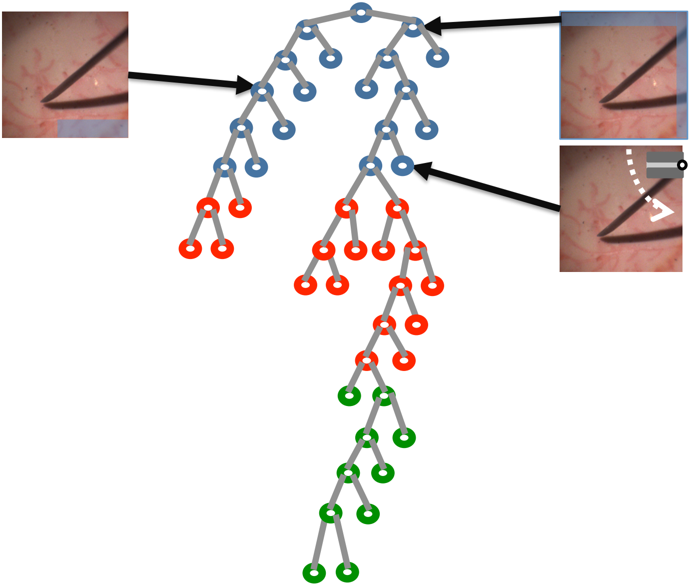
$$X^k_{i,j} : I_{\Lambda_{i,j}} \mapsto R$$

- Answer $Y^k_{i,j}$ is random,

$$Y^k_{i,j} = \begin{cases} f_1(\cdot; i, j) & \text{if } X^* \in \Lambda_{i,j} \\ f_0(\cdot; i, j) & \text{if } X^* \notin \Lambda_{i,j} \end{cases}$$

  (f_1, f_0) are estimated from labeled training data.

Noisy Tool Questions:



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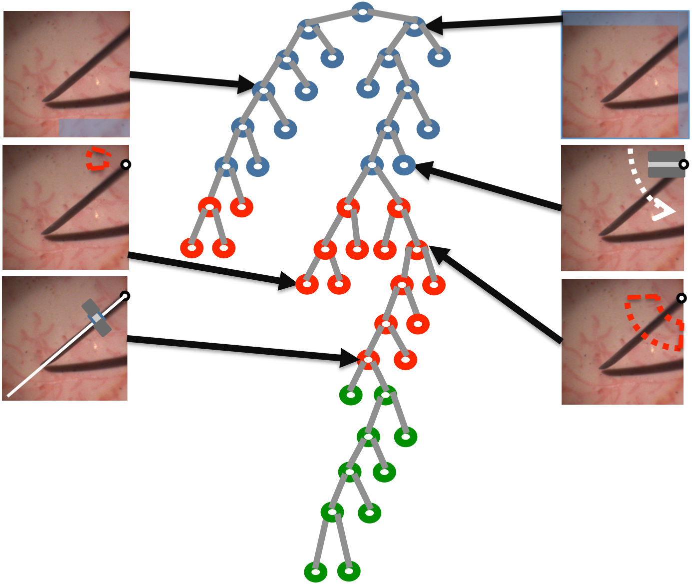

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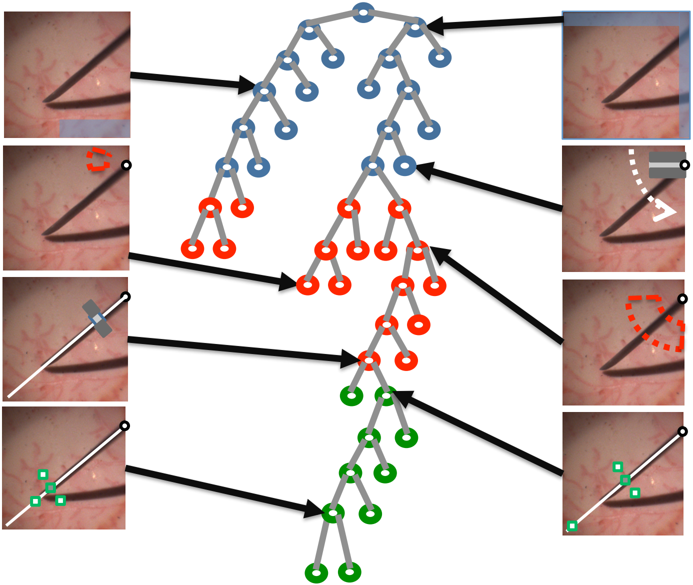
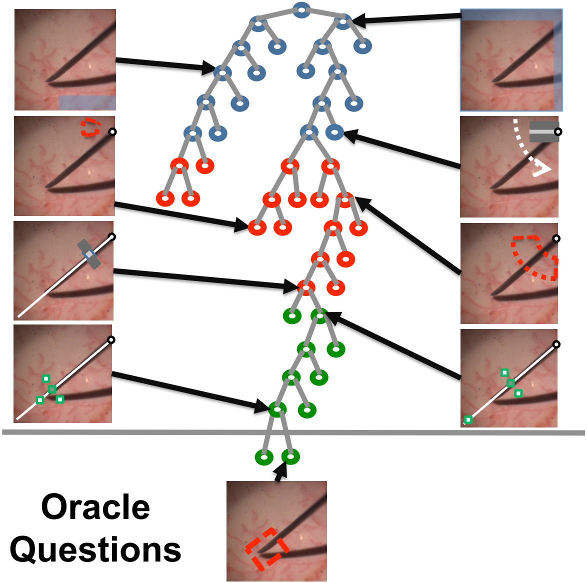


Noisy and Oracle Tool Question:



Oracle Questions

Active Testing Algorithm

1. Initialize node to query: $i = 0, j = 0$
2. Initialize question type: $k = 0$
3. **Repeat**
   3.1 Test: $y = X_{i,j}^k$
   3.2 Update $p_{t+1}(\cdot)$ from y and $p_t(\cdot)$
   3.3 Choose next Question:

$$\{i, j, k\} = \arg\max_{i,j,k} \mathrm{MI}(i, j, k)$$

4. **Until** $H(p_{t+1}) < \epsilon$ or a fixed number of iterations.

Tool Tracking by Active Testing Filtering

- Given an image sequence, $\mathcal{I} = (I^1, \ldots, I^T)$ and a tool dynamics model $P(X_t^*|X_{t-1}^*)$, we can perform Bayesian Filtering:

1. Initialize: $p_0(X^*)$
2. **Repeat**
   2.1 $P_t(X^*|\mathcal{I}^{t-1}) = \int P(X_t^*|X_{t-1}^*)p_{t-1}(X^*)dX^{t-1}$
   2.2 $P_t(X^*|\mathcal{I}^t) = ActiveTesting(I^t, P_t(X^*|\mathcal{I}^{t-1}))$

Examples of runs

Examples of runs

conclusion

The ”20 questions with noise” model offers a framework for ”machine perception”. It is amenable to mathematical analysis through the use of information theory, control theory and probabilities.