4 Algebraically Closed Valued Fields

4.1 Quantifier Elimination for ACVF

We now have developed enough machinery to begin the study of the model theory of algebraically closed valued fields.

Valued fields as structures

The first issue is deciding what kind of structure we are looking at, i.e., what language or signature do we use to study valued fields? There are several natural candidates.

One-sorted structures

We can think of valued fields as pairs $(K, \mathcal{O})$ where K is the field and $\mathcal{O}$ is the valuation ring. In this case the natural language would be the usual language of rings $\{+, -, \cdot, 0, 1\}$ together with a unary predicate $\mathcal{O}$ which picks out the valuation.

Three-sorted structures

We can think of valued fields as three-sorted structures $(K, \Gamma, \boldsymbol{k})$ where we have separate sorts for the field (which we refer to as the *home sort*, the value group and the residue field. On the home sort and on the residue field we will have the $+, -, \cdot, 0$, and 1. On the group we will have $+, -, <, 0$. We also have the valuation map v and the residue map res. [4]

It would also be natural to think of valued fields as two sorted structure (K, Γ) and later we will consider adding more imaginary sorts.

How does this effect definability? It's easy to see that it doesn't.

Lemma 4.1 *In the one-sorted structure $(K, \mathcal{O})$ we can interpret the value group Γ, the residue field $\boldsymbol{k}$ and the maps $v : K^\times \to \Gamma$ and $\mathrm{res} : \mathcal{O} \to \boldsymbol{k}$. Thus any subset of K^n definable in the three-sorted structure is definable in the one-sorted structure. Moreover if $X \subseteq K^l \times \Gamma^m \times \boldsymbol{k}^n$ is definable in the three-sorted structure, then there is $A \subseteq K^{l+m+n}$ definable in $(K, \mathcal{O})$ such that*

$$X = \{(a_1, \ldots, a_l), v(a_{l+1}, \ldots, v(a_{l+m}), \mathrm{res}(a_{l+m+1}), \ldots, \mathrm{res}(a_{l+m+n}) : (a_1, \ldots, a_{l+m+n}) \in A.$$

In the three-sorted structure (K, Γ, v) we can define the value ring $\mathcal{O} = \{x \in K : v(x) \geq 0\}$. Thus any subset of K^n definable in the one-sorted structure is definable in the three-sorted structure.

We will also look at further variants of these languages.

- When studying the p-adic field $\mathbb{Q}_p$, we have already shown in Exercise 2.11 that $\mathbb{Z}_p$ is definable in the field language. Thus any subset of $\mathbb{Q}_p^n$ definable in $(\mathbb{Q}_p, \mathbb{Z}_p)$ is already definable in $\mathbb{Q}_p$ in the field language. The exercises below show that this is not always possible.

[4] Note we should think of they symbols on each sort as being distinct, so while we routinely use $+$ on $K, \boldsymbol{k}$ and Γ, if we were more careful we would think of them as three distinct symbols.

- To prove quantifier elimination for algebraically closed valued fields we will work in the language of divisibility

$$\mathcal{L}_{\mathrm{div}} = \{+, -, \cdot, \mathcal{O}, |, 0, 1\}$$

  where $|$ is a binary function symbol which we interpret

$$(K, \mathcal{O}) \models x|y \text{ if and only if } \exists z \in \mathcal{O}\ xz = y.$$

  The relation $x|y$ is definable in $(K, \mathcal{O})$ thus any subset of K^n definable in the language $\mathcal{L}_{\mathrm{div}}$ is already definable in $(K, \mathcal{O})$.

  Note that once we have added $|$ to the language we could get rid of $\mathcal{O}$ since $x \in \mathcal{O}$ if and only if $1|x$.

- To prove quantifier elimination for $\mathbb{Q}_p$ we will work in the *Macintyre Language* $\mathcal{L}_{\mathrm{Mac}} = \{+, -, \cdot, \mathcal{O}, 0, 1, P_2, P_3, P_4, \dots\}$ where P_n is a unary predicate predicate which we interpret in $(\mathcal{K}, \mathcal{O})$ as the n^{th} powers of K. Since $x \in P_n$ if and only if $K \models \exists y\ y^n = x$, any subset of K^n definable in $\mathcal{L}_{\mathrm{Mac}}$ is already definable using $\mathcal{L}$. Indeed in $\mathbb{Q}_p$ we can define $\mathbb{Z}_p$ in a quantifier free way using P_2 as in Exercise 2.11. Thus we don't really need the predicate for $\mathcal{O}$.

- In the original work of Ax and Kochen it was useful to work in the three-sorted language and add a symbol for $\pi : \Gamma \to K$ a section of the valuation. This is more problematic. We saw in Exercise 2.34 that not every valued field has a section. Moreover we will show that the section map is not definable in the three-sorted language. Thus, while adding the section can be useful, we will end up with new definable sets.

- An *angular component* map is a multiplicative homomorphism $\mathrm{ac} : K^\times \to \boldsymbol{k}^\times$ such that ac agrees with the residue map on the units. For example on $\mathbb{Q}_p$ if $v_p(x) = m$ then $x = a_m p^m + a_{m+1} p^{m+1} + \dots$ and we can let $\mathrm{ac}(x) = a_m$. Similarly, there is an angular component map on $K((T))$.

  If we have a section $\pi : \Gamma \to K$, then we can define an angular component map by $\mathrm{ac}(x) = \mathrm{res}(x/\pi(x))$. But, like sections, angular component maps need not exist and, even when they do exist, may change definability.

  Nevertheless, we will find it useful to work in the three-sorted language $\mathcal{L}_{\mathrm{Pas}}$ where we add a symbol for an angular component map. This is called the *Pas language*

Exercise 4.2 Let $(K, \mathcal{O})$ be a valued field where K is algebraically closed or real closed. Show that $\mathcal{O}$ is not definable in K in the pure field language.

Exercise 4.3 Suppose $\pi : \Gamma \to K$ is a section of the valuation. Show that $\mathrm{ac}(x) = \mathrm{res}(x/\pi(x))$ is an angular component map.

Quantifier Elimination

We will prove quantifier elimination for algebraically closed valued fields in the language $\mathcal{L}_{\text{div}}$. Let ACVF be the $\mathcal{L}_{\text{div}}$-theory such that $(K, \mathcal{O}, |) \models$ ACVF if and only if K is an algebraically closed field with valuation ring $\mathcal{O}$ and $x|y$ if and only if there is $z \in \mathcal{O}$ such that $zx = y$. We will also assume that the valuation is nontrivial so there is $x \in K^\times \setminus \mathcal{O}$.

Theorem 4.4 (Robinson) *The theory of algebraically closed fields with a non-trivial valuation admits quantifier elimination in the language $\mathcal{L}_{\text{div}}$.*[5]

Quantifier elimination will follow from the following proposition.

Proposition 4.5 *Suppose (K, v) and (L, w) are algebraically closed fields with non-trivial valuation and L is $|K|^+$-saturated. Suppose $R \subseteq K$ is a subring, and $f : R \to L$ is an $\mathcal{L}_{\text{div}}$-embedding. Then f extends to a valued field embedding $g : K \to L$.*

Exercise 4.6 Show that the proposition implies quantifier elimination. [Hint: See [22] 4.3.28.]

We will prove the Proposition via a series of lemmas.

Definition 4.7 Suppose R is a subring of K. We say that a ring embedding $f : R \to L$ is an $\mathcal{L}_{\text{div}}$-embedding if for $a, b \in R$,

$$R \models a|b \Leftrightarrow w(f(a)) \leq w(f(b)).$$

First, we show that without loss of generality we can assume R is a field.

Lemma 4.8 *Suppose (K, v) and (L, w) are valued fields, $R \subseteq K$ is a subring and $f : R \to L$ is and $\mathcal{L}_{\text{div}}$-embedding. Then f extends to a valuation preserving embedding of K_0, the fraction field of R into L.*

Proof Extend f to K_0, by $f(a/b) = f(a)/f(b)$. If $x \in K_0$, then x is a unit in (K, v) if and only if $x|1$ and $1|x$ if and only in $f(x)$ is a unit in (L, w). Since the value group is given by $K^\times/U$, addition in the value group is preserved. So we need only show that the order is preserved.

Suppose $x, y \in K_0$. There are $a, b, c \in R$ such that $x = \frac{a}{c}$ and $y = \frac{b}{c}$. Then

$$v(x) \leq v(y) \Leftrightarrow v(a) \leq v(b) \leq R \models a|b \Leftrightarrow L \models f(a)|f(b) \Leftrightarrow w(f(x)) \leq w(f(y)).$$

$\square$

We next show that we can extend embedding from fields to their algebraic closures.

[5] Actually, Robinson only proved model completeness, but his methods extend to prove quantifier elimination.

Lemma 4.9 *Suppose (K, v) and (L, w) are algebraically closed valued fields, $K_0 \subseteq K$ is a field and $f : K_0 \to L$ is a valuation preserving embedding. Then f extends to a valuation preserving embedding of K_0^{alg}, the algebraic closure of K_0 into L.*

Proof It suffices to show that if $x \in K \setminus K_0$ is algebraic over K_0, then we can extend f to $K_0(x)$. Let $K_0(x) \subseteq F \subseteq K$ with F/K_0 normal. There is a field embedding $g : F \to L$ with $g \supset f$ and $g(v)$ gives rise to a valuation on $g(F)$ extending $f(v|K_0)$. Then $g(v|F)$ and $w|g(F)$ are valuations on $g(F)$ extending $f(v|K_0)$ on $f(K_0)$. By Theorem 3.24, there is $\sigma \in \mathrm{Gal}(g(F)/f(K_0))$ mapping $g(v|F)$ to $w|g(F)$. Thus $\sigma \circ g$ is the desired valued field embedding of F into L extending f. □

Thus in proving Proposition 4.5 it suffices to show that if we have (K, v) and (L, w) non-trivially valued algebraically closed fields, L is $|K|^+$-saturated, $K_0 \subset K$ algebraically closed and $f : K_0 \to L$ a valuation preserving embedding, then we can extend f to K. There are three cases to consider.

case 1 Suppose $x \in K$, $v(x) = 0$ and $\overline{x}$ is transcendental over $\boldsymbol{k}_{K_0}$.

We will show that we can extend f to $K_0[x]$, then use Lemmas 4.8 and 4.9 to extend to $K_0(x)^{\mathrm{acl}}$. Since L is $|K|^+$-saturated, there is $y \in L$ such that $\overline{y}$ is transcendental over $\boldsymbol{k}_{f(K_0)}$. We will send x to y.

Suppose $a = m_0 + a_1x + \cdots + m_nx^n$, where $m_i \in K_0$. Suppose m_l has minimal valuation. Then $a = m_l(\sum b_ix^i)$ where $v(b_i) \geq 0$ and $b_l = 1$. Then $v(\sum b_ix^i) \geq 0$. If $v(\sum b_ix^i) > 0$, then taking residues we see that

$$\sum \overline{b}_i\overline{x}^i = 0,$$

but $\overline{b}_l = 1$, so this is a nontrivial polynomial and $\overline{x}$ is algebraic over $\boldsymbol{k}_{K_0}$. Thus $v(\sum b_ix^i) = 0$ and $v(a) = m_l$.

Thus $v(a) = \min\{v(m_i) : i = 0, \ldots, n\}$. Similarly, in L, $w(\sum f(m_i)y^i) = \min\{w(f(m_i)) : i = 0, \ldots, n\}$. Thus the extension of f to $K_0[x]$ is and $\mathcal{L}_d$-embedding.

case 2 Suppose $x \in K$ and $v(x) \notin v(K_0)$.

Let $\gamma = v(x)$. Suppose $a, b \in K_0$, $i < j$ are in $\mathbb{N}$, and $v(a) + i\gamma = v(b) + j\gamma$. Since K_0 is algebraically closed there is $c \in K_0$ such that $c^{j-i} = \frac{a}{b}$, but then $\gamma = v(c) \in v(K_0)$.

Suppose $a \in K_0[x]$ and $a = m_0 + m_1x + \ldots m_nx^n$. Since the $v(m_i) + i\gamma$ are distinct, $v(a) = \min(v(m_i) + i\gamma)$.

Since L is $|K|^+$-saturated, there is $y \in L$ realizing the type

$$\{w(f(a)) < w(y) : a \in K_0, v(a) < v(x)\} \cup \{w(y) < w(f(b)) : v(x) < v(a)\}.$$

Then $v(a)+iv(x) < v(b)+jv(x)$ if and only if $w(f(a))+iw(y) < w(f(b))+jw(y)$ for all $a, b \in K_0$ and the extension of f to $K_0[x]$ sending x to y is and $\mathcal{L}_{\mathrm{div}}$-embedding.

case 3 Suppose $x \in K \setminus K_0$, $v(K_0(x)) = v(K_0)$ and $\boldsymbol{k}_{K_0(x)} = \boldsymbol{k}_{K_0}$, i.e., $K_0(x)$ is an *immediate* extension of K_0.

Let $C = \{v(x-a) : a \in K_0\}$. Since $v(K_0(x)) = v(K_0)$, $C \subseteq v(K_0)$. We claim that C has no maximal element. Suppose $v(b) \in C$ is maximal. Then $v(\frac{x-a}{b}) = 0$ and, since $\boldsymbol{k}_{K_0} = \boldsymbol{k}_{K_0(x)}$, there is $c \in K_0$ such that $\frac{x-a}{b} - c = \epsilon$ where $v(\epsilon) > 0$. But then,

$$v(x - a - bc) = v(b\epsilon) > v(b),$$

a contradiction.

Consider the type

$$\Sigma(y) = \{w(y - f(a)) = w(b) : a, b \in K_0, v(x-a) = v(b).\}$$

We claim that Σ is finitely satisfiable. Suppose $a_1, \ldots, a_n, b_1, \ldots, b_n \in K_0$ and $v(x - a_i) = v(b_i)$. Because f is valuation preserving it suffices to find $c \in K_0$ with $v(c - a_i) = v(b_i)$ for $i = 1, \ldots, n$. Since C has no maximal element, there is $c \in K_0$ such that $v(x-c) > v(b_i)$ for $i = 1, \ldots, n$. Then $v(c - a_i) = v(x - a_i) = v(b_i)$.

By sending x to y we can extend f to a ring isomorphism between $K_0[x]$ and $f(K_0)[y]$. For $a \in K_0(x)$, there is $p(X) \in K_0[X]$ such that $d = p(x)$. Factoring p into linear factors over the algebraically closed field K_0, there is $a_0, \ldots, a_n$ such that

$$d = p(x) = a_0 \prod_{i=1}^{n} (x - a_i).$$

For each i we can find $b_i \in K_0$ such that $v(x - a_i) = v(b_i)$. Thus

$$v(d) = v(a_0) + \sum_{i=1}^{n} v(b_i)$$

By choice of y, we also have

$$w(f(d)) = w(f(a_0)) + \sum_{i=1}^{n} w(f(b_i)),$$

thus f preserves the valuation.

This concludes the proof of Proposition 4.5 and hence the proof that ACVF has quantifier elimination in the language $\mathcal{L}_{\mathrm{div}}$.

The proofs we have given can readily be adapted to prove quantifier elimination in the three-sorted language.

Exercise 4.10 Modify the proofs above to verify that algebraically closed fields have quantifier elimination when viewed as three-sorted structures in the usual language.

4.2 Consequences of Quantifier Elimination

Completions of ACVF

ACVF is not a complete theory. We need to specify the characteristic of the field K and the residue field $\boldsymbol{k}$. If K has characteristic p, then $\boldsymbol{k}$ has characteristic p. If K has characteristic 0, the $\boldsymbol{k}$ may have any characteristic. Let a be either 0 or a prime. If $a = p$ a prime, then $b = p$. If a is zero, then b is either zero or a prime. Let $\mathrm{ACVF}_{a,b}$ be ACVF with additional axioms asserting the field has characteristic a and the residue field has characteristic b.

Corollary 4.11 *Each theory* $\mathrm{ACVF}_{a,b}$ *is complete and these are exactly the completions of* ACVF.

Proof If $(a, b) = (0, 0)$ let $R = (\mathbb{Q}, \mathbb{Q}, |)$. If $(a, b) = (0, p)$ let $R = (\mathbb{Q}, \mathbb{Z}_{(p)}, |)$ and if $(a, b) = (p, p)$, let $R = (\mathbb{F}_p, \mathbb{F}_p, |)$. Suppose $(K, \mathcal{O}_K, |)$ and $(L, \mathcal{O}_L, |)$ are models of $\mathrm{ACVF}_{a,b}$. Then R is a common substructure of both fields. Let ϕ be an $\mathcal{L}_{\mathrm{div}}$-sentence. Then there is a quantifier free $\mathcal{L}_{\mathrm{div}}$-sentence such that

$$\mathrm{ACVF} \models \phi \leftrightarrow \psi.$$

But then, since ψ is quantifier free,

$$K \models \phi \Leftrightarrow K \models \psi \Leftrightarrow R \models \psi \Leftrightarrow L \models \psi \Leftrightarrow K \models \phi.$$

Thus $\mathrm{ACVF}_{a,b}$ is complete.

We have listed the only possibilities for the characteristics of the field and residue field. Thus these are the only possible completions of ACVF.[6] □

Definable subsets of K

In any valued field we can always define open and closed balls and any finite boolean combination of balls.[7] We will show that in an algebraically closed valued field these are the only definable subsets of K.

Lemma 4.12 *Let (K, v) be an algebraically closed valued field. Suppose $f \in K[X]$. Then we can partition K into finitely many sets each of which is a finite boolean combination of balls such that that for each Y in the partition there are $n \geq 1$, $a \in K$ and $\gamma \in \Gamma$ in the value group such that $v(f(x)) = nv(x - a) + \gamma$ for all $x \in Y$.*

Proof Let $f(X) = c(X - a_1) \cdots (X - a_n)$ for $c \in K^\times$ and $a_1, \ldots, a_n \in K$. Then $v(f(x)) = v(c) + \cdots + v(x - a_1) + \cdots + v(x - a_n)$. We will show that we can partition K such that on each set in the partition there is i such that either

[6] Here we are using the assumption that our fields have nontrivial valuations. If we were to also consider the trivial valuation we would have completions saying that I have a trivial valued field of characteristic 0 or p. But these are just the completions of ACF.

[7] Here we allow trivial balls $K = \{x : v(x) < \infty\}$ and $\{a\} = \{x : v(x) = \infty\}$. If we don't want to do this, we should look at boolean combinations of points and balls instead.

$v(x-a_j)=v(x-a_i)$ for each set in the partition or $v(x-a_j)$ is constant on the partition.

For each partition I,J of $\{1,\dots,n\}$ where I is nonempty, let $\widehat{i}$ be the least element of I. Let

$$Y_{I,J}=\{x\in K: v(x-a_i)=v(x-a_{\widehat{i}})>v(x-a_j) \text{ for } i\in I, j\in J\}.$$

Then the sets $Y_{I,J}$ are boolean combinations of balls and they partition K (of course some $Y_{I,J}$ might be empty.

For $j\neq\widehat{i}$ let $\gamma_j=v(a_{\widehat{i}}-a_j)$. Then

• if $v(x-a_{\widehat{i}})<\gamma_j$, then $v(x-a_j)=v(x-a_{\widehat{i}})$

• If $v(x-a_{\widehat{i}})>\gamma_j$, then $v(x-a_j)=\gamma_j$

• We can not have $v(x-a_{\widehat{i}})=\gamma_j$, as then $v(x-a_j)\geq\gamma_j$, contradicting $x\in Y_{I,J}$.

This allows to partition $Y_{I,J}$ into finitely many pieces each of which is a boolean combination of balls, such $v(x-a_j)$ is either $v(x-a_{\widehat{i}})$ or constant on each set in the partition. □

Exercise 4.13 Show that if (K,v) is algebraically closed and $f,g\in K[X]$, then $\{x\in K: v(f(x))\leq v(g(x))\}$ is a finite Boolean combination of balls.

Corollary 4.14 *If* $(K,\mathcal{O})\models$ ACVF *and $X\subseteq K$ is definable, then X is a finite boolean combination of balls.*

Proof By quantifier elimination any definable subset of X is a finite boolean combination of sets of the form $\{x: f(x)=g(x)\}$ and $\{x: f(x)|g(x)\}=\{x: v(f(x))\leq v(g(x))\}$ for $f,g\in K[X]$. □

Definition 4.15 A *swiss cheese* is a definable set of the form $B\setminus(C_1\cup\cdots\cup C_n)$ where $B,C_1,\dots,C_n$ are balls and $C_i\subset B$ (and we allow the possibilities where $B=K$ or $\emptyset$, $n=0$ and some B or C_i is a point.)

Exercise 4.16 a) Show the intersection of two swiss cheese is a finite disjoint union of swiss cheese.

b) Show that the complement of a swiss cheese is a finite disjoint union of swiss cheese.

c) Prove that every definable subset of K can be written in a unique way as a finite union of disjoint swiss cheese.

Corollary 4.17 *i) Any infinite definable subset of K has interior.*

ii) There is no definable section of the value group.

Proof i) Any infinite definable set will contain a swiss cheese $S=B\setminus(C_1\cup\cdots\cup C_m)$, where $B\neq\emptyset$. If $a\in S$, then S contains a ball U with $a\in U$.

ii) The image of the section would be infinite with no interior. □

Exercise 4.18 Suppose K is an algebraically closed valued field and $A\subseteq K^{m+n}$ is definable. For $x\in K^m$ let $A_x=\{y\in K^n:(x,y)\in A\}$. Show that $\{x: A_x$

is finite$\}$ is definable and that there is an N such that if A_x is finite, then $|A_x| \leq N$.

Exercise 4.19 Let $A \subset K$. Show that the model theoretic algebraic closure of A is the field theoretic algebraic closure of A.

In Exercise 5.23 we will characterize definable closure in ACVF.

Exercise 4.20 Let (K, v) be an algebraically closed valued field. Prove that there is no definable angular component map.

NIP

Let $\mathcal{M}$ be a structure. Recall that $\phi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ has the *independence property* if for all k there are $\overline{b}_1, \ldots, \overline{b}_k \in \mathcal{M}^m$ and $(\overline{c}_J : J \subset \{1, \ldots, k\})$ in $\mathcal{M}^n$ such that

$$\mathcal{M} \models \phi(\overline{b}_i, \overline{c}_J) \Leftrightarrow i \in J).$$

In which case we say that ϕ *shatters* $\overline{b}_1 \ldots, \overline{b}_k$. Otherwise we say ϕ has NIP.

We say that a theory has NIP if no formula has the independence property. We need two basic facts about NIP. See [28] 2.9 and 2.11.

Lemma 4.21 *i) T has* NIP *if and only if every formula $\phi(x_1, y_1, \ldots, y_n)$ has NIP.*

ii) A boolean combination of NIP formulas has NIP.

Corollary 4.22 ACVF *has NIP.*

Proof By the lemma above and Corollary 4.14, it suffices to show that no definable family of balls has the independence property. We claim that the family of all balls can not shatter a set of size 3. Suppose a, b and $c \in K$ are distinct and, without loss of generality, $v(a - b) \leq v(a - c), v(b - c)$. Then any ball that contains a and b contains c. Thus the family of all balls does not shatter any three element set. $\square$

Definable subsets of the value group and residue field

To study definable subsets of $\boldsymbol{k}^m$, Γ^n and, more generally $\boldsymbol{k}^m \times \Gamma^n$ we need to apply quantifier elimination in the three-sorted language. We will let variables $x_0, x_1, \ldots$ range over the home sort, while $y_0, y_1, \ldots$ ranges over the residue field and $z_0, z_1, \ldots$ range over the value group. Any atomic formula is equivalent to one in one of the following forms

- $t(x_0, \ldots, x_m) = 0$, where t is a polynomial over $\mathbb{Z}$;
- $t(y_0, \ldots, y_n, \mathrm{res}(x_0), \ldots, \mathrm{res}(x_m)) = 0$, where t is a polynomial over $\mathbb{Z}$;
- $s(z_0, \ldots, z_l, v(x_0), \ldots, v(x_m)) = 0$, where $s(u_0, \ldots, u_{l+m+1}) = \sum r_i u_i$, $r_i \in \mathbb{Z}$;
- $s(z_0, \ldots, z_l, v(x_0), \ldots, v(x_m)) > 0$, where $s(u_0, \ldots, u_{l+m+1}) = \sum r_i u_i$, $r_i \in \mathbb{Z}$;

We say that $A \subseteq \boldsymbol{k}^n \times \Gamma^m$ is a *rectangle* if there is $B \subseteq \boldsymbol{k}^n$ definable in the field structure on $\boldsymbol{k}$ and $C \subseteq \Gamma^m$ definable in the ordered abelian group Γ such that $A = B \times C$.

Corollary 4.23 (Orthogonality) *Every definable subset of $\boldsymbol{k}^n \times \Gamma^m$ is a finite union of rectangles.*

Proof By quantifier elimination, every definable set is a finite union of sets defined by conjunctions of atomic and negated atomic formulas. But atomic formulas defining subsets of $\boldsymbol{k}^n \times \Gamma^m$ only have variables over just the residue field sort or just the value group sort and the definable set is either of the form $\boldsymbol{k}^n \times A$ or $B \times \Gamma^n$ where $A \subseteq \boldsymbol{k}^n$ is already definable in $\boldsymbol{k}$ or $B \subseteq \Gamma^m$ is already definable in Γ. Thus any set defined by a conjunction of atomic and negated atomic formulas is a rectangle and every definable set is a finite union of rectangles. $\square$

Corollary 4.24 *i) Any definable function $f : \boldsymbol{k} \to \Gamma$ has finite image.*
ii) Any definable function $g : \Gamma \to \boldsymbol{k}$ has finite image.

This shows that the residue field and value group are as unrelated as possible. It also shows that the valuation structure induces no additional definability on the residue field and value group.

Corollary 4.25 *i) Any subset of $\boldsymbol{k}^n$ definable in $(K, \Gamma, \boldsymbol{k})$ is definable in the field $\boldsymbol{k}$.*
ii) Any subset of Γ^m definable in $(K, \Gamma, \boldsymbol{k})$ is definable in the ordered abelian group Γ.

In this case $\boldsymbol{k}$ with all induced structure, is just a pure algebraically closed field and hence ω-stable, while Γ with all induced structure, is a divisible ordered abelian group and hence o-minimal.

Definition 4.26 We say that a sort S is *stably embedded* if any subset of S^n that is definable in the full structure is definable using parameters from S.

Corollary 4.27 *The residue field and value group of an algebraically closed field are stably embedded.*

In the next section we give an example of an imaginary sort that is not stably embedded.

Exercise 4.28 Let $A \subset \boldsymbol{k}$. Prove that if $b \in \boldsymbol{k}$ is algebraic over A in the three-sorted valued field structure, then b is algebraic over A in the field $\boldsymbol{k}$.

4.3 Balls

For this section we start by thinking of valued fields as three-sorted structures $(K, \Gamma, \boldsymbol{k})$, but this also makes sense if we think of them as one-sorted structures $(K, \mathcal{O})$.

For any valued field we can introduce two new sorts $\mathcal{B}_o$ and $\mathcal{B}_c$ for open and closed balls. For $\mathcal{B}_o$ define an equivalence relation $\sim$ on $K \times \Gamma$ such that $(a, \gamma) \sim (b, \delta)$ if and only if $\gamma = \delta$ and $\gamma = \delta$ and $v(a - b) > \gamma$. Then

$$(a, \gamma) \sim (b, \gamma) \Leftrightarrow b \in B_\gamma(a) \Leftrightarrow a \in B_\gamma(b).$$

Thus we can identify $(a, \gamma)/\sim$ with $B_\gamma(a)$. Let $\mathcal{B}_o = K \times \Gamma/\sim$. We can indentify $\mathcal{B}_o$ with the open balls of K. There is a definable map $r : \mathcal{B}_o \to \Gamma$ given by $r((a, \gamma)/\sim) = \gamma$, i.e., r assigns each ball it's radius. There is a definable relation R_o on $K \times \mathcal{B}_o$ such that aR_ob if and only if $a \in b$. Replacing $\sim$ by $(a, \gamma) \sim^* (b, \delta)$ on $K \times \Gamma \cup \{\infty\}$ if and only if $\gamma = \delta$ and $v(a - b) \geq \gamma$, we can similarly define the sort of closed balls $\mathcal{B}_c$.

Exercise 4.29 Let $a \in K$ and let $X \subset S$ be the set of all open balls containing a. Prove that X is not definable with parameters from $\mathcal{B}_o$. [Hint: Show that for any finite subset A of $\mathcal{B}_o$ there is an automorphism (possibly of a larger field) fixing A pointwise but moving X.]

While up to this point the construction makes sense in any valued field, henceforth we will assume K is algebraically closed.

Lemma 4.30 *If $X \subseteq \mathcal{B}_c$ is an infinite definable set then either $r|X$ is finite-to-one, or there is an infinite definable $Z \subseteq X$ and a definable surjection $f : Z \to \boldsymbol{k}$.*

Proof If $r|X$ is not finite-to-one, there is $\gamma \in \Gamma$ such that $Y = \{B \in X : r(B) = \gamma\}$ is infinite. Let $A = \bigcup_{B \in Y} B$. Then A is an infinite definable subset of K and if $a \in A$, then $\overline{B}_\gamma(a) \in Y$.

claim There is a closed ball $\overline{B}_\epsilon(a)$ with $\epsilon < \gamma$ such that every closed ball of radius γ in $\overline{B}_\epsilon(a)$ is in Y.

By quantifier elimination A is a finite disjoint union of sets of $W = B \setminus (C_1 \cup \cdots \cup C_m)$, where $B, C_1, \ldots, C_m$ are balls. Since Y is infinite, some B must have radius $\delta < \gamma$. If $a \in W$, then $B_\gamma(a) \subset W$. Let a_i be the center of C_i, then $\delta \leq v(a - a_i) < \gamma$ for all i. Choose ϵ such that $\delta \leq v(a - a_i) < \epsilon < \gamma$. Then $\overline{B}_\epsilon(a) \subset W \subseteq A$. Thus if $b \in \overline{B}_\epsilon(a)$, then $\overline{B}_\gamma(b) \in Y$.

Let Z be the set of closed balls of radius γ contained in $B_\epsilon(a)$. Then Z is an infinite set of closed balls and $Z \subseteq Y$.

If we choose $c \in K$ with $v(c) = -\epsilon$, then $g(x) = c(x-a)$ is a bijection between $\overline{B}_\epsilon(a)$ and $\mathcal{O}$. If $b_1, b_2 \in \overline{B}_\epsilon(a)$ such that $v(b_1 - b_2) \geq \gamma$, then $v(g(b_1) - g(b_2)) = v(b_1 - b_2) - \epsilon > 0$. Thus $\mathrm{res}(g(b_1)) = \mathrm{res}(g(b_2))$. Thus the map $B_\gamma(b) \mapsto \mathrm{res}(g(b))$ is a well defined map from Z onto $\boldsymbol{k}$. □

Corollary 4.31 *Suppose $f : \Gamma \to \mathcal{B}_c$. Let X be the image of f. Then $r|X$ is finite-to-one.*

Proof If not there is an infinite $Z \subseteq X$ and a definable surjection $g : Z \to \boldsymbol{k}$. Let $A = f^{-1}(Z)$. Then $g \circ f|A$ is a definable map from an infinite definable subset of Γ onto k, a contradiction. □

Lemma 4.32 *If $X \subseteq \mathcal{B}_c$ is infinite, there is a definable $f : X \to \Gamma$ with infinite image. In particular, the image of f contains a non-trivial interval.*

Proof First consider the image of X under the radius map. If this is infinite, then we are done. If not, then, without loss of generality we may assume that all balls in X have radius γ. Let $A = \bigcup_{B \in Y} B$. As the proof of Lemma 4.30, there is a closed ball $\overline{B}_\epsilon(a) \subset A$ with $\epsilon < \gamma$. If $x, y \in \overline{B}_\epsilon(a) \setminus \overline{B}_\gamma(a)$ such that $v(x-y) \geq \gamma$, then $v(x-a) = v(y-a)$. Thus we have a well defined function $f : X \to \Gamma$ such that

$$f(B) = \begin{cases} v(x-a) & \text{if } B \subset \overline{B}_\epsilon(a) \setminus \overline{B}_\gamma(a) \text{ and } a \in B \\ 0 & \text{otherwise} \end{cases}.$$

Then the image of f is an infinite subset of Γ. $\square$

We can extend this result to balls in n-spaces. Let $\gamma \in \Gamma$ and let $\mathbf{a} = (a_1, \ldots, a_n) \in \Gamma^n$. Then

$$\overline{B}_\gamma(\mathbf{a}) = \{\mathbf{b} \in K^n : \bigwedge v(a_i - b_i) \geq \gamma\}$$

is the closed ball around $\mathbf{a}$ of radius γ. Let $\mathcal{B}_c^n$ be the collection of all closed balls in K^n. Let $\pi : K^n \to K^{n-1}$ be the projection onto the first $n-1$ coordinates. If $B \in \mathcal{B}_c^n$ is a closed ball of radius δ, then $\pi(B) \in \mathcal{B}_c^{n-1}$ and if $\overline{B}_\delta(a_1, \ldots, a_{n-1}) \in \mathcal{B}_c^{n-1}$ then B is in the fiber $\pi^{-1}(B_1)$ if and only if

$$B = \overline{B}_\delta(a_1, \ldots, a_n) = \overline{B}_\delta(a_1, \ldots, a_{n-1}) \times \overline{B}_\delta(a_n)$$

for some $a_n \in K$. Thus the fiber is in definable bijection with an infinite subset of $\mathcal{B}_c$.

Corollary 4.33 *If $X \subseteq \mathcal{B}_c^n$ is infinite and definable, there is a definable function $f : X \to \Gamma$ with infinite image.*

Proof We proceed by induction on n, knowing the result is true for $n = 1$. Let $X \subset \mathcal{B}_c^{n-1}$. Consider the projection of X to $\mathcal{B}_c^n$. If this is infinite we are done. If not, some fiber is infinite. But this gives rise to an infinite subset of $\mathcal{B}_c$ and we are done. $\square$

Corollary 4.34 *If $X \subseteq K^n$ is infinite and definable, then there is a definable $f : X \to \Gamma$ with infinite image.*

Proof We have a definable injection $\mathbf{a} \mapsto \{\mathbf{a}\} = \overline{B}_\infty(\mathbf{a})$ of K^n into $\mathcal{B}_c^n$. Thus this follows from the previous corollary. $\square$

4.4 Real Closed Valued Fields

We next consider valued fields $(K, \mathcal{O})$ where K is a real closed field and $\mathcal{O}$ is a proper convex subring. We call $\mathcal{O}$ a *real closed ring* and we refer to $(K, \mathcal{O})$ as a real closed valued field. In a series of exercises we will prove the following theorem of Cherlin and Dickmann.

Theorem 4.35 *The theory of theory of real closed valued fields admits quantifier elimination in the language* $\mathcal{L}_{\mathrm{div},<} = \{+, -, \cdot, <, |, 0, 1\}$.

As usual, the theorem will follow from an embedding lemma.

Lemma 4.36 *Let* $(K, \mathcal{O})$ *and* $(L, \mathcal{O}_L)$ *be real closed valued fields such that* L *is* $|K|^+$*-saturated. Let* R *be a subring of* K *and* $f : R \to L$ *is an embedding that preserves both the order and the divisibility relation. Then* f *extends to an order and valuation preserving embedding of* K *into* L.

Let K, L, R and $f : R \to K$ be as in the lemma. We let v denote the valuation on K and v_L denote the valuation on L.

Exercise 4.37 Let K_0 be the fraction field of R. Show that f extends to an order and and valuation preserving embedding of K_0 into L.

Exercise 4.38 Let K_0 be as above and let K_0^{rcl} be the real closure of K_0 inside K. Show that we can extend f to an order and valuation preserving of K_0^{rcl} into K.

Henceforth, we assume that we have K_0 a real closed subfield of K and $f : K_0 \to L$ an order and valuation preserving embedding.

Exercise 4.39 Suppose $x \in K \setminus K_0$, $v(x) = 0$ and $\overline{x}$ is transcendental over k_{K_0}. Show that we can extend f to $K_0(x)$ preserving the ordering and the valuation.

Exercise 4.40 Suppose $x \in K \setminus K_0$, $v(x) \notin v(K_0)$. Show that we can extend f to $K_0(x)$ preserving the ordering and the valuation.

Exercise 4.41 Suppose $x \in K \setminus K_0$ and K/K_0 is immediate. Show that we can extend f to $K_0[x]$ preserving the ordering and the valuation.

Exercise 4.42 Conclude that the theory of real closed rings has quantifier elimination. Show that the theory of real closed valued fields is complete.

Recall that an ordered structure $(M, <, \dots)$ is *weakly o-minimal* if every definable $X \subset M$ is a finite union of points and convex sets.

Exercise 4.43 Show that a real closed ring is weakly o-minimal and NIP.

A partial converse holds ([21]). It T is a theory all of whose models are weakly o-minimal rings, then they are real closed rings or real closed fields.