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MOLLIFIERS FOR GAMES IN NORMAL FORM AND THE HARSANYI-SELTEN VALUATION FUNCTION

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1. INTRODUCTION

Littlechild-Vaidya [1976] defined and studied ratio measures of coalitions' "propensity to disrupt" in an n-person characteristic function game. The attendant difficulties with the choice of ratio measures led to the introduction and development by Charnes-Rousseau-Seiford [1978] of new incremental measures giving rise to a wide variety of "disruption" and "mollifier" solution concepts free of various ratio defects.

Shapley raised the question of the relation of these "mollifier" concepts to the Harsanyi-Selten [1959] modification of von Neumann-Morgenstern's [1953] construction of a characteristic function for games in normal form to take better account of "disruption" or "threat" possibilities.

In this paper, we show for a large class of games that the Harsanyi-Selten construction yields a constant mollifier. In general, it can be non-superadditive when the von Neumann-Morgenstern function is superadditive.

We then extend the "mollification" concept to games in normal form. In the extended theory, the Harsanyi-Selten construct is a constant mollifier with the preceding non-superadditive impediment. It also fails to take account of coalitional sizes. Our extended "homomollifier" concept does and always yields a superadditive constant sum characteristic function.

2. COMPLEMENTS AND MOLLIFIERS

The concepts of complement and mollifier for n-person games in characteristic function form were defined and studied in Charnes-Rousseau-

Seiford [1978]. Since we will require some of these results and also as motivation for our extended theory to games in normal form, we give the following brief summary.

Let (N,v) be a characteristic function game where $N = \{1,2,\ldots,n\}$ is the set of players and v is a characteristic function, i.e., a non-negative function defined on the subsets of N with $v(\emptyset) = 0$. The complement of a game v, denoted $\bar{v}$, is defined by

$$\bar{v}(S) = v(N) - v(N-S).$$

We have immediately that

(i) $\bar{\bar{v}} = v$ (i.e., the complement transformation is involutory).

(ii) $\bar{v}(\emptyset) = 0$ and $\bar{v}(N) = v(N)$.

(iii) For two games u and v

$\overline{u+v} = \bar{u} + \bar{v}$ (i.e., the complement of a sum is the sum of the complements).

While $\bar{v}$ will not necessarily be superadditive, even if v is superadditive, $\bar{v}$ does inherit some of the structure of v.

Theorem 2.1:

(i) If v is monotone, i.e., $A \subseteq B \Rightarrow v(A) \leq v(B)$, then $\bar{v}$ is monotone.

(ii) If g is strategically equivalent to v, i.e.,

$g(S) = r \cdot v(S) + \sum_{i \in S} \alpha_i$ with $r > 0$, then $\bar{g}$ is strategically equivalent to $\bar{v}$.

Theorem 2.2: The Shapley value of a game v and its complement $\bar{v}$ are identical, i.e.,

$$\Phi_i(v) = \Phi_i(\bar{v}) \;, \quad \forall\, i \in N.$$

If we assume that v is superadditive, the structure of $\bar{v}$ becomes more fixed.

Theorem 2.3: If v is superadditive, then

(i) $\bar{v}(S) \geq v(S)$ $(\forall S \subseteq N)$.

(ii) $\bar{v}(s) + \bar{v}(N-S) \geq \bar{v}(N)$. $(\forall S \subseteq N)$.

(iii) $\bar{v}(S \cup T) \geq \bar{v}(S) + v(T)$, whenever $S \cap T = \emptyset$.

(iv) $\bar{v}$ is superadditive iff $\bar{v} = v$.

Corollary 2.3: v is constant sum iff

$\bar{v}(S) = v(S)$ for all $S \subseteq N$.

The value $\bar{v}(S)$ can be considered as a maximum feasible "goal" of coalition S. It is the largest amount that they can reasonably "expect" to get just as v(S) is the least they would "accept."

We, therefore, define a mollifier of a game v as any componentwise convex combination of the function v and its complement $\bar{v}$. In particular, w_μ, a "constant" mollifier of v is defined for $0 \leq \mu \leq 1$ by

$$w_\mu (S) = \mu \, \bar{v}(S) + (1 - \mu) \, v(S),$$

and a coalitional mollifier is defined by

$$w(S) = \mu_S \, \bar{v}(S) + (1 - \mu_S) \, v(S)$$

where $\mu_S \in [0,1]$, $\forall$ S. This allows us to "mollify" different coalitional values to a greater or lesser degree than others. In particular, if $\mu_S = \frac{|S|}{|N|}$, we call the associated w(S) a "homomollifier."

It is again immediate that $w(\emptyset) = 0$ and $w(N) = v(N)$ for any mollifier w of a game v. Mollification also is additive and preserves strategic equivalence.

Theorem 2.4: Let w^u, w^v and w^g be mollifiers of the n-person games u, v and g, respectively, with g strategically equivalent to v. Then

(i) $w^{u+v} = w^u + w^v$.

(ii) w^g is strategically equivalent to w^v.

Constant mollifiers are not necessarily superadditive, but do possess some attractive properties.

<u>Theorem 2.5:</u> Let w_μ be a constant mollifier of a game v.

(i) If v is constant sum, then $w_\mu(S) = v(S)$ for all $\mu\varepsilon[0.1]$.

(ii) If v is superadditive, then

(a) w_μ (S) is linear and monotone non-decreasing in μ.

(b) if w_{μ_1} is superadditive, then w_{μ_2} is superadditive for all $\mu_2 \leq \mu_1$.

(c) the core of w_μ is contained in the core of v.

Coalitional mollifiers, however, are superadditive and constant sum if one imposes some reasonable conditions on the weights μ_S.

<u>Theorem 2.6:</u> Let w be a coalitional mollifier of a superadditive game v.

(i) If the weights μ_S satisfy $\mu_S + \mu_T = \mu_{S\cup T}$ whenever $S \cap T = \emptyset$, then w is a superadditive game.

(ii) If in addition to (i) $\mu_N = 1$, then w is a constant sum game.

3. THE HARSANYI-SELTEN VALUATION FUNCTION

An n-person game in normal form is defined by a set of players $N = \{1,2,\ldots,n\}$ where each player i has a strategy set Π_i and a payoff function M_i defined as a mapping from the product of the strategy sets into the real numbers. Thus, $M_i : \Pi_1 \times \Pi_2 \times \ldots \times \Pi_n \longrightarrow R$. If each player k selects strategy $\pi_k \in \Pi_k$, then each player i receives a payoff $M_i(\pi_1, \pi_2, \ldots, \pi_n)$.

If we assume that the payoffs to each player are in the same transferable utility, then each subset $S \subseteq N$ has a payoff function $\sum_{i \in S} M_i(\pi_1, \pi_2, \ldots, \pi_n)$ where each player k uses strategy $\pi_k \in \Pi_k$. The set of joint strategies for subset $S \subseteq N$ is defined as the product of the strategy sets of the members of S, and is denoted by Π_S ; a particular joint strategy is denoted by π_S.

An n-person game is normal form, denoted (M, Π_N), is constant sum if $\sum_{i \in N} M_i(\pi_N) = c, \forall \pi_N \in \Pi_N$ and is zero sum if c = 0.

For a constant sum normal form game (M, Π_N) the associated von Neumann-Morgenstern characteristic function [1953] is defined on the subsets $S \subseteq N$ by

$$\overset{M}{v}(S) = \underset{\Pi_S}{\text{Max}} \ \underset{\Pi_{N-S}}{\text{Min}} \sum_{i \in S} M_i(\pi_S, \pi_{N-S}).$$

If the game is not constant sum, von Neumann-Morgenstern adjoin a fictitious player whose payoff is the negative of the sum of the payoffs to the other players, thus forming a zero sum game, and restrict the resulting characteristic function to subsets $S \subseteq N$.

For a superadditive characteristic function game (N,v), consider the normal form game (M^v, S_N) where

$S_i = \{T : T \subseteq N,\ i \in T\}$ is the i^{th} player's strategy set,

and $S_N = S_1 \times S_2 \times \ldots \times S_n$. Then, the i^{th} player's payoff function is

$$M_i^v(T_1, T_2, \ldots, T_n) = \begin{cases} \frac{v(T_i)}{|T_i|} & \text{if } T_j = T_i \ \forall j \in T_i, \text{ the } i^{th} \text{ player's choice} \\ v(\{i\}) & \text{otherwise} \end{cases}$$

This is the normal form game constructed in the inverse theorem of von Neumann-Morgenstern. The characteristic function derived from this game is the original characteristic function v, i.e.,

$$v(T) = \underset{S_T}{\text{Max}} \ \underset{S_{N-T}}{\text{Min}} \sum_{i \in T} M_i^v(s_T, s_{N-T}).$$

where $s_T \in S_T$ and $s_{N-T} \in S_{N-T}$.

Thus, for each normal form game (M, Π_N) there is an associated characteristic function game (N, v^M), and for each superadditive characteristic function game (N, v) there is an associated normal form game (M^v, S_N).

For games in normal form a modified concept of characteristic function was advanced by Harsanyi [1959] and Selten [1964] that is supposed to be sentitive to "threats" that the classical max-min definition overlooks.

We denote by M^* the maximal total payoff, i.e.,

$$M^* = \max_{\Pi_N} \sum_{i \in N} M_i(\pi_N).$$

This modified characteristic function, denoted $\hat{h}(S)$, may then be defined by the two conditions

$$(3.1) \qquad \begin{array}{ll} \text{(i)} & \hat{h}(S) + \hat{h}(N-S) = M^* \\ \text{(ii)} & \hat{h}(S) - \hat{h}(N-S) = \Delta_S \end{array}$$

where Δ_S is the minimax value of the two person zero sum game between coalitions S and N-S in which the payoff to S is the difference $\sum_{i \in S} M_i - \sum_{i \in N-S} M_i$.

This characteristic function $\hat{h}$ is obviously constant sum and (as shown later) satisfies $v^M \leq \hat{h} \leq \bar{v}^M$ where v^M is the classical von Neumann-Morgenstern characteristic function and $\bar{v}^M$ is its complement.

A question posed by Shapley is whether there might be a simple, "natural" way to construct a normal form game whose classical characteristic function would be a given (superadditive) function v, and whose modified characteristic function $\hat{h}$ would be a mollifier. The following theorem shows that this is indeed possible using the construction given by the von Neumann-Morgenstern inverse theorem. Moreover, the resultant modified characteristic function is in fact a constant mollifier.

Theorem 3.1: Let (N,v) be a superadditive characteristic function game and (M^V, S_N) the associated normal form game. Then

(i) the classical characteristic function of (M^V, S_N) is V, and (ii) $\hat{h}$, the Harsanyi-Selten modified characteristic function of (M^v, S_N) is a constant mollifier of v with $\mu = \frac{1}{2}$, i.e.,

$$\hat{h}(T) = \tfrac{1}{2}\,\bar{v}\,(T) + \tfrac{1}{2}\,v\,(T),\ \forall\, T \subseteq N$$

Proof: Since $\hat{h}(N) = \underset{S_N}{\text{Max}} \sum_{i \varepsilon N} M_i^v\,(s_N) = M^{v*} = v(N)$, it follows from (3.1) that

$$\hat{h}(T) = \frac{v(N) + \Delta_T}{2}$$

and

$$\hat{h}(N-T) = \frac{v(N) - \Delta_T}{2}\,.$$

The proof will be complete if we show that

$$\Delta_T = v(T) - v(N-T) \quad (\equiv \bar{v}(T)).$$

Let $\bar{s}_T$ be the joint strategy where each player $i \varepsilon T$ chooses strategy T. Then

$$\sum_{i \varepsilon T} M_i^v\,(\bar{s}_T,\ s_{N-T}) = \sum_{i \varepsilon T} \frac{v(T)}{|T|} = v(T) \text{ for any strategy } s_{N-T}.$$

Considering next this fixed $\bar{s}_T$, we see that

$$\underset{s_{N-T}\,\varepsilon\, S_{N-T}}{\text{Max}} \sum_{i \varepsilon N-T} M_i^v(\bar{s}_T,\ s_{N-T}) = v(N-T)$$

Thus,

$$\begin{aligned}
v(T) - v(N-T) &= v(T) - \underset{S_{N-T}}{\text{Max}} \sum_{i \varepsilon N-T} M_i^v\,(\bar{s}_T,\ s_{N-T}) \\
&= \underset{S_{N-T}}{\text{Min}} \left[v(t) - \sum_{i \varepsilon N-T} M_i^v\,(\bar{s}_T,\ s_{N-T}) \right] \\
&= \underset{S_{N-T}}{\text{Min}} \left[\sum_{i \varepsilon T} M_i^v\,(\bar{s}_T,\ s_{N-T}) - \sum_{i \varepsilon N-T} M_i^v\,(\bar{s}_T,\ s_{N-T}) \right] \\
&\leq \underset{S_T}{\text{Max}}\ \underset{S_{N-T}}{\text{Min}} \left[\sum_{i \varepsilon T} M_i^v\,(s_T,\ s_{N-T}) - \sum_{i \varepsilon N-T} M_i^v\,(s_T,\ s_{N-T}) \right] \\
&= \Delta_T
\end{aligned}$$

Similarly, $v(N-T) - v(T) \leq \Delta_{N-T}$.

Since $\Delta_{N-T} = -\Delta_T$, we have $v(T) - v(N-T) = \Delta_T$. Q.E.D.

Theorem 3.1 shows that for a large class of games in normal form the characteristic function implied by Harsanyi-Selten's definition is in fact the average of the von Neumann-Morgenstern characteristic function and its complement. Thus, a game with a superadditive characteristic function can have a non-superadditive Harsanyi-Selten valuation function as the following example demonstrates.

Consider the characteristic function $v(1) = v(3) = 5$, $v(2) = 14$, $v(12) = v(23) = 20$, $v(13) = 10$, $v(123) = 30$ whose complement is $\bar{v}(1) = \bar{v}(3) = 10$, $\bar{v}(2) = 20$, $\bar{v}(12) = \bar{v}(23) = 25$, $\bar{v}(12) = 16$, and $\bar{v}(123) = 30$. Even though v is superadditive (and has non-empty core), the Harsanyi-Selten modified characteristic function is not superadditive since

$$h_{\frac{1}{2}}(1) + h_{\frac{1}{2}}(2) = \frac{15}{2} + \frac{34}{2} = \frac{49}{2} > \frac{45}{2} = h_{\frac{1}{2}}(12).$$

In fact, for this example, h_μ is not superadditive for any $\mu > 1/6$.

In the next section, we extend the mollifier concepts of section 2 to games in normal form, and show that the homomollifier in this extended theory, in contrast to the Harsanyi-Selten construct, is both constant sum and superadditive.

4. PAYOFF MOLLIFIERS FOR NORMAL FORM GAMES

For a general sum n-person game in normal form (M, Π_N), the associated two person game between coalitions S and N-S will also be a general sum game. In such a case, a coalition's desire to make a gain may be tempered by its wish to inflict a loss. That is, should coalition S choose its joint strategy according to $\underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}} \sum_{i\varepsilon S} M_i\,(\pi_S, \pi_{N-S})$ or

$$\underset{\Pi_S}{\text{Min}}\ \underset{\Pi_{N-S}}{\text{Max}} \sum_{i \varepsilon N-S} M_i\ (\pi_S,\ \pi_{N-S})\ ?$$

Alternatively, coalitions S and N-S could choose to cooperate and form the grand coalition. This possibility for cooperation should be taken into account.

As before, let M* denote the maximal total payoff, i.e., $M^* = \underset{\Pi_N}{\text{Max}} \sum_{i \varepsilon N} M_i\ (\pi_N)$. Then the maximal share of M* that coalition S could legitimately claim as payment for cooperation with N-S is given by

$$(4.1) \qquad \underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}} \left[M^* - \sum_{i \,\varepsilon\, N-S} M_i\ (\pi_S,\ \pi_{N-S}) \right]$$

We therefore define the <u>characteristic function</u> of $(M,\ \Pi_N)$, an n-person game in normal form, by

$$(4.2) \qquad \overset{M}{v}(S) = \underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}} \sum_{i \varepsilon S} M_i\ (\pi_S,\ \pi_{N-S})$$

and the <u>complement characteristic function</u> of $(M,\ \Pi_N)$ by

$$(4.3) \qquad \overset{M}{\bar{v}}(S) = \underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}} \left[M^* - \sum_{i \varepsilon N-S} M_i\ (\pi_S, \pi_{N-S}) \right].$$

In section 2 the complement $\bar{v}$ of a characteristic function v was defined by $\bar{v}(S) = v(N) - v(N-S)$.

The following theorem shows that this definition is equivalent to (4.3), hence the relations between v and $\bar{v}$ proved in Charnes-Rousseau-Seiford [1978] are valid for v^M and $\bar{v}^M$ defined by (4.2)and (4.3).

<u>Theorem 4.1:</u> $\bar{v}^M(S) = v^M(N) - v^M(N-S)$

Proof: $$\bar{v}^M(S) \equiv \underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}} \left[M^* - \sum_{i\in N-S} M_i\ (\pi_S, \pi_{N-S}) \right]$$

$$= M^* - \underset{\Pi_S}{\text{Min}}\ \underset{\Pi_{N-S}}{\text{Max}} \sum_{i\in N-S} M_i\ (\pi_S, \pi_{N-S})$$

$$= \underset{\Pi_N}{\text{Max}} \sum_{i\in N} M_i(\pi_N) - \underset{\Pi_S}{\text{Min}}\ \underset{\Pi_{N-S}}{\text{Max}} \sum_{i\in N-S} M_i\ (\pi_S, \pi_{N-S})$$

$$= v^M(N) - \underset{\Pi_S}{\text{Min}}\ \underset{\Pi_{N-S}}{\text{Max}} \sum_{i\in N-S} M_i\ (\pi_S, \pi_{N-S}).$$

Since $v^M(N\text{-}S) \equiv \underset{\Pi_{N-S}}{\text{Max}}\ \underset{\Pi_S}{\text{Min}} \sum_{i\in N-S} M_i\ (\pi_S, \pi_{N-S})$, the proof is completed by applying the minimax theorem to the two person zero sum game between S and N-S with payoff (to N-S) $\sum_{i\in N-S} M_i\ (\pi_S, \pi_{N-S})$. Q.E.D.

As in section 2, we remark that any reasonable "goal" of coalition S should lie between $v^M(S)$ and $\bar{v}^M(S)$. Thus, any characteristic function which attempts to model a game in normal form should lie between v^M and $\bar{v}^M$. By theorem 4.1, if we mollify v^M and v^{-M} as characteristic functions, we would obtain results identical to those of section 2, and our theory would have failed to capture the "normal form" structure of the game.

Therefore, for games in normal form, we first mollify the payoffs and then construct a characteristic function from these mollified payoffs.

As motivation for the general case, we first reexamine Harsanyi-Selten's modified characteristic function (C_η with $\eta = \frac{1}{2}$, in Selten's notation) given by

(4.4) $$\hat{h}(S) = \tfrac{1}{2}M^* + \tfrac{1}{2}\Delta_S$$

or equivalently by

$$(4.5) \qquad \hat{h}(S) = \underset{\Pi_S}{\text{Max}} \underset{\Pi_{N-S}}{\text{Min}} \left[\tfrac{1}{2} \sum_{i\varepsilon S} M_i \;(\pi_S, \pi_{N-S}) + \tfrac{1}{2} \;(M^* - \sum_{i\varepsilon N-S} M_i \;(\pi_S, \pi_{N-S})) \right]$$

For conciseness, we frequently will omit the arguments π_S and π_{N-S} from the payoff functions; our meaning, however, should be clear.

Note in (4.5) that $\hat{h}$ is obtained from the average of the two payoffs used in defining v^M and $\bar{v}^M$. It can be shown that $\hat{h}(S)$ is the Nash arbitrated threat solution for the two person game between coalitions S and N-S with payoff functions $\sum_{i\varepsilon S} M_i$ and $\sum_{i\varepsilon N-S} M_i$, under the assumption of linearly transferable utility between the two players at the rate 1:1. This rate does not seem reasonable; a dollar should be worth more to a smaller coalition that to a larger one. A more reasonable assumption might be a linear transfer of utility between S and N-S at the rate $|S| : |N-S|$. Under this assumption, the Nash solution is given by

$$(4.6) \qquad \underset{\Pi_S}{\text{Max}} \underset{\Pi_{N-S}}{\text{Min}} \frac{n}{2(n-s)} \left[\left(\frac{n-s}{n}\right) \sum_{i\varepsilon S} M_i + \left(\frac{s}{n}\right)\left(M^* - \sum_{i\varepsilon N-S} M_i\right) \right]$$

where $s = |S|$

Observe that the payoffs used in defining v^M and $\bar{v}^M$ are themselves mollified in (4.6). We, therefore, define a <u>payoff mollifier</u>, denoted $h(S)$, as any characteristic function which results from the mollification (convex combination) of the payoffs used in defining v^M and $\bar{v}^M$. In particular, a <u>constant payoff mollifier</u> is defined by

$$(4.7) \qquad h_\mu(S) = \underset{\Pi_S}{\text{Max}} \underset{\Pi_{N-S}}{\text{Min}} \left[(1-\mu) \sum_{i\varepsilon S} M_i + \mu(M^* - \sum_{i\varepsilon N-S} M_i) \right]$$

where $\mu \varepsilon [0,1]$,

and a <u>coalitional payoff mollifier</u> is defined by

$$(4.8) \qquad h(S) = \underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}} \left[(1-\mu_S) \sum_{i\varepsilon S} M_i + \mu_S \left(M^* - \sum_{i\varepsilon N-S} M_i\right) \right]$$

where $\mu_S \,\varepsilon\, [0,1]$, $\forall$ S.

Since the payoff functions used in defining a payoff mollifier lie between those used in the definitions of v^M and $\bar{v}^M$, we immediately have the following theorem.

Theorem 4.2: Let h(S) be any payoff mollifier for the n-person normal form game (M, Π_N).

Then (i) $v^M(\emptyset) = h(\emptyset) = \bar{v}^M(\emptyset) = 0$

(ii) $v^M(N) = h(N) = \bar{v}^M(N) = M^*$

(iii) $v^M(S) \leq h(S) \leq \bar{v}^M(S)$, $\forall\ S \subseteq N$.

We note from (4.5) that Selten's construct $\hat{h}(S)$ is a constant mollifier ($=h_{\frac{1}{2}}(S)$) and is not generally superadditive. In view of this and the unreasonable transfer rate for utility assumed, we examine more closely the payoff mollifier suggested by (4.6).

We define a payoff homomollifier as the payoff mollifier for which the weights are given by $\mu_S = \frac{|S|}{|N|}$. This payoff homomollifier will be the Nash arbitrated threat solution under the assumption of linear transferability of utility at the rate $|S| : |N-S|$. In addition, as shown by the following two theorems, the payoff homomollifier will always be a constant sum superadditive characteristic function.

Theorem 4.3: Let h(S) be a coalitional payoff mollifier of (M, Π_N). If the weights are additive for disjoint coalitions, i.e.,

$\mu_S + \mu_T = \mu_{S \cup T}$ whenever $S \cap T = \emptyset$, then h(S) is superadditive.

Proof:

$$h(S) + h(T) = \underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}} \left[(1-\mu_S) \sum_{i\varepsilon S} M_i + \mu_S \left(M^* - \sum_{i\varepsilon N-S} M_i\right)\right]$$

$$+ \underset{\Pi_T}{\text{Max}}\ \underset{\Pi_{N-T}}{\text{Min}} \left[(1-\mu_T) \sum_{i\varepsilon T} M_i + \mu_T \left(M^* - \sum_{i\varepsilon N-T} M_i\right)\right]$$

$$= (\mu_S + \mu_T)\ M^* + \underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}} \left[(1-\mu_S) \sum_{i\varepsilon S} M_i - \mu_S \sum_{i\varepsilon N-S} M_i\right]$$

$$+ \underset{\Pi_T}{\text{Max}}\ \underset{\Pi_{N-T}}{\text{Min}} \left[(1-\mu_T) \sum_{i\varepsilon T} M_i - \mu_T \sum_{i\varepsilon N-T} M_i\right]$$

$$h(S \cup T) = \underset{\Pi_{S\cup T}}{\text{Max}}\ \underset{\Pi_{N-S\cup T}}{\text{Min}} \left[(1-\mu_{S\cup T}) \sum_{i\varepsilon S\cup T} M_i + \mu_{S\cup T}\left(M^* - \sum_{i\varepsilon N-S\cup T} M_i\right)\right]$$

$$= \mu_{S\cup T} M^* + \underset{\Pi_{S\cup T}}{\text{Max}}\ \underset{\Pi_{N-S\cup T}}{\text{Min}} \left[(1-\mu_{S\cup T}) \sum_{i\varepsilon S\cup T} M_i - \mu_{S\cup T} \sum_{i\varepsilon N-S\cup T} M_i\right].$$

Now let $M_S = \left[(1-\mu_S) \sum_{i\varepsilon S} M_i - \mu_S \sum_{i\varepsilon N-S} M_i\right]$

$M_T = \left[(1-\mu_T) \sum_{i\varepsilon T} M_i - \mu_T \sum_{i\varepsilon N-T} M_i\right]$

and $M_{S\cup T} = \left[(1-\mu_{S\cup T}) \sum_{i\varepsilon S\cup T} M_i - \mu_{S\cup T} \sum_{i\varepsilon N-S\cup T} M_i\right]$.

Then $M_{S\cup T} = M_S + M_T$.

If (π_S^*, π_{N-S}^*), $(\bar{\pi}_T, \bar{\pi}_{N-T})$ and $(\hat{\pi}_{S\cup T}, \hat{\pi}_{N-S\cup T})$ are optimal pairs of joint strategies, we then have

$$\underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}}\ M_S(\pi_S, \pi_{N-S}) + \underset{\Pi_T}{\text{Max}}\ \underset{\Pi_{N-T}}{\text{Min}}\ M_T(\pi_T, \pi_{N-T})$$

$$= M_S(\pi_S^*, \pi_{N-S}^*) + M_T(\bar{\pi}_T, \bar{\pi}_{N-T})$$

$$\leq M_S(\pi_S^*, \bar{\pi}_T, \hat{\pi}_{N-S\cup T}) + M_T(\bar{\pi}_T, \pi_S^*, \hat{\pi}_{N-S\cup T})$$

$$= M_{S\cup T}(\pi_S^*, \bar{\pi}_T, \hat{\pi}_{N-S\cup T})$$

$$\leq M_{S\cup T}(\hat{\pi}_{S\cup T}, \hat{\pi}_{N-S\cup T}) \equiv \underset{\Pi_{S\cup T}}{\text{Max}}\ \underset{\Pi_{N-S\cup T}}{\text{Min}}\ M_{S\cup T}(\pi_{S\cup T}, \pi_{N-S\cup T})$$

Therefore, $h(S) + h(T) \leq h(S \cup T)$. Q.E.D.

Theorem 4.4: If in addition to the assumptions of Theorem 4.3 $\mu_N = 1$, then the coalitional payoff mollifier h(S) is constant sum.

Proof:

$$h(S) + h(N-S) = \underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}} \left[(1-\mu_S) \sum_{i \in S} M_i + \mu_S \left(M^* - \sum_{i \in N-S} M_i\right)\right]$$

$$+ \underset{\Pi_{N-S}}{\text{Max}}\ \underset{\Pi_S}{\text{Min}} \left[(1-\mu_{N-S}) \sum_{i \in N-S} M_i + \mu_{N-S} \left(M^* - \sum_{i \in S} M_i\right)\right]$$

$$= \mu_S M^* + \underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}} \left[(1-\mu_S) \sum_{i \in S} M_i - \mu_S \sum_{i \in N-S} M_i\right]$$

$$+ \mu_{N-S} M^* + \underset{\Pi_{N-S}}{\text{Max}}\ \underset{\Pi_S}{\text{Min}} \left[(1-\mu_{N-S}) \sum_{i \in N-S} M_i - \mu_{N-S} \sum_{i \in S} M_i\right]$$

$$= (\mu_S + \mu_{N-S}) M^* + \underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}} \left[(1-\mu_S) \sum_{i \in S} M_i - \mu_S \sum_{i \in N-S} M_i\right]$$

$$+ \underset{\Pi_{N-S}}{\text{Max}}\ \underset{\Pi_S}{\text{Min}} \left[\mu_S \sum_{i \in N-S} M_i - (1-\mu_S) \sum_{i \in S} M_i\right]$$

$$= (\mu_S + \mu_{N-S}) M^* + \underset{\Pi_S}{\text{Max}}\ \underset{\Pi_{N-S}}{\text{Min}} \left[(1-\mu_S) \sum_{i \in S} M_i - \mu_S \sum_{i \in N-S} M_i\right]$$

$$- \underset{\Pi_{N-S}}{\text{Min}}\ \underset{\Pi_S}{\text{Max}} \left[(1-\mu_S) \sum_{i \in S} M_i - \mu_S \sum_{i \in N-S} M_i\right]$$

$$= \mu_N M^* = M^* = h(N). \qquad \text{Q.E.D.}$$

We note that the construction of payoff mollifiers for a normal form game differs from the construction of mollifiers for the characteristic function derived from the game. In the former, the max-min operator is applied to a convex combination of the two payoff functions, while in the latter, one takes a convex combination of the two max-min values. Though, in general, these two constructions will lead to different values, the following theorem provides a partial answer as to when the constructions coincide.

Theorems: Let (N, v) be a superadditive characteristic function game and (M^v, S_N) the associated von Neumann-Morgenstern normal form game. Then both mollifier constructions coincide, i.e.,

$$w(S) = h(S) \qquad \forall \; S \subseteq N$$

Proof: The key to the proof lies in the observation that the construction of M^v is such that the payoff to a particular coalition depends only on the strategies of the members of that coalition. Thus, for a particular $\bar{\pi}_S \in \Pi_S$

$$M_S{}^v (\bar{\pi}_S) \equiv \sum_{i \varepsilon S} M_i{}^v (\bar{\pi}_S, \pi_{N-S}) \text{ is constant for all } \pi_{N-S} \varepsilon \, \Pi_{N-S}.$$

Hence, it follows that for any S

$$h(S) = \underset{\Pi_S}{\text{Max}} \; \underset{\Pi_{N-S}}{\text{Min}} \left[(1 - \mu_S) \sum_{i \varepsilon S} M_i{}^v + \mu_S \,(M^{v*} - \sum_{i \varepsilon N-S} M_i{}^v)\right]$$

$$= \underset{\Pi_S}{\text{Max}} \; \underset{\Pi_{N-S}}{\text{Min}} \left[(1 - \mu_S)\, M_S{}^v (\pi_S) + \mu_S \left(M^{v*} - M^v_{N-S} (\pi_{N-S})\right)\right]$$

$$= \underset{\Pi_S}{\text{Max}} \; (1 - \mu_S)\, M_S{}^v (\pi_S) + \underset{\Pi_{N-S}}{\text{Min}} \; \mu_S \,(M^{v*} - M^v_{N-S} (\pi_{N-S}))$$

$$= (1 - \mu_S)\, v(S) + \mu_S \, \bar{v}\, (S)$$

$$= w(S).$$

Q.E.D.

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