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Application of Connectivity On Graph Theory

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Abstract: The applications of graph theory have become an exciting research topic in recent years. Some basic definitions and notations are presented. Digraph, Complete graph, Connected or Weakly connected, Strongly connected, Cut Vertex, Cut Set and bridge, Cut Edges and Bonds are discussed. The definitions of Connectivity, Edge Connectivity and Vertex Connectivity are expressed. Moreover, Examples, Theorems, proposition related to Connectivity are mentioned.

Keywords: Digraph, Cycle, Complete graph, Edge Connectivity, Vertex Connectivity.

I. INTRODUCTION

Graph Theory is delightful playground for the exploration of proof technique in discrete mathematics and its results have applications in many areas of the computing Sciences, social Sciences and natural Sciences. Many textbooks have been written about graph theory will be studied, but the surface of some parts of discrete mathematics can be only scratched. Some basic definitions and notations are described. It contains Graph, Digraph, Complete graph, Connected Graph or weakly connected Graph, Unilaterally connected Graph, Strongly connected Graph, Cut vertex Graph, Cut set Graph and bridge Graph, Cut edges and Bonds. The definitions of connectivity, Edge connectivity and Vertex connectivity are expressed. Finally, examples, theorems and propositions related to connectivity are dealt.

II. SOME BASIC DEFINITION AND NOTATION DEFINITIONS

A graph **G** is defined to be a pare (V(G),E(G)), where V(G) is a non-empty finite set of elements called vertices (or points or notes) and E(G) is a finite family of unordered pare of elements of V(G) called edges (or lines). We use the symbols $v(G)$ and $\varepsilon(G)$ to denote the numbers of vertices and the number of edges in a graph G. Two vertices v_1, v_2 of G are adjacent, if they are joined by an edge of G. An edge is incident to its endpoints. For example, in Fig.1,e_1 is incident to v_1 and v_2 .v_1 and v_2 are adjacent. An empty graph is one with no edges. An edge with distinct ends is called a link. The order of graph, denoted by n(G), is the number of vertices, and the **size** of a graph, denoted by m(G), is the number of edges. Graphs are finite or infinite according to their order; however the graphs we consider are all finite. A graph is finite if both its vertex set and edge set are finite. An edge with identical ends is called a **loop**. In Fig.1, v_1,v_3,v_5 are loops. A graph is simple if it has no loops and no two of its links join the same pare of vertices [1].

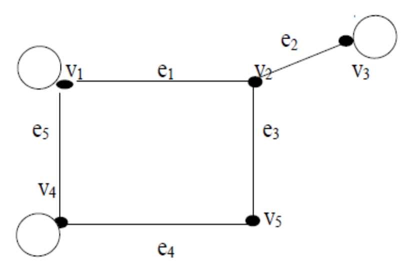
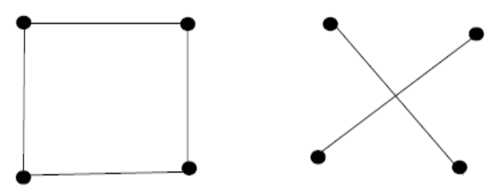


Figure1. Simple Graph.

The complement G^c of G is the simple graph which has V(G) as its vertex-set, and in which two vertices are adjacent if and only if they are not adjacent in G. It is shown in Fig.2.



(a) G (b) Complement G^c

Figure2. Graph G and Complement G^c.

A graph G has n vertices and m edges. Then complement of G has n vertices and $\frac{n(n-1)}{2} - m$ edges.

III. CUT VERTEX, CUT EDGE, DIGRAPH, COMPLETE GRAPH

A. Complete Graphs

A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph. The complete graph on n vertices is usually denoted by K_n. K_n has $\frac{n(n-1)}{2}$ edges. It is shown in Fig3.

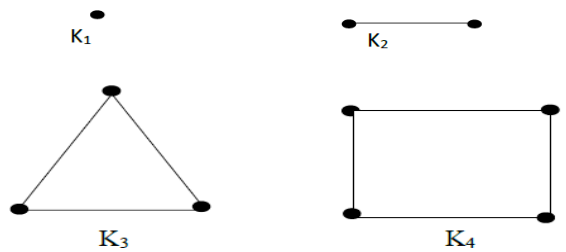


Figure3. Complete Graphs.

B. Digraph ((directed Graph)

A digraph G=(V,E) is a graph in which each edge e=(i,j) has a direction from its "initial point" I to its "termal point" j.

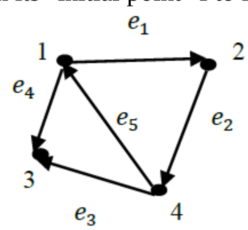
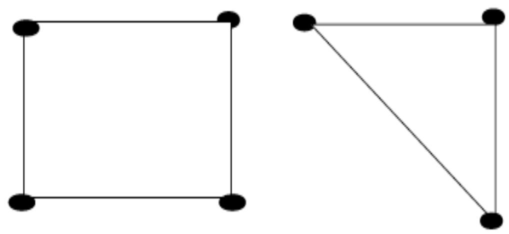


Figure4. Digraph.

C. Subgraphs



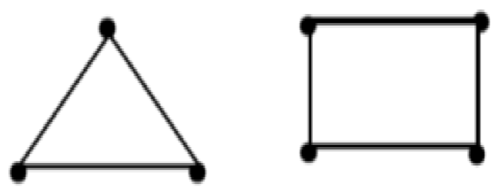
(a)A graph G **(b) A subgraph H**

Figure5. A graph G and a subgraph H.

A subgraph of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It is shown in Fig-5.

D. Cycle

The n –cycle for n ≥ 3, denoted by Cn, is the graph with n vertices, $v_1, v_2, \dots, v_n$ and edge set E={$v_1v_2, v_2v_3, \dots, v_{n-1}v_n \geq$,vnv1.C_n has n vertices and n edges. A cycle is odd (even) cycle if its length is odd (even). A k-cycle is a cycle of length k.



(a) Odd cycle, **(b) Even cycle**

Figure6. Cycles.

E. Tree

An acyclic graph is one that contains no cycles. A tree is a connected acyclic graph.

Example: The tree on six vertices are shown in Fig-7.

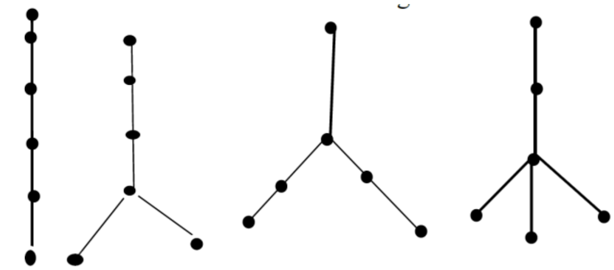


Figure7. Tree.

F. Spanning tree

A spanning tree of G is a spanning subgraph of G that is a tree.

G. Cut vertex, Cut Set and Bridge

Sometimes the removal of a vertex and all edges incident with it produces a subgraph with more connected components. A cut vertex of a connected graph G is a vertex whose removal increases the number of components clearly if v is a cut vertex of a connected graph G, G-v is disconnected. A cut vertex is also called a cut point. Analogously, an edge whose removal produces a graph with more connected components then the original graph is called a cut edge or bridge. The set of all minimum number of edges of G whose removal disconnects a graph G is called a cut set of G. Thus a set S of a satisfy the following.

- S is a subset of the edge set E of G.
- Removal of edges from a connected graph G disconnected G
- No proper Subset of G satisfy the condition

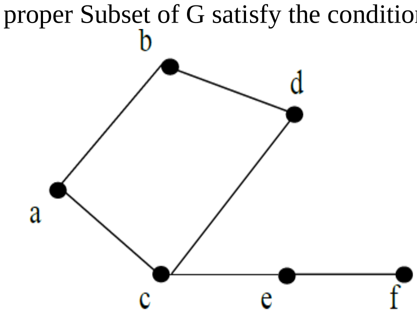


Figure8. Cut vertex, Cut Set and Bridge.

In the graph in figure above, each of the sets [{b, d} , c, d}, {c, e}] and [{e, f}] is a cut set. The edge {e, f} is the only bridge. The singleton set consisting of a bridge is always a cut of set of G.

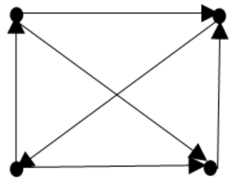
H. Connected or weakly connected

A directed graph is called connected at weakly connected if it is connected as an undirected graph in which each directed edge is converted to an undirected graph.

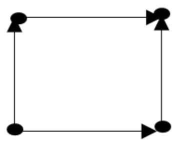
i) Unilaterally Connected: A simple directed graph is said to be unilaterally connected if for any pair of vertices of the graph at least one of the vertices of the pair is reachable from other vertex.

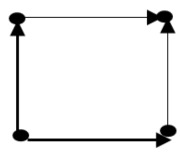
ii) Strongly Connected: A directed graph is called strongly connected if for any pair of vertices of the graph both the vertices of the pair are reachable from one another.

For the digraphs is Fig.9 the digraph in (a) is strongly connected, (b) it is weakly connected, while in (c) it is unilaterally connected but not strongly connected.



(a)Strongly connected





(b)Weakly connected **(c) Unilateraly connected**

Figure9. Connectivity in Digraph.

Note that a unilaterally connected digraph is weakly connected but a weakly connected digraph is not necessarily unilateraly connected. A strongly connected digraph is both unilaterally and weakly connected.

I. Cut edges and Bonds

The cut edge e of G is an edge e such that $\omega(G-e) > \omega(G)$. The graph of Fig.10 has the three cut edges indicate.

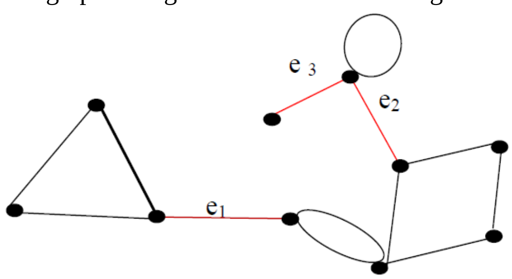


Figure10. The edge cut of a graph.

Theorem-1: An edge e of G is a cut edge of G if and only if e is contained in no cycle of G.

Proof: Let e be a cut edge of G. Since $\omega(G-e) > \omega(G)$, there exist vertices u and v of G that are connected in G but not in G-e. Then there is some (u,v)- path P in G which necessarily traverses e. Suppose that x and y are the ends of e; and that x precedes y on P. In G-e, u is connected to x by a section of P and y is connected to v by a section of P. If e were in a cycle C, x and y would be connected in G-e by the path C-e. Thus, u and v would be connected in G-e and it is a contradiction. Conversely, suppose that e=xy is not a cut edge of G; thus, $\omega(G-e) = \omega(G)$.Since there is an (x , y)-path (namely xy in G, x and y are in the same component of G. It follows that x and y are in the same component of G-e. Hence, there is an (x , y) -path P in G-e. But then e is in the cycle P+e of G.

Example-2: In the following graph, the cut edge is [(c , e)]. By removing the edge (c ,e) from the graph, it becomes a disconnected graph.

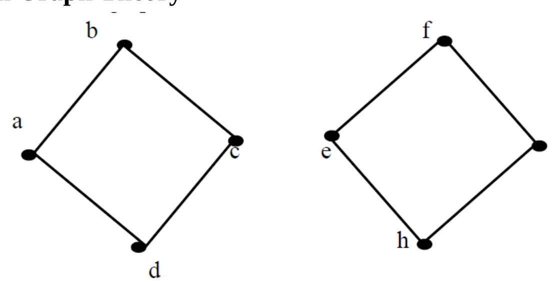
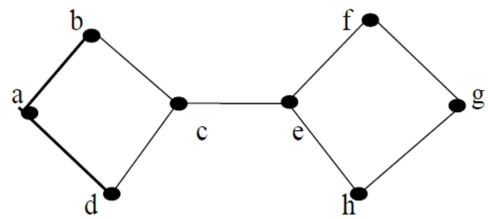


Figure11. Connected Graph G.

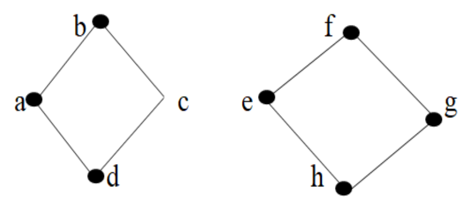
In the below graph, removing the edge (c, e) breaks the graph into two which is nothing but a disconnected graph. Hence, the edge (c ,e) is a cut edge of the graph.

Note – Let 'G' be a connected graph with 'n' vertices , then

- A cut edge e ∈ G iff the edge 'e' is not a part of any cycle in G.
- The maximum number of cut edges possible in 'n-1'
- Whenever cut edges exist, cut vertices also exists because at least one vertex of a cut edge is a cut vertex.
- If a cut vertex exists, then a cut edge may or may not exist



(a) Connected Graph G



(b) Disconnected Graph G

Figure12a)Connected Graph G,b)Disconnected Graph G.

J. Cut set of a Graph

Let 'G' =(V , E) be a connected graph. A subset E' of E is called a cut set of G if deletion of all the edges of E' from G makes G disconnect. If deleting a certain number of edges from a graph makes it disconnected, then those deleted edges are called the cut set of the graph.

Example-1: Take a look at the following graph . Its cut set is $E_1 = \{e_1, e_3, e_5, e_8\}$.

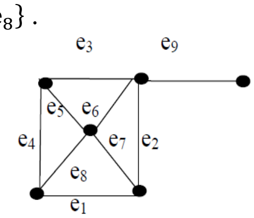


Figure13. Connected Graph G.

After removing the cut set E1 from the graph, it would appear as follow-

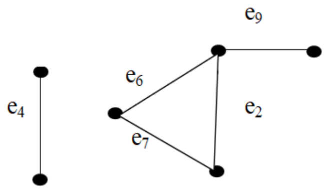


Figure14. Disconnected Graph.

Similarly there are other cut sets that can disconnect the graph-

- $E_3 = \{e_9\}$ – Smallest cut set of the graph.
- $E_4 = \{e_3, e_4, e_5\}$

IV. CONNECTIVITY OF A GRAPH

A. Connectivity

To study the measure of connectedness of a graph G we consider the minimum number of vertices and edges to be removed from the graph in order to disconnect it.

Example-1: In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverses from vertex 'a' to vertex 'e' using the path 'a-b-e'

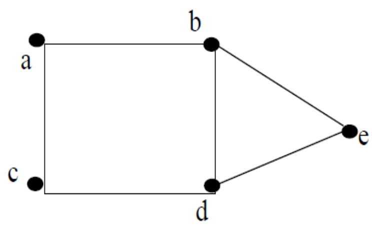


Figure15. Connected Graph G.

Example-2: In the following example, traversing from vertex 'a' to vertex 'f' is not possible because there is no path between them directly or indirectly. Hence it is a disconnected graph.

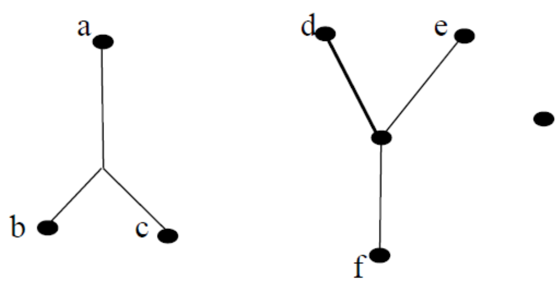


Figure16. Disconnected Graph.

B. Edge Connectivity

Let G be a connected graph. The edge connectivity of G is the minimum number of edges whose removal results in a disconnected or trivial graph G is denoted by E(G)

(i) If G is a disconnected graph , then E(G) = 0
(ii) Edge connectivity of a connected of a connected graph G with a bridge is 1.

Example1: Take a look at the following graph. By removing two minimum edges, the connected graph becomes disconnected. Hence, its edge connectivity E(G) is 2.

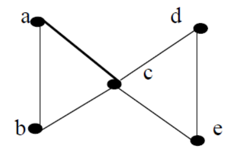


Figure17. Connected Graph.

Here are the four ways to disconnected the graph by removing two edge-

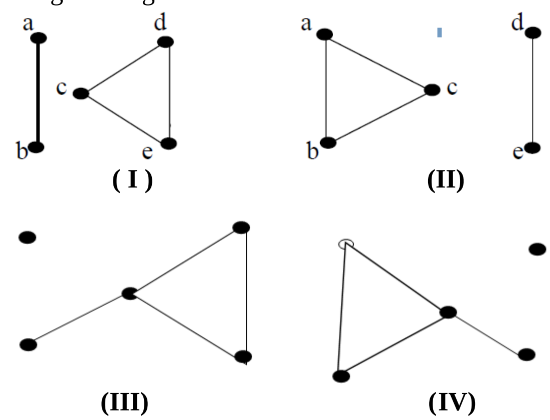


Figure18. Disconnected Graph.

C. Vertex Connectivity

Let G be a connected graph. The vertex connectivity of G is the minimum number of vertices whose removal results in a disconnected or trivial graph. The vertex connectivity of a connected graph is denoted by V(G) .

(i) If G is a disconnected graph, then E(G) =0.
(ii) Edge connectivity of a connected graph G with a bridge is 1.
(iii) The complete graph K_n cannot be disconnected by removing any number of vertices, but the removal of n-1 vertices results in a trivial graph. Hence $k(k_n)$=n-1.
(iv) The vertex connectivity of graph of order at least there is one if and only if has a cut vertex.
(v) Vertex connectivity of a path is one and that of cycle $c_n(n \geq 4)$, is two

Example1:
Find the (i) vertex sets of components, (ii) cut-vertices and (iii) cut- edges (iii)cut-edge of the graph given below.

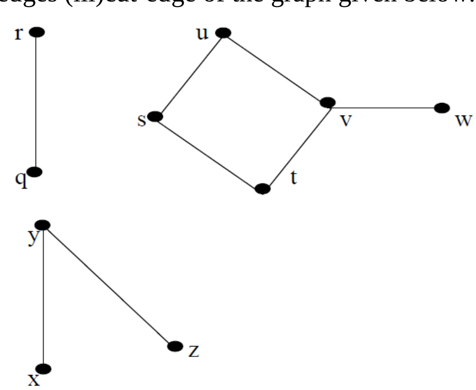


Figure19. The vertex connectivity of G.

Solution: The graph has three components. The vertex set of the components are $\{q, r\}$, $\{s, t, u, v, w\}$ and $\{x, y, z\}$. The cut vertices of the graph are t and y. It cut-edges are qr ,st , xy and yz.

Example2: Is the directed graph given below strongly connected?

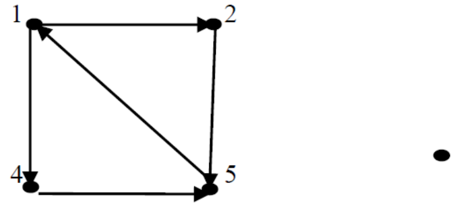


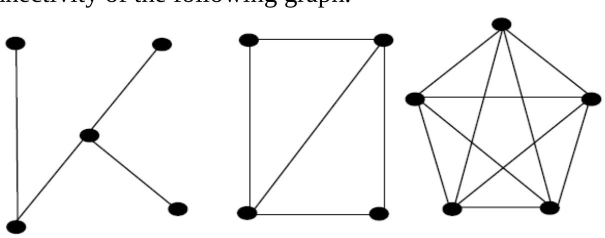
Figure20. Connected Graph G.

Solution: The possible pairs of vertices and the forward and backward paths between them are shown below for the given graph.

Pairs of vertices	Forward path	Backward path
(1,2)	1-2	2-3-1
(1,3)	1-2-3	3-1
(1,4)	1-4	4-3-1
(2,3)	2-3	3-1-2
(2,4)	2-3-1-4	4-3-1-2
(3,4)	4-3	4-3

Therefore, We see that between every pair of distinct vertices of the given graph there exists a forward as wall as backward path, and hence it is strongly connected.

Example-3: Find the vertex connectivity and Edge connectivity of the following graph.



G_1 G_2 G_3

Figure21. Connected Graph G.

G1:vertex connectivity = 1
edge connectivity = 1
G2:vertex connectivity = 2
edge connectivity =2
G3:vertex connectivity = 4
edge connectivity = 4

Theorem-1: vertex connectivity $\leq$ edge connectivity $\leq$ minimum degree of vertices.

Let G be any given graph. First we show that $k' \leq \delta$.

If G is trivial, k = 0.Therefore, k' = δ .Suppose that G is non-trivial. The edges incident to a vertex v of minimum degree from an edge cut of G. Since k' is number of element of smallest edge cut of G. $k'(G) \leq \delta(G)$,that is $k' \leq \delta$. (1)

It remains to show that $k(G) \leq k'(G)$. We shall prove by induction on k'. The result is true if k' = 0, Since then G must be either trivial or disconnected. Then also k = 0. So, k = k'. Suppose that the result holds for all graphs with edge connectivity less than n where $n > 0$. Let G be a graph with $k' = k'(G) = n$. Let e be an edge in a n-edge cut of G.

Putting H= G-e, we have $k'(H) = n-1$ and so, by induction hypothesis, $k(H) \leq k'(H) = n-1$.

Case (i): If H contains a complete graph as spanning subgraph, then so does G. Then both H and G have no vertex cut and $k(G) = k(H) \leq k'(H) = n - 1 < n = k'(G)$

$$k(G) < k'(G).$$

Case(ii): If H does not contain a complete graph as a spanning subgraph. Then H has at least a vertex cut. Let S be a vertex cut of H with k (H) elements.

Then either(a) G – S is disconnected (or)

(b) G – S is connected and e is a cut edge of G – S

If (a)holds, S is also a vertex cut of G which has k(H) element. So, $k(G) \leq k'(H) = n - 1 < n = k'(G)$.

Hence $k(G) < k'(G)$.

If (b) holds, there are two possibilities either(b_1) $v(G–S)=2$ (or) (b_2) G –S has a 1 vertex cut {v}.

If (b1) holds, since $v(G – S) = 2$,

we have $v(G) - v(S) = 2$

$v(G) = k(H) + 2$

Therefore $v(G) - 1 = k(H) + 1$

Then

$k(G) \leq v(G)-1=k(H) + 1 \leq n - 1 + 1 = n = k'(G)$.

$k(G) \leq k'(G)$.

If (b) holds, then $S \cup \{v\}$ is vertex cut of G and

$k(G) \leq k(H)+1 \leq n - 1 + 1 = n = k'(G)$.

By induction, thus, $k(G) \leq k'(G)$.

For all cases, we have $k \leq k'$. (2)

By Equation (1) and Equation (2), $k \leq k' \leq \delta$.

Example-3: Calculate E(G) and k(G) for the following graph.

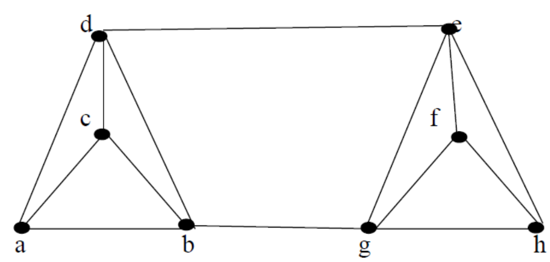


Figure22. Connected Graph G.

Solution:

From the graph,

$$\delta(G) = 3$$

$$k(G) \leq k'(G) \leq \delta(G) = 3 \quad (1)$$

$$k(G) \geq 2 \quad (2)$$

Deleting the edges $\{d, e\}$ and $\{b, h\}$, we can disconnect G.

Proposition.

Let G be connected with $v \geq 3$. we show that

(a) If G be has a cut edge, ten G has a vertex v such that

$\omega G - v > \omega(G)$.

(b)The converse of (a) is not necessarily is true.

Proof: (a) Let e = uv be a cut edge of G. Then G – e is disconnected.

So,u and v lie in different conponents of G, say u lies in G and v lies G2.

There is no path between any vertex of g1 and any vertex of G2.

Clearly G – v or G – uis also disconnected and $\omega G - v > \omega(G)$.

(b) The converse of (a) is not necessary true. See the following figure23.

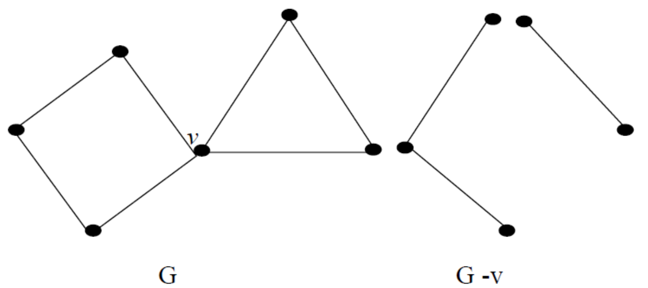


Figure23. Connected Graph G.

V. CONCLUSION

Applications are the popular on graph theory. There are many different applications on different fields of graph theory. Among them, application of connectivity that was presented is distinct on human society. Some basic definitions and notations are described. Graph, Digraph, Complete graph, Connected Graph or weakly connected Graph, Unilaterally connected Graph, Strongly connected Graph, Cut vertex Graph, Cut set Graph and bridge Graph, Cut edges and Bonds are expressed. The definitions of connectivity, Edge connectivity and Vertex connectivity are studying. Finally, examples, theorems and propositions related to connectivity are dealt. I were continue studying connectivity of a graph.

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