DOUBLE SUBFRACTIONAL INTEGRALS AND MOLLIFIER APPROXIMATION

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We consider double Riemann-Stieltjes integrals as approximations to double Stratonovich subfractional integrals. The convergence, in mean square and uniformly on compact time intervals, to the double Stratonovich integral is shown for the mollifiers associated with the even and the odd fractional Brownian motions and for integrands that are continuous or given by bimeasures.

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1. INTRODUCTION

In recent years there was an increasing interest for processes that possess long-range dependence and are self-similar, both for their applications in telecommunication network, economics, hydrology, etc., and also for their intrinsic mathematical interest. The (classical) fractional Brownian motion (cfBm) for short) widely appears as the main mathematical object in these studies. Besides its long-range dependence and self-similarity properties, cfBm has stationary increments.

In the present note we consider two Gaussian processes with long-range dependence and which are self-similar, closely related to fBm, but without stationary increments. These processes are *the even fractional Brownian motion* (efBm), and *the odd fractional Brownian motion* (ofBm). Both processes have increments over non-overlaping intervals more weakly correlated and their covariances decays at a higher rate in comparison with cfBm. In some applications (such as turbulence) cfBm is an adequate model for small increments, but it seems to be inadequate for large increments. For this reason, efBm and ofBm may be an alternative to cfBm in some stochastic models.

Sometimes, efBm is called *subfractional Brownian motion* and appears naturally as weak limits of occupation time fluctuations of branching systems.

An example of important nonlinear functionals associated with the above mentioned processes are multiple Stratonovich integrals.

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Multiple Stratonovich integrals for cfBm are studied in [3], [4] by using the reproducing kernel Hilbert space theory, and in [8] by using fractional integrals and derivatives for deterministic functions of several variables and a transfer principle from the multiple integrals with respect to Brownian motion.

The case of multiple integrals with respect to efBm and ofBm is treated in [11].

Strong approximations in mean square of multiple Stratonovich integrals are given in [2], [5] for the case of Brownian motion, in [10], [11] for the case of cfBm and in [11], [12] for efBm, and ofBm and Wong-Zakai and series expansion approximations.

In the present paper we consider the problem of approximation of the double Stratonovich fractional integrals with respect to efBm and ofBm (also we call them *double Stratonovich subfractional integrals*) by ordinary double integrals with respect to the mollifier approximation.

For deterministic integrands which are continuous or given by bimeasures, we show the convergence in mean square and uniformly on every compact time interval of the Riemann-Stieltjes integrals with respect to the mollifiers, to the double Stratonovich subfractional integral (Theorems 2.7).

The method we use in order to obtain the results is based on the explicit expression of the second moment of the double Stratonovich subfractional integrals for continuous integrands and also on the explicit form of the double Stratonovich subfractional integrals for integrands associated with bimeasures.

2. MAIN RESULT

Consider $H \in (\frac{1}{2}, 1)$ and a probability space $(\Omega, \mathcal{F}, P)$ (all processes are defined on this space).

Let $\left(B^{H,e}\right)_{t\in[0,1]}$ and $\left(B_t^{H,o}\right)_{t\in[0,1]}$ be two centered Gaussian processes starting from 0, with covariances

$$C^{H,e}(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H}\right) - \frac{1}{4}\left((s+t)^{2H} + |s-t|^{2H}\right), \quad 0 \le s,t \le 1,$$

$$C^{H,o}(s,t) = \frac{1}{4}\left((s+t)^{2H} - |s-t|^{2H}\right), \quad 0 \le s,t \le 1.$$

We call $B^{H,e}$ (resp. $B^{H,o}$) the *even* (resp. *odd*) *fractional Brownian motion* (efBm and ofBm for short).

Let $\rho : R \to R$ be a mollifier, i.e., ρ is a nonnegative C^∞-function, with support contained in $[0,1]$, and $\int_R \rho(x)\mathrm{d}x = 1$. For $\varepsilon > 0$ define

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right),$$

and introduce the *mollifier approximation* $\left(B_t^{H,i,\varepsilon}\right)_{t\in[0,1]}$ *of the* fBm $B^{H,i}$ *as*

$$B_t^{H,i,\varepsilon} = \left(B_{\cdot\wedge 1}^{H,i} * \rho_\varepsilon\right)(t) = \int_0^\infty B_{s\wedge 1}^{H,i}\rho_\varepsilon(s-t)\mathrm{d}s = \int_0^\varepsilon B_{(s+t)\wedge 1}^{H,i}\rho_\varepsilon(s)\mathrm{d}s.$$

Lemma 2.1. *We have*

$$\sup_{\varepsilon>0} E\left(\sup_{0\le t\le 1}\left|B_t^{H,i,\varepsilon}\right|^{2m}\right) < \infty,\ \lim_{\varepsilon\to 0} E\left[\sup_{0\le t\le 1}\left|B_t^{H,i,\varepsilon} - B_t^{H,i}\right|^{2m}\right] = 0,\ \forall m \ge 1.$$

Proof. It is a standard result in Analysis that

$$\sup_{0\le t\le 1}\left|B_t^{H,i,\varepsilon} - B_t^{H,i}\right| \to 0 \quad \text{as } \varepsilon \to 0.$$

Now, the conclusion follows easily since $B^{H,i}$ being Gaussian, for all $p \ge 1$ we have

$$E\left(\sup_{0\le t\le 1}\left|B_t^{H,i}\right|^p\right) < \infty,$$

hence

$$E\left(\sup_{0\le t\le 1}\left|B_t^{H,i,\varepsilon}\right|^p\right) \le$$

$$\le E\left(\sup_{0\le t\le 1}\left|B_t^{H,i}\right|^p\right)\int_0^\varepsilon \rho_\varepsilon(s)\mathrm{d}s = E\left(\sup_{0\le t\le 1}\left|B_t^{H,i}\right|^p\right) < \infty. \quad \square$$

The concept of multimeasure is useful. For simplicity we consider only the case of bimeasures and give a few properties that are used in the sequel (for more details see [7]). Let $(X_i, \mathcal{B}_i)_{i=1,2}$ be two measurable spaces.

Definition 2.2. A mapping $\mu : \mathcal{B}_1 \times \mathcal{B}_2 \to R$ is a bimeasure if it is a signed measure in each component. Let $\Pi_i = \left\{A_1^i, \ldots, A_N^i\right\}$ be a partition of X_i. The *Fréchet variation* of the bimeasure μ is defined by

$$\|\mu\|_{FV} = \sup_{\varepsilon_{i_1},\varepsilon_{i_2}=\pm 1}\ \sup_{\Pi_1,\Pi_2}\ \sum_{i_1,i_2=1}^{N} \varepsilon_{i_1}\varepsilon_{i_2}\mu\left(A_{i_1}^1 \times A_{i_2}^1\right) < \infty.$$

Remark 2.3. The integral with respect to a bimeasure is defined similarly to that with respect to a signed measure. In particular, if f_1, f_2 are measurable and bounded, then the integral $\int_{X_1\times X_2} f_1(t_1)f_2(t_2)\mu(\mathrm{d}t_1,\mathrm{d}t_2)$ is well defined, is independent of the order of integration and

$$\left|\int_{X_1\times X_2} f_1(t_1)f_2(t_2)\mu(\mathrm{d}t_1,\mathrm{d}t_2)\right| \le \|f_1\|_\infty \|f_2\|_\infty \|\mu\|_{FV}. \tag{3.1}$$

Next, we consider the case $(X_i, \mathcal{B}_i) = \left([0,1], \mathcal{B}_{[0,1]}\right)$.

Definition 2.4. A measurable application $f : [0,1]^2 \to R$ is given by a bimeasure μ if

$$f(t_1,t_2) = \mu\left((t_1,1],(t_2,1]\right) \quad \text{for a.e. } (t_1,t_2) \in [0,1]^2.$$

Remark 2.5. If f is given by a bimeasure, then necessarily f is right continuos (see [7, Proposition 2.5]).

Let $\Delta : 0 = t_0 < t_1 < \cdots < t_{r(\Delta)-1} = 1$ be a partition of $[0,1]$. Put

$$\Delta_j = (t_j, t_{j+1}], \quad |\Delta_j| = t_{j+1} - t_j, \quad \|\Delta\| = \max_j |\Delta_j|,$$

$$B^{H,i}(\Delta_j) = B^{H,i}_{t_{j+1}} - B^{H,i}_{t_j}.$$

The next concept of Stratonovich integrability is introduced in [9] for the case of Brownian motion.

Definition 2.6. We say that $f_2 \in L^2_s([0,1]^2)$ (f square integrable and symmetric) is *Stratonovich integrable with respect to* $B^{H,i}$, $i \in \{e,o\}$, if the limit in $L^2(\Omega)$

$$J^{H,i}_2(f_2) = \lim_{\|\Delta\|\to 0} \sum_{k,l=0}^{r(\Delta)-1} \left(\frac{1}{|\Delta_k|\,|\Delta_l|} \int_{\Delta_k\times\Delta_l} f_2(u,v)\mathrm{d}u\mathrm{d}v \right) B^{H,i}(\Delta_k)\, B^{H,i}(\Delta_l)$$

exists.

Let us denote

$$J^{H,i,\varepsilon}_2(g) = \int_{[0,1]^2} g(u,v)\mathrm{d}B^{H,i,\varepsilon}_u \mathrm{d}B^{H,i,\varepsilon}_v.$$

We have the following main result.

THEOREM 2.7. (a) *Assume that the function* $f : [0,1]^2 \to R$ *is continuous and symmetric. Then*

$$\lim_{\varepsilon\to 0} \sup_{0\le t\le 1} E\left(\left| J^{H,i,\varepsilon}_2(f 1_{[0,t]^2}) - J^{H,i}_2(f 1_{[0,t]^2}) \right|^2 \right) = 0.$$

(b) *Assume that the function* $f : [0,1]^2 \to R$ *is symmetric and given by a bimeasure* μ. *Then*

$$\lim_{\varepsilon\to 0} E\left(\sup_{0\le t\le 1} \left| J^{H,i,\varepsilon}_2(f 1_{[0,t]^2}) - J^{H,i}_2(f 1_{[0,t]^2}) \right|^{2m} \right) = 0, \quad \forall m \ge 1.$$

Proof. (a) As in [10, Proposition 4.2], we have the equation

$$J^{H,i,\varepsilon}_2(f 1_{[0,t]^2}) = \tag{3.2}$$

$$= \int_{[0,\varepsilon]^2} \rho_\varepsilon(v_1)\rho_\varepsilon(v_2) J^{H,i}_2\left(f_2((.-v_1)\vee 0, (.-v_2)\vee 0)\right) \mathrm{d}v_1\mathrm{d}v_2.$$

Denote

$$\varphi^0(s,t) = \frac{H(2H-1)}{2}\left[|s-t|^{2H-2} + (s+t)^{2H-2}\right],$$

$$\varphi^e(s,t) = \frac{H(2H-1)}{2}\left[|s-t|^{2H-2} - (s+t)^{2H-2}\right].$$

By [11, Proposition 3.7], we have

$$E\left[\left|J_2^{H,i}(g)\right|^2\right] = \left\{\int_{[0,1]^2} g(s,t)\varphi^i(s,t)\mathrm{d}s\mathrm{d}t\right\}^2 + \tag{3.3}$$

$$+2\int_{[0,1]^4} g(s,u)g(t,v)\varphi^i(s,t)\varphi^i(u,v)\mathrm{d}s\mathrm{d}t\mathrm{d}u\mathrm{d}v,$$

for g measurable and bounded (in fact the equation holds for more general $g's$).

Now, by (3.2) we have

$$J_2^{H,i,\varepsilon}\left(f1_{[0,t]^2}\right) - J_2^{H,i}\left(f1_{[0,t]^2}\right) =$$

$$= \int_{[0,\varepsilon]^2} \rho_\varepsilon(v_1)\rho_\varepsilon(v_2)\left[J_2^{H,i}(f_2((.-v_1)\vee 0,(.-v_2)\vee 0))) - J_2^{H,i}\left(f1_{[0,t]^2}\right)\right]\mathrm{d}v_1\mathrm{d}v_2,$$

and from (3.3) we deduce that

$$E\left[\left|J_2^{H,i,\varepsilon}\left(f1_{[0,t]^2}\right) - J_2^{H,i}\left(f1_{[0,t]^2}\right)\right|^2\right] \le \varepsilon^2 \int_{[0,\varepsilon]^2} \rho_\varepsilon^2(v_1)\rho_\varepsilon^2(v_2)\times$$

$$\times E\left[\left|J_2^{H,i}\left(f_2((.-v_1)\vee 0,(.-v_2)\vee 0))\right) - J_2^{H,i}\left(f1_{[0,t]^2}\right)\right|^2\right]\mathrm{d}v_1\mathrm{d}v_2 \le$$

$$\le \varepsilon^{-2}\|f\|_\infty^4 \int_{[0,\varepsilon]^2} E\left[\left|J_2^{H,i}(f_2((.-v_1)\vee 0,(.-v_2)\vee 0))) - J_2^{H,i}\left(f1_{[0,t]^2}\right)\right|^2\right]\mathrm{d}v_1\mathrm{d}v_2$$

$$\le C \sup_{|u_i-v_i|\le\varepsilon,\, i=1,2} |f(u_1,u_2) - f(v_1,v_2)|^2 \left[\int_{[0,1]^2}\varphi^i(u,v)\mathrm{d}u\mathrm{d}v\right]^2 =$$

$$= C \sup_{|u_i-v_i|\le\varepsilon, i=1,2} |f(u_1,u_2) - f(v_1,v_2)|^2 \left\{E\left[\left|B_1^{H,i}\right|^2\right]\right\}^2 \to 0 \text{ as } \varepsilon \to 0.$$

(b) First, integration by parts with respect to each variable implies the equations

$$J_2^{H,i,\varepsilon}\left(f1_{[0,t]^2}\right) = \int_{[0,t]^2} B_u^{H,i,\varepsilon} B_v^{H,i,\varepsilon}\mu_t(\mathrm{d}u,\mathrm{d}v), \tag{3.4}$$

$$J_2^{H,i}\left(f1_{[0,t]^2}\right) = \int_{[0,t]^2} B_u^{H,i} B_v^{H,i}\mu_t(\mathrm{d}u,\mathrm{d}v), \tag{3.5}$$

where μ_t is a bimeasure such that $\|\mu_t\|_{FV} \leq \|\mu\|_{FV}$ (see [1, Theorem 3.1]). Then (3.4), (3.5) and Lemma 2.1 imply

$$E\left(\sup_{0\leq t\leq 1}\left|J_2^{H,i,\varepsilon}\left(f1_{[0,t]^2}\right)-J_2^{H,i}\left(f1_{[0,t]^2}\right)\right|^{2m}\right)\leq$$

$$\leq 2^{2m-1}\|\mu\|_{FV}\left\{E\left(\sup_{0\leq t\leq 1}\left|B_t^{H,i,\varepsilon}\right|^{4m}\right)+E\left(\sup_{0\leq t\leq 1}\left|B_t^{H,i}\right|^{4m}\right)\right\}\times$$

$$\times E\left(\sup_{0\leq t\leq 1}\left|B_t^{H,i,\varepsilon}-B_t^{H,i}\right|^{4m}\right)\leq CE\left(\sup_{0\leq t\leq 1}\left|B_t^{H,i,\varepsilon}-B_t^{H,i}\right|^{4m}\right)\to 0 \text{ as } \varepsilon\to 0. \square$$

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