INTERPLAY BETWEEN HOMOLOGICAL DIMENSIONS OF A COMPLEX AND ITS RIGHT DERIVED SECTION

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Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, $\mathfrak{a}$ be a proper ideal of R and M be an R-complex in $\mathrm{D}(R)$. We prove that if $M \in \mathrm{D}^f_{\sqsubset}(R)$ (respectively, $M \in \mathrm{D}^f_{\sqsupset}(R)$), then $\mathrm{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{id}_R M$ (respectively, $\mathrm{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{fd}_R M$). Next, it is proved that the right derived section functor of a complex $M \in \mathrm{D}_{\sqsubset}(R)$ (R is not necessarily local) can be computed via a genuine left-bounded complex $G \simeq M$ of Gorenstein injective modules. We show that if R has a dualizing complex and M is an R-complex in $\mathrm{D}^f_{\square}(R)$, then $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M$ and $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gid}_R M$. Also, we show that if M is a relative Cohen-Macaulay R-module with respect to $\mathfrak{a}$ (respectively, Cohen-Macaulay R-module of dimension n), then $\mathrm{Gfd}_R \mathbf{H}^{\mathrm{ht}_M \mathfrak{a}}_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M + n$ (respectively, $\mathrm{Gid}_R \mathbf{H}^n_{\mathfrak{m}}(M) = \mathrm{Gid}_R M - n$). The above results generalize some known results and provide characterizations of Gorenstein rings.

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1. INTRODUCTION

Throughout this paper, R is a commutative Noetherian ring, $\mathfrak{a}$ is a proper ideal of R and M is an R-complex. The category of R-complexes is denoted $\mathrm{C}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$ and $\square$ to denote genuine boundedness conditions. So, $\mathrm{C}_{\sqsupset}(R)$ is the full subcategory of $\mathrm{C}(R)$ of bounded below complexes (see [2, Definition 2.1.1]). Also, the derived category is denoted $\mathrm{D}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$ and $\square$ to denote homological boundedness conditions (see [2, Definition 4.1.15]). The symbol $\simeq$ is the sign for isomorphism in $\mathrm{D}(R)$ and quasiisomorphisms in $\mathrm{C}(R)$. We also use superscript f to signify that the homology modules are degreewise finitely generated. An R-complex I is semiinjective if the functor $\mathrm{Hom}_R(-, I)$ converts injective quasiisomorphisms into surjective quasiisomorphisms. A semiinjective resolution of an R-complex M is a semiinjective complex I and a quasiisomorphism $M \xrightarrow{\simeq} I$. For an R-complex M the injective dimension $\mathrm{id}_R M$ is defined as

$$\mathrm{id}_R M = \inf \left\{ \ell \in \mathbb{Z} \mid \substack{\exists \text{ semiinjective } R-\text{complex } I \text{ such that} \\ M \simeq I \text{ in } \mathrm{D}(R) \text{ and } I_v = 0 \text{ for all } v < -\ell} \right\}.$$

Several of the main results of this paper involve the hypothesis that R has a dualizing complex. A complex $D \in \mathrm{D}^f_\square(R)$ is dualizing for R if it has finite injective dimension and the canonical morphism $\chi^R_M : R \to \mathbf{R}\mathrm{Hom}_R(D, D)$ is an isomorphism in $\mathrm{D}(R)$. If R has a dualizing complex D, we may consider the functor $-^\dagger = \mathbf{R}\mathrm{Hom}_R(-, D)$. The notion of Gorenstein injective module was introduced by E.E. Enochs and O.M.G. Jenda in [6]. An R-module M is said to be Gorenstein injective, if there exists a $\mathrm{Hom}_R(\mathcal{I}, -)$ exact acyclic complex E of injective R-modules such that $M = \mathrm{Ker}(E_0 \to E_{-1})$. The Gorenstein injective dimension, $\mathrm{Gid}_R M$, of $M \in \mathrm{D}_\sqsubset(R)$ is defined to be the infimum of the set of integers n such that there exists a complex $G \in \mathrm{C}_\sqsubset(R)$ consisting of Gorenstein injective modules satisfying $M \simeq G$ and $G_n = 0$ for $n < -\ell$. Also, an R-complex F is semiflat if the functor $- \otimes_R F$ preserves injective quasiisomorphisms. For an R-complex M the flat dimension $\mathrm{fd}_R M$ is defined as

$$\mathrm{fd}_R M = \inf \left\{ n \in \mathbb{Z} \mid \substack{\exists \text{ semiflat } R-\text{complex } F \text{ such that} \\ F \simeq M \text{ in } \mathrm{D}(R) \text{ and } F_v = 0 \text{ for all } v > n} \right\}.$$

An R-module M is said to be Gorenstein flat, if there exists an $\mathcal{I} \otimes_R -$ exact acyclic complex F of of flat R-modules such that $M = \mathrm{Ker}(F_0 \to F_{-1})$. The Gorenstein flat dimension, $\mathrm{Gfd}_R M$, of $M \in \mathrm{D}_\sqsupset(R)$ is defined to be the infimum of the set of integers n such that there exists a complex $F \in \mathrm{C}_\sqsupset(R)$ consisting of Gorenstein flat modules satisfying $M \simeq F$ and $G_n = 0$ for $n < \ell$. Let M be an R-complex in $\mathrm{D}(R)$. The right derived section functor of the complex M is defined as $\mathbf{R}\Gamma_\mathfrak{a}(M) = \Gamma_\mathfrak{a}(E)$, where E is a semiinjective resolution of M (see [8] and [16]). If $\underline{x} = x_1, \cdots, x_r$ is a generating set for the ideal $\mathfrak{a}$ and $\check{C}_{\underline{x}}$ the corresponding Čech complex, then $\mathbf{R}\Gamma_\mathfrak{a}(M) \simeq M \otimes^{\mathbf{L}}_R \check{C}_{\underline{x}}$ (see [15, Theorem 1.1(iv)]).

It has been shown in [7, Theorem 6.5] that the right derived section functor (with support in any ideal $\mathfrak{a}$) sends complexes of finite flat dimension (respectively, finite injective dimension) to complexes of finite flat dimension (respectively, finite injective dimension). In Section 2, we prove that if $(R, \mathfrak{m})$ is a local ring and $M \in \mathrm{D}^f_\sqsubset(R)$, then $\mathrm{id}_R \mathbf{R}\Gamma_\mathfrak{a}(M) = \mathrm{id}_R M$ (see Theorem 2.2). It shows that the following statements are equivalent:

(i) R is Gorenstein;

(ii) $\mathrm{id}_R \mathbf{R}\Gamma_\mathfrak{a}(R) = \dim(R)$ for any ideal $\mathfrak{a}$ of R;

(iii) $\mathrm{id}_R \mathbf{R}\Gamma_\mathfrak{a}(R) < \infty$ for some ideal $\mathfrak{a}$ of R.

This provides a characterization of Gorenstein rings, which recovers [20, Corollary 2.7]. Next, in 2.4, we prove that if $(R, \mathfrak{m})$ is a local ring and M is an R-complex in $\mathrm{D}^f_\sqsubset(R)$ with $\mathrm{amp}\mathbf{R}\Gamma_\mathfrak{a}(M) = 0$, then $\mathrm{id}_R \mathbf{H}_\mathfrak{a}^{-\inf \mathbf{R}\Gamma_\mathfrak{a}(M)}(M) =$

$\mathrm{id}_R M + \inf \mathbf{R}\Gamma_{\mathfrak{a}}(M)$. Notice that, this result is a generalization of [20, Theorem 2.5]. Also, a flat version of 2.2 is demonstrated. Indeed, it is shown, in Theorem 2.7, that if $(R, \mathfrak{m})$ is a local ring and $M \in \mathrm{D}^f_\square(R)$, then $\mathrm{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{fd}_R M$.

It has been proved in [4, Theorem 5.9] that if $M \in \mathrm{D}_\square(R)$, then

$$\mathrm{Gfd}_R M < \infty \Rightarrow \mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty.$$

Moreover, if R has a dualizing complex, the above implication may be reversed if $\mathfrak{a}$ is in the Jacobson radical of R and $M \in \mathrm{D}^f_\square(R)$. We show that if $(R, \mathfrak{m})$ is a local ring and $M \in \mathrm{D}^f_\square(R)$, then

$$\mathrm{Gfd}_R M < \infty \Rightarrow \mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M, \textit{ and}$$

$$R \textit{ admits a dualizing complex } \Rightarrow \mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M.$$

Then, as a corollary, we prove that if M is a relative Cohen-Macaulay R-module with respect to $\mathfrak{a}$, where is defined as in [20], and that $n = \mathrm{grade}(\mathfrak{a}, M)$, then $\mathrm{Gfd}_R \mathbf{H}^n_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M + n$.

In Section 3, first we prove that if C is an $\Gamma_{\mathfrak{a}}$-acyclic R-complex in $\mathrm{C}_\sqsubset(R)$, then $\mathbf{R}\Gamma_{\mathfrak{a}}(C) \simeq \Gamma_{\mathfrak{a}}(C)$ (see Theorem 3.5). It implies that the right derived section functor of a complex $M \in \mathrm{D}_\sqsubset(R)$ can be computed via a genuine left-bounded complex $G \simeq M$ of Gorenstein injective modules.

Also, as a main result, we show that if $(R, \mathfrak{m})$ is a local ring admitting a dualizing complex and M is an R-complex in $\mathrm{D}^f_\square(R)$, then $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gid}_R M$ (see Theorem 3.9). It shows that the following statements are equivalent:

(i) R is Gorenstein;
(ii) $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(R) = \dim(R)$ for any ideal $\mathfrak{a}$ of R;
(iii) $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(R) < \infty$ for some ideal $\mathfrak{a}$ of R.

This provides a characterization of Gorenstein rings, which improves [20, Corollary 3.10] and [19, Theorem 2.6], that is, we may prove them without assuming that R is Cohen-Macaulay. Next, in Theorem 3.10, we prove a complex version of 2.2, which improves [20, Theorem 3.8]. As a corollary, in 3.11, we deduce that $\mathrm{Gid}_R \mathbf{H}^n_{\mathfrak{m}}(M) = \mathrm{Gid}_R M - n$, wherever $(R, \mathfrak{m})$ is a local ring and M is a Cohen-Macaulay R-module with $\dim_R M = n$.

2. RIGHT DERIVED SECTION FUNCTOR, INJECTIVE DIMENSION AND (GORENSTEIN) FLAT DIMENSION

The following lemma, which is an immediate consequence of [10, Corollary 3.4.4] and [10, Proposition 3.2.2], determines the i-th Bass number $\mu^i_{R_\mathfrak{p}}((\mathbf{R}\Gamma_{\mathfrak{a}}(M))_\mathfrak{p})$ of $(\mathbf{R}\Gamma_{\mathfrak{a}}(M))_\mathfrak{p}$ (see [2, Definition 6.1.18]).

Lemma 2.1. *Let* $(R,\mathfrak{m},k)$ *be a local ring, and let* M *be an* R*-complex in* $\mathrm{D}(R)$. *Then* $\mu_R^i(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) = \mu_R^i(M)$ *for all* $i \in \mathbb{Z}$. *In particular, for every* $\mathfrak{p} \in \mathbf{V}(\mathfrak{a})$ *there is an equality* $\mu_{R_{\mathfrak{p}}}^i((\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}}) = \mu_{R_{\mathfrak{p}}}^i(M_{\mathfrak{p}})$ *for all* $i \in \mathbb{Z}$.

The following theorem, which is one of the main results of this section, provides a comparison between the injective dimensions of a complex and its right derived section functor.

Theorem 2.2. *Let* $(R,\mathfrak{m},k)$ *be a local ring, and let* M *be an* R*-complex in* $\mathrm{D}_{\sqsubset}^f(R)$. *Then* $\mathrm{id}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{id}_R M$.

Proof. Let $s := \mathrm{id}_R M < \infty$. Then, in view of [2, Lemma 6.1.19] and Lemma 2.1, $\mu_{R_{\mathfrak{p}}}^{i+s}((\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and for all $i > 0$. Therefore, it follows from [2, Lemma 6.1.19] that $\mathrm{id}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) \leq s$.

For the opposite inequality, let $t := \mathrm{id}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Then, by [2, Theorem 5.1.6], $\inf\mathbf{R}\mathrm{Hom}_R(T, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) \geq -t$ for all cyclic R-modules T. Hence, in view of [10, Proposition 3.2.2], there are isomorphisms

$$\mathbf{H}_{-t-i}(\mathbf{R}\mathrm{Hom}_R(k, M)) \cong \mathbf{H}_{-t-i}(\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\Gamma_{\mathfrak{a}}(M))) \cong 0$$

for all $i > 0$. Therefore $-\inf\mathbf{R}\mathrm{Hom}_R(k, M) \leq t$, and so $\mathrm{id}_R M \leq t$ by [2, Theorem 6.1.13]. □

The following corollary, which recovers [20, Corollary 2.7], is an immediate consequence of the previous Theorem.

Corollary 2.3. *Let* $(R,\mathfrak{m})$ *be a local ring. Then the following statements are equivalent:*

(i) *R is Gorenstein;*

(ii) $\mathrm{id}_R\mathbf{R}\Gamma_{\mathfrak{a}}(R) = \dim(R)$ *for any ideal* $\mathfrak{a}$ *of* R*;*

(iii) $\mathrm{id}_R\mathbf{R}\Gamma_{\mathfrak{a}}(R) < \infty$ *for some ideal* $\mathfrak{a}$ *of* R.

The following theorem, which is an immediate consequence of Theorem 2.2, is a generalization of [20, Theorem 2.5].

Theorem 2.4. *Let* $(R,\mathfrak{m})$ *be a local ring. Suppose that* M *is an* R*-complex in* $\mathrm{D}_{\square}^f(R)$ *such that* $\mathrm{amp}\mathbf{R}\Gamma_{\mathfrak{a}}(M) = 0$. *Then*

$$\mathrm{id}_R\mathbf{H}_{\mathfrak{a}}^{-\inf\mathbf{R}\Gamma_{\mathfrak{a}}(M)}(M) = \mathrm{id}_R M + \inf\mathbf{R}\Gamma_{\mathfrak{a}}(M).$$

Proof. Let $n := -\inf\mathbf{R}\Gamma_{\mathfrak{a}}(M)$. Since $\mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$, there is an equality

$$\mathrm{id}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{id}_R\Sigma^n\mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) + n.$$

But $\Sigma^n\mathbf{R}\Gamma_{\mathfrak{a}}(M)$ is equivalent to the module $\Sigma^n\mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$ in the category of R-modules. So, we may identify $\Sigma^n\mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$ with $\mathbf{H}_{\mathfrak{a}}^n(M)$. Hence

$\mathrm{id}_R\mathbf{H}^n_{\mathfrak{a}}(M) = \mathrm{id}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) - n$. The desired equality now follows from Theorem 2.2. □

In the following we use the notion of a semifree resolution. A semifree resolution of an R-complex M is a semifree complex F (see [2, Definition 3.1.1]) and a quasiisomorphism $F \xrightarrow{\simeq} M$.

LEMMA 2.5. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be an R-complex in* D(R). *Then*

$$k \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq k \otimes^{\mathbf{L}}_R M.$$

Proof. Let F be a semifree resolution of the residue field k, and let $\underline{x} = x_1, \cdots, x_r$ be a generating set for the ideal $\mathfrak{a}$ and $\check{C}_{\underline{x}}$ be the Čech complex with respect to $\underline{x}$. But, as in the proof of [11, Lemma 2.4], there exists a quasiisomorphism $F \otimes_R \check{C}_{\underline{x}} \simeq F$ in C(R). The result now follows, since $k \otimes^{\mathbf{L}}_R \check{C}_{\underline{x}} \simeq k$. □

The following lemma, which is an immediate consequence of [10, Corollary 3.4.4] and 2.5, determines the i-th Betti number $\beta^{R_{\mathfrak{p}}}_i((\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}})$ of $(\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}}$ (see [2, Definition 6.1.14]).

LEMMA 2.6. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be an R-complex in* D(R). *Then $\beta^R_i(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) = \beta^R_i(M)$ for all $i \in \mathbb{Z}$; In particular, for every $\mathfrak{p} \in \mathbf{V}(\mathfrak{a})$ there is an equality $\beta^{R_{\mathfrak{p}}}_i((\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}}) = \beta^{R_{\mathfrak{p}}}_i(M_{\mathfrak{p}})$ for all $i \in \mathbb{Z}$.*

THEOREM 2.7. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let $\mathfrak{a}$ be a proper ideal of R. Suppose that M is an R-complex in $\mathrm{D}^f_{\sqsupset}(R)$. Then $\mathrm{fd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{fd}_R M$.*

Proof. Let $s := \mathrm{fd}_R M < \infty$. Then, in view of [2, Lemma 6.1.15] and Lemma 2.6, $\beta^{R_{\mathfrak{p}}}_{i+s}((\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and for all $i > 0$. Therefore, it follows from [2, Lemma 6.1.15] that $\mathrm{fd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) \leq s$.

For the opposite inequality, let $t := \mathrm{fd}_R\mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Then, by [2, Theorem 5.1.9], $\sup T \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M)) \geq t$ for all cyclic R-modules T. Hence, in view of Lemma 2.5, there are isomorphisms

$$\mathbf{H}_{t+i}(k \otimes^{\mathbf{L}}_R M) \cong \mathbf{H}_{t+i}(k \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M)) \cong 0$$

for all $i > 0$. Therefore $\sup k \otimes^{\mathbf{L}}_R M \leq t$, and so $\mathrm{fd}_R M \leq t$ by [2, Theorem 5.2.13]. □

The following theorem, which is an immediate consequence of Theorem 2.7, is a flat version of 2.4.

THEOREM 2.8. *Let $(R, \mathfrak{m})$ be a local ring. Suppose that M is an R-complex in $\mathrm{D}^f_{\square}(R)$ such that $\mathrm{amp}\mathbf{R}\Gamma_{\mathfrak{a}}(M) = 0$. Then*

$$\mathrm{fd}_R\mathbf{H}^{-\inf\mathbf{R}\Gamma_{\mathfrak{a}}(M)}_{\mathfrak{a}}(M) = \mathrm{fd}_R M - \inf\mathbf{R}\Gamma_{\mathfrak{a}}(M).$$

Proof. Straightforward verification similar to the proof of Theorem 2.4. □

In the rest of this section, we make a comparison between the Gorenstein flat dimensions of a complex and its right derived section functor.

PROPOSITION 2.9. *Suppose that M is an R-complex in $\mathrm{D}_{\square}(R)$. Then*

$$\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \leq \mathrm{Gfd}_R M.$$

Proof. Notice that if $\mathrm{Gfd}_R M = \infty$, then there is nothing to prove. So, we may assume that $\mathrm{Gfd}_R M < \infty$. Hence, it follows from [4, Theorem 5.9] that $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Now, by [9, Theorem 8.8], there exists $\mathfrak{p} \in \mathbf{V}(\mathfrak{a})$ such that

$$\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{depth} R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} (\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}}.$$

But $\mathrm{depth}_{R_{\mathfrak{p}}} (\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}} = \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. It follows, again by [9, Theorem 8.8], that

$$\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{depth} R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \mathrm{Gfd}_R M$$

as desired. □

PROPOSITION 2.10. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be an R-complex in $\mathrm{D}^f_{\square}(R)$ such that $\mathrm{Gfd}_R M < \infty$. Then*

$$\mathrm{Gfd}_R M \leq \mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M).$$

Proof. By [4, Theorem 5.9], $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Hence, by [9, Theorem 8.7], there are equalities

$$\begin{aligned}\sup(E(k) \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M)) &= \mathrm{depth} R - \mathrm{depth}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \\ &= \mathrm{depth} R - \mathrm{depth}_R M = \sup(E(k) \otimes^{\mathbf{L}}_R M).\end{aligned}$$

Since $M \in \mathrm{D}^f_{\square}(R)$, $\sup(E(k) \otimes^{\mathbf{L}}_R M) = \mathrm{Gfd}_R M$. The result now follows from the fact that $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \geq \sup(E(k) \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M))$ (see [4, Corollary 3.6]). □

The following theorem is a Gorenstein flat version of Theorem 2.7.

THEOREM 2.11. *Let $(R, \mathfrak{m})$ be a local ring, and let M be an R-complex in $\mathrm{D}^f_{\square}(R)$.*

(i) *If $\mathrm{Gfd}_R M < \infty$, then $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M$.*

(ii) *If R admits a dualizing complex, then $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M$.*

Proof. (i) A straightforward application of Proposition 2.9 and Proposition 2.10.

(ii) In view of part (i), we may assume that $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Hence, by [4, Theorem 5.9], $\mathrm{Gfd}_R M < \infty$. The desired equality now follows from Proposition 2.9 and Proposition 2.10. □

COROLLARY 2.12. *Let $(R, \mathfrak{m})$ be a local ring. Suppose that M is relative Cohen-Macaulay with respect to $\mathfrak{a}$ and that $n = \operatorname{grade}(\mathfrak{a}, M)$. Then* $\operatorname{Gfd}_R \mathbf{H}^n_{\mathfrak{a}}(M) = \operatorname{Gfd}_R M + n$.

Proof. Notice that $(\widehat{R}, \widehat{\mathfrak{m}})$ is a local ring admitting a dualizing complex and $M \otimes_R \widehat{R}$ is a relative Cohen-Macaulay $\widehat{R}$-module with respect to $\mathfrak{a}\widehat{R}$ and that $\operatorname{grade}(\mathfrak{a}\widehat{R}, M \otimes_R \widehat{R}) = n$. Hence, in view of [3, Theorem 4.27], we may assume that R is complete; and so it has a dualizing complex. The result therefore follows from Theorem 2.11. □

3. RIGHT DERIVED SECTION FUNCTOR AND GORENSTEIN INJECTIVE DIMENSION

In this section, the category of R-modules is denoted $\mathcal{C}(R)$. Recall from [1, Exercise 4.1.2] that the local cohomology modules of R-module M with respect to $\mathfrak{a}$ can be calculated by an $\Gamma_{\mathfrak{a}}$-acyclic resolution of M. First, we prove the complex version of it.

Definition 3.1. (see [17, 5.7.9]) Let $\mathrm{F} : \mathcal{C}(R) \to \mathcal{C}(R)$ be a left exact functor, and assume that M is an R-complex in $\mathrm{C}_{\sqsubset}(R)$. If $0 \to M \to C_{*,0} \to C_{*,1} \to \cdots \to C_{*,q} \to$ is a Cartan-Eilenberg injective resolution of M, where is defined as in [13, §10.5], define $\mathbb{R}^i(\mathrm{F}M)$ to be $\mathbf{H}_i(\operatorname{Tot}(\mathrm{F}C))$.

LEMMA 3.2. *Let M and $\acute{M}$ be two R-complexes in $\mathrm{C}_{\sqsubset}(R)$, and let $\zeta : M \to \acute{M}$ be a morphism of R-complexes. Suppose that $0 \to M \to C_{*,0} \to C_{*,1} \to \cdots \to C_{*,q} \to$ and $0 \to \acute{M} \to \acute{C}_{*,0} \to \acute{C}_{*,1} \to \cdots \to \acute{C}_{*,q} \to$ are Cartan-Eilenberg injective resolutions of M and $\acute{M}$, respectively. Then there exists a sequence $\{\zeta_{*,q}\}_{q \in \mathbb{N}_0}$ of morphisms $\zeta_{*,q} : C_{*,q} \to \acute{C}_{*,q}$ of R-complexes over ζ.*

Proof. A straightforward application of [12, Theorem 19]. □

LEMMA 3.3. *Let $\mathrm{F} : \mathcal{C}(R) \to \mathcal{C}(R)$ be a left exact functor, and let M and $\acute{M}$ be two R-complexes. Then*

(i) *Any quasiisomorphism $\zeta : M \to \acute{M}$ induces isomorphism*

$$\mathbb{R}^i(\mathrm{F}M) \cong \mathbb{R}^i(\mathrm{F}\acute{M})$$

for all $i \in \mathbb{Z}$; and

(ii) *If M is F-acyclic R-complex in $\mathrm{C}_{\sqsubset}(R)$, that is, M_i is F-acyclic for all $i \in \mathbb{Z}$, then*

$$\mathbb{R}^i(\mathrm{F}M) = \mathbf{H}_i(\mathrm{F}M)$$

for all $i \in \mathbb{Z}$.

Proof. Straightforward verification similar to the proof of [17, Corollary 5.7.7]. $\square$

The following theorem, which is one of the main results of this section, enables us to prove some interesting results.

THEOREM 3.4. *Let* $\mathrm{F} : \mathcal{C}(R) \to \mathcal{C}(R)$ *be a left exact functor, and let* M *be an* F*-acyclic* R*-complex in* $\mathrm{C}_{\sqsubset}(R)$*. Assume that* I *is* F*-acyclic and* $\mathrm{F}(I)$ *is injective for every injective* R*-module* I*. Then* $\mathrm{F}(M) \simeq \mathrm{F}(E)$*, for every semiinjective resolution* $E \in \mathrm{C}_{\sqsubset}(R)$ *of* M*.*

Proof. Let E be a semiinjective resolution of M with $E_v = 0$ for $v > \sup M$, and let $\zeta : M \xrightarrow{\simeq} E$ be an quasiisomorphism. By [13, Theorem 10.45], there exist Cartan-Eilenberg injective resolutions $0 \to M \to C_{*,0} \to C_{*,1} \to \cdots \to C_{*,q} \to$ and $0 \to E \to \acute{C}_{*,0} \to \acute{C}_{*,1} \to \cdots \to \acute{C}_{*,q} \to$. Hence, in view of Lemma 3.2, there is a sequence $\{\zeta_{*,q}\}_{q \in \mathbb{N}_0}$ of morphisms of R-complexes such that the diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & E & \longrightarrow & \acute{C}_{*,0} & \longrightarrow & \acute{C}_{*,1} & \longrightarrow & \cdots & \longrightarrow & \acute{C}_{*,q} & \longrightarrow \\
 & & \uparrow \zeta & & \uparrow \zeta_{*,0} & & \uparrow \zeta_{*,1} & & & & \uparrow \zeta_{*,q} & \\
0 & \longrightarrow & M & \longrightarrow & C_{*,0} & \longrightarrow & C_{*,1} & \longrightarrow & \cdots & \longrightarrow & C_{*,q} & \longrightarrow
\end{array}$$

commutes in $\mathrm{C}(R)$. By Lemma 3.3(ii), $\mathbb{R}^p(\mathrm{F}(M)) = \mathbf{H}_p(\mathrm{F}(M))$ for all $p \in \mathbb{Z}$, so that the natural morphism $\mathrm{F}(M) \to \mathrm{Tot}(\mathrm{F}(C))$ is a quasiisomorphism. Similarly, $\mathbb{R}^p(\mathrm{F}(E)) = \mathbf{H}_p(\mathrm{F}(E))$ for all $p \in \mathbb{Z}$, so that the natural morphism $\mathrm{F}(E) \to \mathrm{Tot}(\mathrm{F}(\acute{C}))$ is a quasiisomorphism. Thus, by [2, Proposition 3.3.5(a)], there exists a quasiisomorphism $\mathrm{Tot}(\mathrm{F}(\acute{C})) \to \mathrm{F}(E)$, since $\mathrm{F}(E)$ is injective.

But, by Lemma 3.3(i), there are isomorphisms $\mathbb{R}^p(\mathrm{F}(M)) \cong \mathbb{R}^p(\mathrm{F}(E))$ for all $p \in \mathbb{Z}$. Hence the morphism $\zeta_* : \mathrm{Tot}(\mathrm{F}(C)) \longrightarrow \mathrm{Tot}(\mathrm{F}(\acute{C}))$ is a quasiisomorphism, where

$$\zeta_n = \sum_{p+q=n} \mathrm{F}(\zeta_{p,q}) : \mathrm{Tot}(\mathrm{F}(C))_n \to \mathrm{Tot}(\mathrm{F}(\acute{C}))_n$$

for all $n \in \mathbb{Z}$. Therefore, there are quasiisomorphisms

$$\mathrm{F}(M) \xrightarrow{\simeq} \mathrm{Tot}(\mathrm{F}(C)) \xrightarrow{\simeq} \mathrm{Tot}(\mathrm{F}(\acute{C})) \xrightarrow{\simeq} \mathrm{F}(E).$$

Now $\mathrm{F}(M) \simeq \mathrm{F}(E)$ as desired. $\square$

The next theorem, which offers an application of the previous theorem, is a complex version of [1, Exercise 4.1.2].

THEOREM 3.5. *Let* C *be an* $\Gamma_{\mathfrak{a}}$*-acyclic* R*-complex in* $\mathrm{C}_{\sqsubset}(R)$*. Then* $\mathbf{R}\Gamma_{\mathfrak{a}}(C) \simeq \Gamma_{\mathfrak{a}}(C)$*.*

The next corollary shows that if $M \in \mathrm{D}_{\sqsubset}(R)$, then $\mathbf{R}\Gamma_{\mathfrak{a}}(M)$ can be computed via a genuine left-bounded complex $G \simeq M$ of Gorenstein injective modules. Also, notice that [14, Theorem 3.4] is an immediate consequence of this fact.

Corollary 3.6. *Let M be an R-complex in $\mathrm{D}_{\sqsubset}(R)$, and let $G \in \mathrm{C}_{\sqsubset}(R)$ be an R-complex of Gorenstein injective modules such that $M \simeq G$. Then $\mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq \Gamma_{\mathfrak{a}}(G)$.*

Proof. Since by [18, Lemma 1.1] every Gorenstein injective module is $\Gamma_{\mathfrak{a}}$-acyclic, the result follows from Theorem 3.5. □

It has been proved in [14, Corollary 3.3] that if R admits a dualizing complex and $M \in \mathrm{D}_{\sqsubset}(R)$, then $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \leq \mathrm{Gid}_R M$. The following proposition together with [4, Theorem 5.9] recover this result.

Proposition 3.7. *Suppose that M is an R-complex in $\mathrm{D}_{\sqsubset}(R)$ such that $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Then*

$$\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \leq \mathrm{Gid}_R M.$$

Proof. Notice that if $\mathrm{Gid}_R M = \infty$, then there is nothing to prove. So, we may assume that $\mathrm{Gid}_R M < \infty$. By [5, Theorem 2.2], there exists $\mathfrak{p} \in \mathbf{V}(\mathfrak{a})$ such that

$$\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{depth} R_{\mathfrak{p}} - \mathrm{width}_{R_{\mathfrak{p}}} (\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}}.$$

But $\mathrm{width}_{R_{\mathfrak{p}}} (\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}} = \mathrm{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. It follows, again by [5, Theorem 2.2], that

$$\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{depth} R_{\mathfrak{p}} - \mathrm{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \mathrm{Gid}_R M$$

as desired. □

Proposition 3.8. *Let $(R, \mathfrak{m})$ be a local ring, and let M be an R-complex in $\mathrm{D}^f_{\square}(R)$ such that $\mathrm{Gid}_R M < \infty$. Then*

$$\mathrm{Gid}_R M \leq \mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M).$$

Proof. By [4, Proposition 6.3], there is an inequality

$$\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \geq \mathrm{depth} R - \mathrm{width}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M).$$

But $\mathrm{width}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{width}_R M$. Thus, we have

$$\begin{aligned} \mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) &\geq \mathrm{depth} R - \mathrm{width}_R M \\ &= \mathrm{depth} R - \inf M. \end{aligned}$$

The result therefore follows from [5, Corollary 2.3]. □

The following theorem, which is a Gorenstein injective version of Theorem 2.2, is one of the main results of this section.

Theorem 3.9. *Let* $(R, \mathfrak{m})$ *be a local ring admitting a dualizing complex, and let* M *be an* R*-complex in* $\mathrm{D}^f_\square(R)$*. Then* $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gid}_R M$*.*

Proof. A straightforward application of [4, Theorem 5.9], Proposition 3.7 and Proposition 3.8. □

The next theorem, which is a Gorenstein injective version of Theorem 2.4, recovers [20, Theorem 3.8].

Theorem 3.10. *Let* $(R, \mathfrak{m})$ *be a local ring admitting a dualizing complex. Suppose that* M *is an* R*-complex in* $\mathrm{D}^f_\square(R)$ *such that* $\mathrm{amp}\mathbf{R}\Gamma_{\mathfrak{a}}(M) = 0$*. Then*

$$\mathrm{Gid}_R \mathbf{H}_{\mathfrak{a}}^{-\inf \mathbf{R}\Gamma_{\mathfrak{a}}(M)}(M) = \mathrm{Gid}_R M + \inf \mathbf{R}\Gamma_{\mathfrak{a}}(M).$$

Proof. Follows from Theorem 3.9 (similar to the proof of Theorem 2.4). □

The following corollary, which improves [20, Corollary 3.9], is a consequence of the previous Theorem.

Corollary 3.11. *Let* $(R, \mathfrak{m})$ *be a local ring, and let* M *be a Cohen-Macaulay* R*-module with* $\dim_R M = n$*. Then the following statements hold.*

(i) $\mathrm{Gid}_{\widehat{R}} \mathbf{R}\Gamma_{\mathfrak{a}\widehat{R}}(M \otimes_R \widehat{R}) = \mathrm{Gid}_R M$*.*
(ii) $\mathrm{Gid}_R \mathbf{H}^n_{\mathfrak{m}}(M) = \mathrm{Gid}_R M - n$*.*

Proof. Notice that $(\widehat{R}, \widehat{\mathfrak{m}})$ is a local ring admitting a dualizing complex and $M \otimes_R \widehat{R}$ is a Cohen-Macaulay $\widehat{R}$-module of dimension n.

(i) A straightforward application of Theorem 3.9 and [3, Theorem 3.24].

(ii) There is an inequality

$$\mathrm{Gid}_{\widehat{R}} \mathbf{R}\Gamma_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R}) = \mathrm{Gid}_{\widehat{R}} \mathbf{H}^n_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R}) + n.$$

But, in view of [14, Lemma 3.6], $\mathrm{Gid}_{\widehat{R}} \mathbf{H}^n_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R}) = \mathrm{Gid}_R \mathbf{H}^n_{\mathfrak{m}}(M)$. The result now follows from part (i). □

The next corollary provides a characterization of Gorenstein rings, which together with Corollary 2.3 show that [19, Theorem 2.6] and [20, Corollary 3.10] hold without assuming that R is Cohen-Macaulay.

Corollary 3.12. *Let* $(R, \mathfrak{m})$ *be a local ring admitting a dualizing complex. Then the following statements are equivalent:*

(i) *R is Gorenstein;*
(ii) $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(R) = \dim(R)$ *for any ideal* $\mathfrak{a}$ *of* R*;*
(iii) $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(R) < \infty$ *for some ideal* $\mathfrak{a}$ *of* R*.*

Proof. A straightforward application of Theorem 3.9 and [3, Proposition 3.11]. □

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