Arcsine Laws And Its Simulation And Application

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1 Introduction

As we all know, in probability theory, a random variable X on $[0,1]$ is arcsin-distributed if its cumulative distribution $F(x)$ satisfies

$$F(x) = P(X \leq x) = \frac{2}{\pi} \arcsin(\sqrt{x})$$

In Brownian motion theory, there are at least three arcsin laws which are a collection of results for one-dimensional Brownian motion. In this project, firstly we will introduce three kinds of arcsin law and give some proofs of them. Then, we will use matlab software to simulate them and check the result in the theoretical proofs. At last,in order to see the importance of the arcsine laws, we will give an application of arcsine laws in the finance market.

2 Statements of The Laws

Throughout we suppose that (B_t) $0 \leq t \leq 1 \in \mathbb{R}$ is the one-dimensional Brownian motion on $[0,1]$.

Theorem 2.1. *Let $T_+ =\mid \{t \in [0,1], B_t > 0\} \mid$ be the measure of the set of times in $[0,1]$ at which the Brownian motion is positive. Then T_+ is arcsine distributed, i.e.*

$$P(T_+ \leq x) = \frac{2}{\pi} \arcsin(\sqrt{x}) \ \forall x \in [0,1]$$

Proof: We can rewrite T_+ as $\int_0^t \mathbf{1}_{\{B_s>0\}} ds$, where t=1 in our theorem. We prove a general situation where t is a fixed time point on $(0,\infty)$. Applying Feynmann-Kac Formula[1], we define

$$f(x) = \mathbb{E}_x \int_0^\infty [exp(-\alpha t - \beta \int_0^t \mathbf{1}_{\{B_s>0\}} ds)]dt$$

Then f(x) solves the following differential equation:

$$\alpha f(x) = \frac{1}{2} f''(x) - \beta f(x) + 1, x \geq 0$$
$$\alpha f(x) = \frac{1}{2} f''(x) + 1, x \leq 0$$

[1]Introduction to Stochastic Processes, chapter 9, Lawer

Then the solution for the differential equation are given by

$$f(x) = Ae^{-x\sqrt{2(\alpha+\beta)}} + \frac{1}{\alpha+\beta}, x \geq 0$$
$$f(x) = Be^{x\sqrt{2\alpha}} + \frac{1}{\alpha}, x \leq 0$$

Relying on the continuity of f and f' at zero, we get

$$A = \frac{\sqrt{\alpha+\beta} - \sqrt{\alpha}}{(\alpha+\beta)\sqrt{\alpha}}, B = \frac{\sqrt{\alpha} - \sqrt{\alpha+\beta}}{\alpha\sqrt{\alpha+\beta}}$$

Then, let $x = 0$,

$$f(0) = \int_0^\infty [e^{-\alpha t}\mathbb{E}_0(e^{-\beta T_+})]dt = \frac{1}{\sqrt{\alpha+\beta}}$$

We can invert the laplace transform using the identity

$$\int_0^\infty e^{-\alpha t}(\int_0^t \frac{e^{-\beta u}}{\pi\sqrt{u(t-u)}}du)dt = \frac{1}{\sqrt{\alpha+\beta}}$$

and the density of T_+ is obtained

$$P(T_+ \in ds) = \frac{ds}{\pi\sqrt{s(t-s)}}, 0 < s < 1$$

Hence, let $t = 1$

$$P(T_+ \leq x) = \int_0^x \frac{ds}{\pi\sqrt{s(1-s)}} = \frac{2}{\pi}\arcsin(\sqrt{x})\ \forall x \in [0, 1]$$

The first theorem is proved.

Theorem 2.2. *Let $L = sup\{t \in [0, 1] \mid B_t = 0\}$, the last time when the Brownian motion changes its sign. Then L is arcsine distributed, i.e*

$$P(L \leq x) = \frac{2}{\pi}\arcsin(\sqrt{x})\ \forall x \in [0, 1]$$

Lemma 2.3. *Define $E_t = \{\exists s \in [a, t], B_s = 0\}$, then,*

$$P(E_t) = 1 - \frac{2}{\pi}\arctan(\sqrt{\frac{a}{t-a}})$$

Proof: By the symmetries of Brownian motion, Let us say that $B_a = b > 0$, then E_t is realized iff

$$\exists s \in [a, t], s.t.\ B_s - B_a \leq -b \Longleftrightarrow \exists s \in [0, t-a], s.t. B_{a+s} - B_a \leq -b$$

Hence, if $I_x = inf\{B_{a+s} - B_a, s \in [0, x]\}$, then E_t is realized iff $I_{t-a} \leq -b$, and I_{t-a} is independent of B_a by simple Markov property.

$$P(E_t) = 2\int_0^\infty P(E_t|B_a = b)P(B_a \in [b, b+db])$$

$$= 2\int_0^\infty P(I_{t-a} \le -b | B_a = b) P(B_a \in [b, b+db])$$
$$= 2\int_0^\infty P(I_{t-a} \le -b) P(B_a \in [b, b+db])$$
$$= 2\int_0^\infty P(S_{t-a} \ge -b) P(B_a \in [b, b+db])$$
$$= 2\int_0^\infty P(|B_{t-a}| \ge -b) P(B_a \in [b, b+db])$$
$$= 4\int_0^\infty P(B_{t-a} \ge -b) P(B_a \in [b, b+db])$$
$$= \frac{2}{\pi\sqrt{a(t-a)}} \int_0^\infty \int_b^\infty e^{\frac{-x^2}{2(t-a)}} dx e^{\frac{-b^2}{2a}} db$$

The substitution $u = \frac{x}{\sqrt{t-a}} v = \frac{b}{\sqrt{a}}$ in the inside integral reduce this integral to

$$\frac{2}{\pi} \int_0^\infty \int_{\frac{\sqrt{a}v}{\sqrt{t-a}}}^\infty e^{-\frac{u^2+v^2}{2}} du dv$$

This integral can be computed using polar coordinates. Note that the region $\{0 < b < \infty, b(t-a)^{1/2} < u < \infty\}$ corresponds to the polar region $\{\arctan(\sqrt{a/(t-a)}) < \sigma < \pi/2, 0 < r < \infty\}$. Hence the probability equals

$$1 - \frac{2}{\pi} \arctan\sqrt{\frac{a}{t-a}}$$

Here we have proved the important lemma, then we will use it to prove Theorem 2.2.

Proof *of theorem* 2.2: Note that for $x \in [0,1], L \le x$ means that from time x to time 1, the Brownian motion does not reach 0, which implies that $E_1 = \{\exists s \in [x,1], B_s = 0\}$ does not realize. Hence by the last lemma,

$$P(L \le x) = 1 - P(E_1) = 1 - (1 - \frac{2}{\pi}\arctan\sqrt{\frac{x}{1-x}}) = \frac{2}{\pi}\arctan\sqrt{\frac{x}{1-x}}$$

Then, it suffices to show that

$$\frac{2}{\pi}\arctan\sqrt{\frac{x}{1-x}} = \frac{2}{\pi}\arcsin(\sqrt{x})$$

In fact, let $f(x) = \frac{2}{\pi}\arctan\sqrt{\frac{x}{1-x}} - \frac{2}{\pi}\arcsin(\sqrt{x})$, then it is easy to compute that $f'(x) = 0$ for all $x \in [0,1]$. And $f(0) = 0$, hence we get that $f(x) \equiv 0$ on the interval [0,1), which implies that

$$P(L \le x) = \frac{2}{\pi}\arcsin(\sqrt{x}) \ \forall x \in [0,1]$$

Here we finish the proof!

Theorem 2.4. *Let $T =$ the point $t \in [0,1]$, where $B_t = sup\{B_s, s \in [0,1]\} = S_1$, the time when brownian motion achieves its maximum. Then T is arcsine distributed, i.e*

$$P(T \le x) = \frac{2}{\pi}\arcsin(\sqrt{x}) \ \forall x \in [0,1]$$

Proof: For any $t \in [0,1]$, let $X_r = B_{t-r} - B_t, Y_s = B_{t+s} - B_t$, then $(X_r, 0 \le r \le t)$ is a Brownian motion starting from zero and is $\mathcal{F}_t$ measurable, and $(Y_s, 0 \le s \le 1-t)$ is also a Brownian motion starting from zero and is independent of $\mathcal{F}_t$, with this setup, we have

$$\begin{aligned} P(T \le t) &= P(\underset{[0,t]}{max}\, B_u > \underset{[t,1]}{max}\, B_u) \\ &= P(\underset{[0,t]}{max}(B_u - B_t) > \underset{[t,1]}{max}(B_u - B_t)) \\ &= P(\underset{[t,1]}{max}(B_{t-r} - B_t) > \underset{[0,1-t]}{max}(B_{t+s} - B_t)) \\ &= P(\underset{[t,1]}{max}\, X_r > \underset{[0,1-t]}{max}\, Y_s) \\ &= P(|X_t| > |Y_{1_t}) \\ &= P(\sqrt{t}|N_1| > \sqrt{1-t}|N_2|) \\ &= P(\frac{|N_2|}{\sqrt{N_1^2 + N_2^2}} < t) \end{aligned}$$

Note that if we treat $Q = (N_1, N_2)$ as a point on the xy plane, then $\frac{N_2}{\sqrt{N_1^2+N_2^2}} = sin\theta$, where θ denotes the angle between OQ and the X axis. What's more, since $N_1, N_2 \sim N(0,1)$ are independent, θ is uniformly distributed on $[0, 2\pi)$, hence the probability equals

$$P(|sin\theta| < \sqrt{t}) = \frac{2}{\pi}\arcsin(\sqrt{t})$$

Hence, $P(T \le x) = \frac{2}{\pi}\arcsin(\sqrt{x})\ \forall x \in [0,1]$, the theorem is proved.
Note: As we can see, Theorem 2.4 is not valid unless the maximum point is unique on [0,1]. In fact, we can prove that T is uniquely determined [2]*, but considering the limitation of length, we don't write down here. Any way, it is right and Theorem 2.4 is valid!*

Up to here, we have proved the three kinds of arcsine laws. As you can see, apart from the proof of the first theorem, the other two proofs are quite friendly, which is not hard to think. But the first one is quite complicated ,where we even apply Feynmann-Kac Formula, which is a result in stochastic integration! But our main aim is not proving the laws in mathematical symbols, in the next section, we will simulate the process and verify the results in this section. Instead of the painful mathematical symbols, we will use figures to demonstrate the three laws!

3 Simulation

3.1 How to simulate Brownian motion

As we have known, we can treat the Brownian motion as a limit of random walk. Based on the idea, it is easy to simulate the process on the computer. Choose a small number of Δt. We can simulate the Brownian motion by a simple random walk with time increments Δt, and the space increments $\sqrt{\Delta t}$. To do this let $Y_1, Y_2, \dots$ be independent random variables with

$$\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = \frac{1}{2}$$

[2]Morters, Peter and Peres, Yuval, Brownian Motion, Chapter 2

Consider the standard Bronian motion, where $B_0 = 0$. For $n > 0$,

$$B_{n\Delta t} = B_{(n-1)\Delta t} + \sqrt{\Delta t} Y_n$$

In our codes in the appendix, we make the increments normal, which is more usual in the practice. i.e. If $Z_1, Z_2, \dots$ are independent standard unit normals,then we have

$$B_{n\Delta t} = B_{(n-1)\Delta t} + \sqrt{\Delta t} Z_n$$

That is our basis idea of this simulation!

There is another thing to mention, in the process of simulation, apart from the basic idea, the most important thing is that how to find the point which satisfies the the theorems respectively in the last section. Next, we will explain them respectively.

3.2 Simulation of Theorem 2.1

If you have read the proof in the last section carefully, you will find that Theorem 2.1 is the hardest to prove, but in the simulation, it is the easiest, I think. Recall that the first theorem states that the proportion of the time that the Brownian motion spends beyond zero. So we try to find the points which are bigger than zero. It is easy to achieve on the computer if we count when the point is Larger that zero, and then we calculate the proportion. That is the way we get an realization of T_+. Fig 1 and 2 show the results, where the left hand side show the probability density function(pdf) and the right hand side is cumulative distribution function(cdf). Note the the SLIM LINEs are the real pdf and cdf.

As the figures illustrated, the histograms fit amazingly well, which indicates that our first theorem is absolutely right!

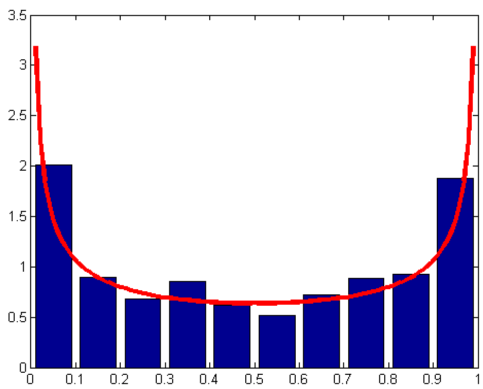

Fig 1: *pdf*

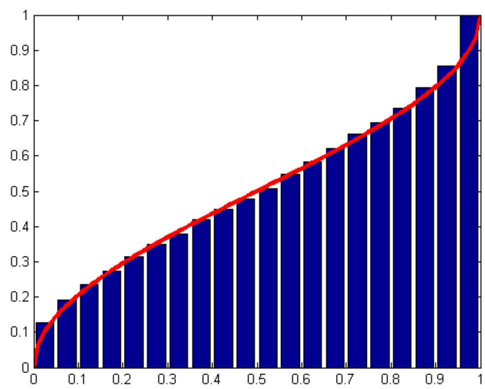

Fig 2: *cdf*

3.3 Simulation of Theorem 2.3

When facing the simulation of this theorem initially, we think that it is easy to simulate. Because we just have to find the the last zero out. But this is the difference between theoretical proof and simulation, in the simulation, there is no zero points almost surely since you just find finite points and the jump is discrete. So we give up finding the zero points, and we try to find the the point which is close to zero. Once again, we failed, because it is hard to decide the "how close is close". In mathematical words, we can not find ϵ properly. At last, considering the continuity of Brownian motion, we product the adjacent points, if the outcome is smaller than zero, which implies that the Brownian motion attains zero between the two points. That is way we find the point satisfies Theorem 2.3. Fig 3 and 4 show the results!

The same as the last subsection, as the figures illustrated, the histograms fit amazingly well, which indicates that our second theorem is absolutely right!

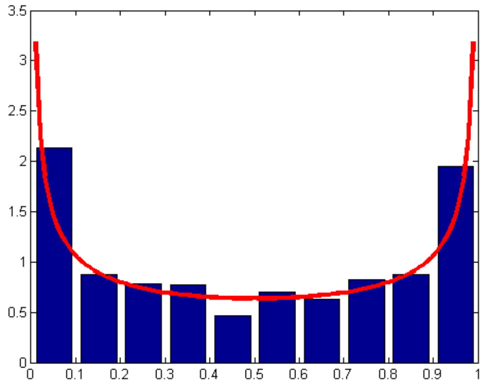

Fig 3: *pdf*

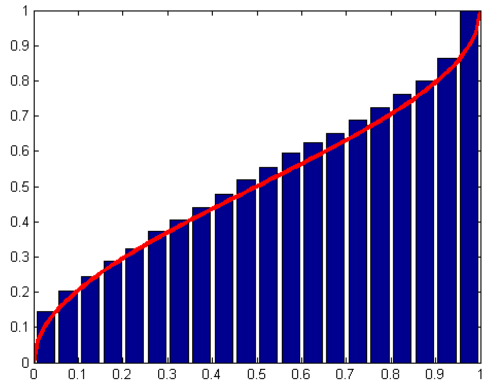

Fig 4: *cdf*

3.4 Simulation of Theorem 2.4

In the last simulation, our aim is to find the maximum point, which is quite easy to realize on the computer. Hence we show the results directly(Fig 5 and 6).

Once again, without any accidence, the histograms fit both pdf and cdf well. So the last theorem is quite right!

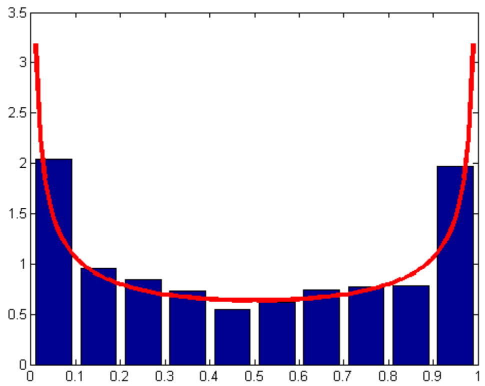

Fig 5: *pdf*

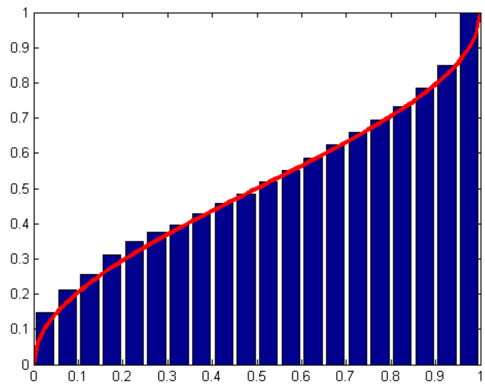

Fig 6: *cdf*

Up to now, we believe that nobody is skeptical about the rightness of the three arcsine laws. Now that the laws are right, we will see an applications of the first acsine law in stocks in the next section, showing that it is not the toy that the mathematicians plays.

4 Application

4.1 The application of the first arc-sine law

We extend the first arc-sine to an asymmetry walk model,and gets its limit behavior. With the arc-sine law, we compare the Shanghai and Hong Kong stock market from a view that focus on the proportion of win time to the whole investment period.

4.1.1 Introduction

In the earlier time, the investor just pay attention to the asset price change and only the price increasing will bring the profit. But whatever we focus on ,the asset profit or volatility, we have ignored an important problem-the proportion of win time to the whole investment period. And the investment process is completed by investors, so the knowledge and psychology circumstance of the investor will decide the investment behavior and the investment strategy. If a investor lose the money for a long time , it will have a influence on the opinion about the market surely. In conclusion, its necessary to analyze the market from the view-the win time in the whole investment period. This is the reason that we try to compare Shanghai and HongKong stock market from this point view.

4.1.2 Detailed explanation

Here, we try to use the arc-sine law to observe the securities market. Assume that we have known the any day purchase index before the stock market closed. Over the next n days, if the day of the closing index point higher than the purchase index, it means the investor is in the win time. So, what we consider is the proportion of cumulative win time to the whole investment time. If this proportion is so lower, the investor will lose money in a long time. And the investor concerned are not only whether losing money but also the loss duration. Usually when we mention the loss , we will care about a time point of view on this issue and ignore overlooking the circumstance of the entire investment range. Under a long loss time, the psychology and expected situation of the investor will change. For instance, the most normal person is the risk aversion . As stated in the prospect theory, when we related to the loss, we prefer to pursue the risk.

4.1.3 Data calculation and simulation

In our practical example, we collect one years data from Shanghai and HongKong stock market. For the Shanghai stock market, we use the Shanghai composite index(000001) from 6/23/2014 to 6/12/2015(Fig 7). And for the Hongkong stock market, we use the Hang Seng index from 06/03/2014 to 06/15/2015(Fig 8).

Assume that there are T days closing index $Ii, i = 1$. The investor use the closing price buy the indexation fund on the j-th day and calculate the number(k_i) of index higher than Ii betweenI_{j+1} and I_{j+n-1}

Def:$P_{tj} = \frac{K_j}{n}$

Which is the proportion of the positive profit to the whole investment period. Due to

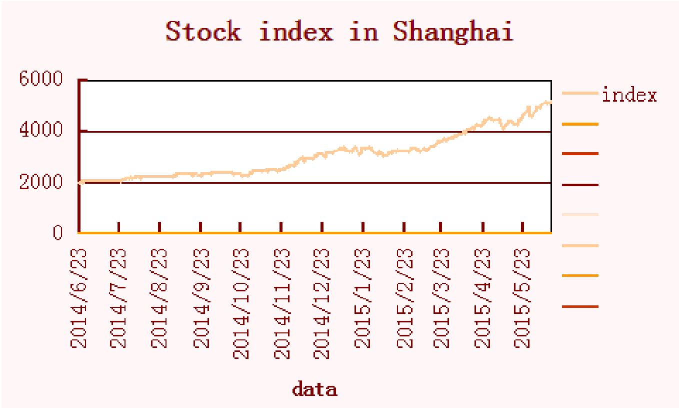

Fig 7: The stock index(000001)data in Shanghai

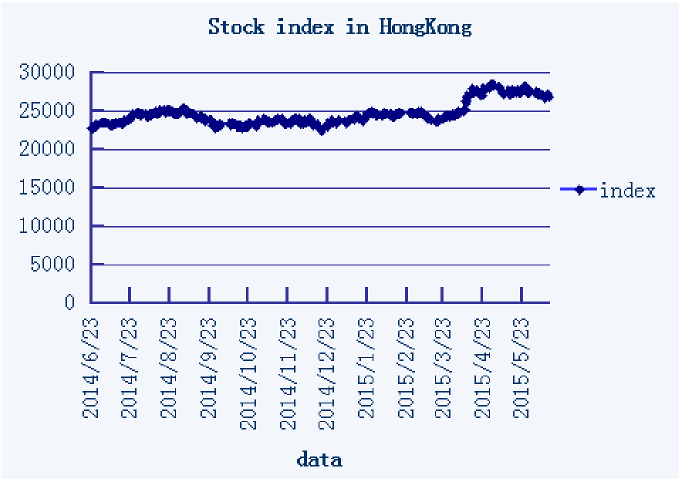

Fig 8: The stock index(000001)data in Hong Kong

our collective data , we get $(T - n + 1)$ samples to calculate the proportion of the n-th day. According to the first arc-sine law, we can get the arc-sine distribution , which can compare to our realistic data and prove the rationality of our assumption. The table as followed:

	Arc-sine	5 days	10 days	20 days	60 days	125 days	250 days
$P_{SH}(\leq 0.05)$	0.1436	0.2609	0.1935	0.2224	0.1793	0.1211	0.0590
$P_{HK}(\leq 0.05)$	0.1436	0.2369	0.1661	0.1558	0.0943	0.0777	0.0642
$P_{SH}(\geq 0.95)$	0.1436	0.2937	0.2289	0.2490	0.2129	0.1881	0.2180
$P_{HK}(\geq 0.95)$	0.1436	0.2935	0.2215	0.2534	0.2373	0.2307	0.3588

Fig 9: SH and HK stock market the win time proportions probability

4.1.4 The analysis of data result

From the table 1,we can find some facts as followed:

• According to the standard arc-sine distribution, we can get the probability of over-5% and low-5% are both 0.1436. When the continue investment time of this two stock market is about 20 days,we can find that the realistic probability that the win time proportion is over -5% or low-5% is bigger than the arc-sine probability. On the other hand, by the chi-square test, we will reject the assumption that two stock market win time proportion is discrete arc-sine distribution under the sufficient lever 1%, which means that the random walk hypothesis does not hold in those two stock market. Practical example shows that the probability of continued loss and earning may be large. Compare with the fair gamble, its easier to happen the situation. So,if we just choose to get in the stock market in a short time, we should have a good psychology situation to dace to the long time loss.And it will have a impact on the later investment strategy.

• When the investment period is long enough, we will find that the probability in the table is less than before. For the HongKong market, in the case of 250 days or longer, the probability of the situation that the market happen the longtime loss is so small. This suggests that the expected value of the day profit is positive for the HongKong market, which means that the probability of the positive return id bigger than the probability of the negative return or the rise rate id larger than the decline rate. The phenomenon is consist with the idea of long-term appreciation, but also reflect the fairness for the investor.

• For the Shanghai stock market, compared with the HK, the win time proportion probability is larger than the HK. It shows that it will exist a big probability to appear a long time loss situation. The shanghai market is not so mature compared with the HK stock market.

4.1.5 Conclusion

We use the method derived from the arc-sine law to observe the Shanghai and HK stock market. An we find that the probability of continued loss and earning are quite large in both stock market during the short-term investment. This shows a greater risk of short-term investment. With the increase in the investment during the period, the continued loss probability are getting smaller and smaller. At last, for most of investors, the factors considered should not just include only the aspect of revenue and earning volatility, but should also include the duration of the loss or profit.

5 Conclusion

From theoretical proof to visual simulation, then to important application in our real life, we have demonstrated the three arcsine laws from three different aspects, and we know that the seemingly useless laws(which are just the toys that the mathematicians play) play an important role in finance market. Maybe it is one of the reasons that "boring" Math exists all the time and a lot of people, like us, go for it!

6 Appendix

The first three following functions are used to find the points we want in the theorem respectively!

Code 1

$function[P] = Pf1(n)$

```
B = zeros(n);
P = 0;
B(1) = normrnd(0,1)/(sqrt(n));
for i = 2 : n
B(i) = B(i-1) + normrnd(0,1)/(sqrt(n));
end
for j = 1 : n
if B(j) > 0
P = P + 1;
else
P = P;
end
end
P = P/n;
end
```

Code 2

```
function[P] = Pf2(n)
B = zeros(n);
P = 0;
B(1) = normrnd(0,1)/(sqrt(n));
for i = 2 : n
B(i) = B(i-1) + normrnd(0,1)/(sqrt(n));
end
for j = n : -1 : 2
if B(j) * B(j-1) <= 0;
break;
end
end
P = j/n;
end
```

Code 3

```
function[P] = Pf3(n)
B = zeros(n,1);
P = 0;
B(1) = normrnd(0,1)/(sqrt(n));
for i = 2 : n
B(i) = B(i-1) + normrnd(0,1)/(sqrt(n));
end
P = find(B == max(B));
P = P/n;
end
```

The last two functions are used to get pdf and cdf.

Code 4

```
function[ ] = pdff(m,n)
```

```
P = zeros(m,1);
for i = 1 : m
P(i) = Pf3(n);
end
[e,f] = hist(P);
bar(f,10*e/sum(e))
hold on;
x = linspace(0,1,100);
y = 1./(pi.*(x.*(1-x)).^(1/2));
plot(x,y,'-r','LineWidth',3)
end
```

Code 5

```
function[ ] = cdff(m,n)
P = zeros(m,1);
for i = 1 : m
P(i) = Pf2(n);
end
[cumP,f] = hist(P,20);
cumP = cumsum(cumP./m);
axis([0 1 0 1])
bar(f,cumP)
hold on;
x = linspace(0,1,100);
y = (2/pi).*asin(x.^(1/2));
axis([0 1 0 1])
plot(x,y,'-r','LineWidth',3)
end
```