Non-standard Models of Arithmetic and Their Standard Systems

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A Brief History of Axiomatization of $\mathbb{N}$

1860s, Mathematicians realized that a proper axiomatization of natural numbers was in need.

1881, Charles Sanders Peice (American mathematician/statistician, philosopher, 'the father of pragmatism') provided an axiomatization of $\mathbb{N}$.

1888, Richard Dedekind (German mathematician, who invented 'Dedekind cuts') gave a different axiomatization.

1889, Giuseppe Peano (Italian mathematician, linguist) simplified Dedekind's axiomatization, and the resulted system is called Dedekind-Peano axioms or Peano's axioms.

Now Peano Arithmetic (PA) is usually referred to the first order part of Peano's axioms.

The Language of Arithmetic

L_A, the language of arithmetic, is a first order language with two constants $(0, 1)$, two binary operations $(+, \times)$ and a binary relation $(<)$.

So L_A is just the language of rings, augmented by a binary relation; or just the language of ordered rings.

PA^-, the Ordered Ring Part of PA

PA^- is the part of PA, saying that a desired model is the non-negative part of a discrete ordered ring.

So PA^- contains the some ring axioms as follows,

- $0 + x = x$; (i.e., $(\forall x(0 + x = x))$)
- $x + (y + z) = (x + y) + z$;
- $x + y = y + z$;
- $0 \times x = 0$;
- $1 \times x = x$;
- $x \times (y \times z) = (x \times y) \times z$;
- $x \times y = y \times x$;
- $x \times (y + z) = x \times y + x \times z$.

Here we follow usual conventions in algebra, like omitting $\forall$.

PA^-, the Ordered Ring Part of PA

PA^- also contains axioms saying that $<$ is linear with a least element 0,

- $x \leq y \rightarrow y \not< x$;
- $x = y \vee x < y \vee x > y$;
- $x < y \wedge y < z \rightarrow x < z$;
- $0 \leq x$;

and the order is discrete,

- $x < y \rightarrow x + 1 \leq y$;

and $+, \times$ obey the ordering to some extent,

- $x < y \rightarrow \exists z(x + z = y)$;
- $x \leq y \rightarrow x + z \leq y + z$;
- $x \leq y \rightarrow xz \leq yz$.

$\mathrm{PA} = \mathrm{PA}^- +$ Inductions

Recall: fix M a first order model (of some language), $B \subseteq M$, $X \subseteq M^m$ is B-definable in M, iff there exist a formula $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ and $b_1, \ldots, b_n \in B$ s.t.

$$X = \{\vec{a} \in M^n : M \models \varphi(\vec{a}, \vec{b})\} = \varphi(M, \vec{b}).$$

PA consists of PA^- and the induction scheme saying that mathematical induction holds for every definable set of numbers, i.e.,

$$\varphi(0, \vec{y}) \wedge \forall x(\varphi(x, \vec{y}) \to \varphi(x+1, \vec{y})) \to \forall x \varphi(x, \vec{y}),$$

for all formula $\varphi(x, \vec{y})$.

The above instance of induction for φ is denoted by $I\varphi$.

Standard and Non-standard Models of PA

$\mathbb{N} = (\mathbb{N}, 0, 1, +, \times, <)$ is the standard model of PA.

The following set of formulas

$$T = \mathrm{PA} \cup \{c > 0, c > 1, c > 1 + 1, c > 1 + (1 + 1), \ldots\}$$

is finitely satisfiable in $\mathbb{N}$. So by Compactness Theorem, there exists $M \models T$. M contains $\mathbb{N}$ as an initial segment (i.e., in M, if $a < n \in \mathbb{N}$ then $a \in \mathbb{N}$) and also an infinite element c^M (the interpretation of c in M).

A non-standard model of PA is a model of PA different from $\mathbb{N}$, like M.

Every non-standard model contains $\mathbb{N}$ as an initial segment.

The Order-Type of A Non-standard Model

Let M be a non-standard model of PA. For $X \subset M$ and $a \in M$, let

$$a + X = \{a + b : b \in X\}.$$

We define an equivalence relation on M as follows,

$$a \sim b \Leftrightarrow b \in a + \mathbb{N} \text{ or } a \in b + \mathbb{N}.$$

Let $[a]$ be the equivalence class of a and let $[a] < [b]$ iff $a < b$ and $a \not\sim b$.

If $[a] < [b]$ then $b - a = 2c$ or $2c + 1$ for some c. Clearly, $[a] < [c] < [b]$.

If $[a] > [0] = \mathbb{N}$ then $[2a] > [a]$.

So $\sim$ induces a dense linear order L with a least element $[0]$. And the order-type of M under $<$ is of the form

$$\mathbb{N} + \mathbb{Z} \times L.$$

If M is countable then the order-type is $\mathbb{N} + \mathbb{Z} \times \mathbb{Q}$.

Standard Systems

Let $M \models \mathrm{PA}$ be non-standard. Every $a \in M$ can be identified as a binary sequence, let $(a)_i$ be the i-th bit.

The standard system of M is the following set of subsets of $\mathbb{N}$

$$\mathrm{SSy}(M) = \{\{i \in \mathbb{N} : M \models (a)_i = 1\} : a \in M - \mathbb{N}\}.$$

We say that the set $\{i \in \mathbb{N} : M \models (a)_i = 1\}$ is coded by a in M.

But there are other ways to present $\mathrm{SSy}(M)$. Let p_i denote the i-th prime number and let $p_i|a$ denote the formula saying that p_i divides a. Then

$$\begin{aligned}\mathrm{SSy}(M) &= \{\{i \in \mathbb{N} : M \models p_i|a\} : a \in M - \mathbb{N}\} \\ &= \{A \cap \mathbb{N} : A \text{ is definable in } M\}.\end{aligned}$$

Computability of Standard Systems

Theorem (Scott, 1962)

If M is a non-standard model then $\mathrm{SSy}(M)$ satisfies the following.

1. *If $Y \leq_T X \in \mathrm{SSy}(M)$ then $Y \in \mathrm{SSy}(M)$;*
2. *If X and Y are both in $\mathrm{SSy}(M)$ then*

$$X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\} \in \mathrm{SSy}(M);$$

3. *If T is an infinite binary tree computable in $X \in \mathrm{SSy}(M)$ then there exists $Y \in \mathrm{SSy}(M)$ whose characteristic function as a countable binary sequence is an infinite path on T (we write $Y \in [T]$).*

A Turing ideal is a set $\mathcal{A} \subseteq 2^{\mathbb{N}}$ satisfying (1-2) above, and a Scott set is a Turing ideal satisfying also (3). So $\mathrm{SSy}(M)$ is a Scott set for every non-standard model M.

Computability of Standard Systems

Finding Paths On A Coded Tree

We sketch a proof of the 3rd clause above.

By the 1st clause and by identifying finite binary sequences with natural numbers, we fix an infinite tree $T \in \mathrm{SSy}(M)$. So there is $a \in M - \mathbb{N}$ s.t.

$$T = \{\sigma \in 2^{<\mathbb{N}} : M \models (a)_\sigma = 1\}.$$

Let

$$X = \{n \in M : M \models \exists\sigma(|\sigma| = n \wedge (a)_\sigma = 1 \wedge \forall\rho(\rho \prec \sigma \to (a)_\rho) = 1))\}.$$

where $|\sigma|$ is the length of σ and $\rho \prec \sigma$ means ρ being an initial segment of σ. Since T is an infinite tree, $\mathbb{N} \subseteq X$; by induction, X contains some $b \in M - \mathbb{N}$!

So in M, there is some 'finite' binary sequence τ with length b and satisfying the matrix of the formula defining X. So every TRUE finite initial segment of τ is on T, and τ codes a path of T in $\mathrm{SSy}(M)$.

Theorem (Tennenbaum, 1959)

If $(M, 0, 1, +, \times, <) \models \mathrm{PA}$ *is non-standard and countable then* $(M, +)$ *is NOT computable.*

By the 1st claus of Scott's theorem, every computable set is in $\mathrm{SSy}(M)$. By standard computability construction, there is a computable infinite binary tree T which has NO computable infinite path. So $T \in \mathrm{SSy}(M)$ and $\mathrm{SSy}(M)$ contains a path X of T which is not computable.

Let $a \in M - \mathbb{N}$ be s.t.

$$X = \{i \in \mathbb{N} : M \models p_i | a\}.$$

If $(M, +)$ were computable, the following algorithm would decide $i \in X$. Given a positive $i \in \mathbb{N}$, find $b \in M$ and $r < p_i$ s.t. $a = bp_i + r$. If $r = 0$ then $i \in X$; otherwise $i \notin X$.

Models Coded By Their Standard Systems

Recursive Types and Recursively Saturated Models

Let M be a (first-order) model. If $A \subseteq M$ then

$$\mathrm{Th}_A(M) = \{\varphi(\vec{a}) : \vec{a} \text{ from } A, M \models \varphi(\vec{a})\}.$$

A type of M over a set of parameters $A \subseteq M$ is a set $p(\vec{x})$ of formulas, s.t. each formula is in a fixed set $\vec{x}$ of variables and may have parameters from A, and $p(\vec{x}) \cup \mathrm{Th}_A(M)$ is satisfiable (equivalently, finitely satisfiable in M).

If $\vec{x}$ and $A = \vec{a}$ are finite and

$$\{\varphi(\vec{x}, \vec{y}) : \varphi(\vec{x}, \vec{a}) \in p\}$$

is recursive, then p is a recursive type.

M is recursively saturated, iff every recursive type $p(\vec{x})$ of M over some finite $\vec{a}$ from M is realized in M, i.e., there exists $\vec{b}$ in M s.t. $M \models \varphi(\vec{b}, \vec{a})$ for all $\varphi(\vec{x}, \vec{a}) \in p$.

Proposition

- *Every countable model has a recursively saturated elementary extension.*
- *If $M \models \mathrm{PA}$ is recursively saturated then it is $\mathrm{SSy}(M)$-recursively saturated, i.e., if $X \in \mathrm{SSy}(M)$ and p is a X-recursive type of M over a finite set of parameters then p is realized in M, where X-recursive type is a straight forward relativization of recursive type.*

Theorem (H. Friedman, 1973)

Let M and N be countable recursively saturated non-standard models of PA. If M and N are elementarily equivalent and $\mathrm{SSy}(M) \subseteq \mathrm{SSy}(N)$ then M can be elementarily embedded into N. Moreover, if $\mathrm{SSy}(M) = \mathrm{SSy}(N)$ then $M \cong N$.

Tailor-made Standard Systems

Theorem (Scott, 1962)

For each countable Scott set $\mathcal{S}$ and each consistent extension Γ of PA in $\mathcal{S}$, there exists a non-standard $M \models \Gamma$ s.t. $\mathrm{SSy}(M) = \mathcal{S}$.

Note that the construction of a completion Λ of a consistent Γ with Henkin's property can be viewed as finding an infinite path on a Γ-computable binary tree. So, since $\Gamma \in \mathcal{S}$, there is such a $\Lambda \in \mathcal{S}$ and thus a Henkin model M_0 of Γ exists in $\mathcal{S}$.

Then $\mathrm{SSy}(M_0) \subset \mathcal{S}$. If $X \in \mathcal{S} - \mathrm{SSy}(M_0)$, then let

$$\Gamma_X = \text{ The elementary diagram of } M_0 \\ \cup \{(c)_i = 1 : i \in X\} \cup \{(c)_i = 0 : i \notin X\},$$

which is consistent and in $\mathcal{S}$. By similar construction, in $\mathcal{S}$ we can find $M_1 \models \Gamma_X$, so $M_0 \prec M_1$ and $X \in \mathrm{SSy}(M_1) \subset \mathcal{S}$.

So we can have $M_0 \prec M_1 \prec \ldots$ and finally $M = \bigcup_n M_n$ is as desired.

The Scott Set Problem

Question

Does every Scott set (countable or uncountable) equal to the standard system of some non-standard model of PA*?*

The Scott Set Problem Under CH

Theorem (Knight and Nadel, 1982)

Every Scott set of cardinality ω_1 equals to the standard system of some non-standard $M \models \mathrm{PA}$.
So, under ZF + CH, the Scott set problem has an affirmative answer.

There are several known proofs of this theorem.

The original proof has two parts:

1. Nadel proved that if M is a recursively saturated model of $\mathrm{Pr}' = \mathrm{Th}(\mathbb{Z}, +, 1)$ and $|M| = \aleph_1$ then M can be expanded to a recursively saturated model N of PA. Clearly, $\mathrm{SSy}(N) = \mathrm{SSy}(M)$.
2. Knight and Nadel proved that every Scott set is the standard system of some $M \models \mathrm{Pr}'$.

The Scott Set Problem Under CH

A New Proof

In 2015, Alf Dolich, Julia Knight, Karen Lange and David Marker gave a new proof as follows.

Let $\mathcal{S}$ be a Scott set of cardinality ω_1. Pick a consistent completion $T \in \mathcal{S}$ of PA. We can find countable Scott sets $(\mathcal{S}_\alpha : \alpha < \omega_1)$ s.t.

$$T \in \mathcal{S}_0 \subseteq \mathcal{S}_\alpha \subset \mathcal{S}_\beta \ (\alpha < \beta), \ \mathcal{S} = \bigcup_\alpha \mathcal{S}_\alpha.$$

By Scott's Theorem, for each α, let $M_\alpha \models T$ be countable, recursively saturated with $\mathrm{SSy}(M_\alpha) = \mathcal{S}_\alpha$.

By Friedman's Embedding Theorem, for $\alpha < \beta$, there exists an elementary embedding $f_{\alpha,\beta} : M_\alpha \to M_\beta$. With some additional care, we can ensure that

$$\alpha < \beta < \gamma \to f_{\alpha,\gamma} = f_{\beta,\gamma} \circ f_{\alpha,\beta}.$$

The limit of this elementary chain is a desired model.

Lemma (Ehrenfeucht)

Suppose that $\mathcal{S}$ is a Scott set, $X \in \mathcal{S}$ and $M \models \mathrm{PA}$ is countable with $\mathrm{SSy}(M) \subseteq \mathcal{S}$. Then there exists N s.t. $M \prec N$, $X \in \mathrm{SSy}(N) \subseteq \mathcal{S}$.

It is not hard to see that Ehrenfeucht's Lemma implies the above theorem of Knight and Nadel.

And it happens that Ehrenfeucht's Lemma could also be proved by applications of Friedman's Embedding Lemma. (see Gitman, 2008)

But I shall present another proof of this lemma here.

Let $M \models \mathrm{PA}$.

A cofinal extension of M is a super-model $N \supseteq M$ s.t. for every $b \in N$ there exists $a \in M$ with $N \models b < a$. We write $M \subseteq_{\mathrm{cof}} N$ iff N is a cofinal extension of M, and $M \preceq_{\mathrm{cof}} N$ iff $M \subseteq_{\mathrm{cof}} N$ and $M \preceq N$.

An end extension of M is a super-model $N \supseteq M$ s.t. $b > N$ whenever $b \in N - M$. We write $M \subseteq_{\mathrm{e}} N$ iff N is an end extension of M, and $M \preceq_{\mathrm{e}} N$ iff $M \subseteq_{\mathrm{e}} N$ and $M \preceq N$.

Theorem (Gaifmann, 1972)

If $M \preceq N \models \mathrm{PA}$ *and* $M' = \{b \in N : \exists a \in M (N \models b < a)\}$ *then*

$$M \preceq_{\mathrm{cof}} M' \preceq_{\mathrm{e}} N.$$

Moreover, $\mathrm{SSy}(M') = \mathrm{SSy}(N)$.

So to construct extensions whose standard systems have certain properties, it suffices to construct cofinal extensions.

Another New Proof of Knight-Nadel Theorem

Extending a model of PA

PA has definable Skolem functions, i.e., each formula $\varphi(\vec{x}, y)$ corresponds to another formula $\psi(\vec{x}, y)$ s.t.

- $\mathrm{PA} \models \psi$ defines a function F_φ;
- $\mathrm{PA} \models \forall \vec{x}(\exists y \varphi(\vec{x}, y) \to \varphi(\vec{x}, F_\varphi(\vec{x})))$.

By the above property and Compactness, if $M \models \mathrm{PA}$ and $p(x)$ is a type of M, then there exists N, s.t. $M \prec N$, $p(x)$ is realized in N, and

$$N = \{F(b, \vec{a}) : \vec{a} \subset M, F \text{ is a function definable in } M\}.$$

So by constructing types, we can construct elementary extensions of given models.

And to construct cofinal elementary extensions, we can pick $a \in M - \mathbb{N}$ and then only construct types $p(x) \vdash x < a$. (Is this sufficient to get cofinal extensions? Why?)

Another New Proof of Knight-Nadel Theorem

The Construction of A Type

Let $\mathcal{S}$ be a Scott set, $M \models \mathrm{PA}$ be countable with $\mathrm{SSy}(M) \subseteq \mathcal{S}$, $X \in \mathcal{S}$.
We need N s.t. $M \prec N$, $X \in \mathrm{SSy}(N) \subseteq \mathcal{S}$.

We build a type $p(x)$ over M, s.t.,

(c1) if b realizes $p(x)$ (in an extension of M) then b codes X;

(c2) for each definable function F in parameters $\vec{d} \subset M$, $F(b, \vec{d})$ codes a set in $\mathcal{S}$ (NEVER GO OUTSIDE).

Fix $a \in M - \mathbb{N}$.

We begin with $p_0(x)$ consisting of conjunctions of formulas below

$$x \in 2^a, \ x(i) = X(i),$$

where 2^a denotes the set of binay sequences of length a, $X(i) = 1$ if $i \in X$ and $X(i) = 0$ if $i \notin X$.

As a is an 'infinite natural number', $p_0(x)$ is finitely realizable in M.

So, if b realizes a type $p(x) \supseteq p_0(x)$ then b satisfies (c1) (i.e., it codes X).

Another New Proof of Knight-Nadel Theorem

The Construction of A Type: coding only insiders

Let $F(x, \vec{d})$ be the first definable function ($\vec{d} \subset M$). We need to ensure if b realizes our final type $p(x)$ then $F(b, \vec{d})$ codes something inside $\mathcal{S}$.

Consider the following tree $T \subseteq 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$: $(\sigma, \tau) \in T$, iff $|\sigma| = |\tau|$ ($|\cdot|$ being the length), and the following subset of M is non-empty

$$\left\{x \in 2^a : \forall i < |\sigma| \left(\sigma(i) = x(i) \wedge \tau(i) = (F(x, \vec{d}))_i\right)\right\}.$$

Then, $T \in \mathrm{SSy}(M)$ and T is infinite (why?).

$T \oplus X$ computes another infinite tree: $\sigma \in T^X$ iff $(X \upharpoonright |\sigma|, \sigma) \in T$ (again, why infinite?).

So, T^X has an infinite path $Y \in \mathcal{S}$. Let $p_1(x)$ be the set of conjunctions of formulas from below

$$p_0(x) \cup \{(F(x, \vec{d}))_i = Y(i) : i \in \mathbb{N}\}.$$

So $p_1(x)$ is a type and if b realizes $p_1(x)$ then $F(b, \vec{d})$ codes Y.

Another New Proof of Knight-Nadel Theorem

The Construction of A Type: coding only insiders

Suppose that $p_n(x)$ is defined, $X, Y_k \in \mathcal{S}$, $F_k(x, \vec{d_k})$'s are M-definable functions ($k < n$), s.t., $p_n(x)$ consists of the following formulas

$$x \in 2^a,\ x(i) = X(i),\ (F_k(x, \vec{d_k}))_i = Y_k(i)\ (i \in \mathbb{N}, k < n).$$

$F_n(x, \vec{d_n})$ is a next M-definable function. Let T be the infinite tree in $\mathrm{SSy}(M)$ of $(\sigma, \tau_0, \dots, \tau_n)$, s.t. $|\sigma| = |\tau_k|$, in M there exists $b \in 2^a$ s.t.

$$i < |\sigma| \to \sigma(i) = b(i) \wedge \bigwedge_{k \leq n} (F_k(b, \vec{d_k}))_i = \tau_k(i).$$

As in defining p_1, the projection of T along $X, Y_0, \dots, Y_{n-1}$ is again an infinite tree in $\mathcal{S}$, so it has an infinite path $Y_n \in \mathcal{S}$, and we can define

$$p_{n+1}(x) = p_n(x) \cup \{(F_n(x, \vec{d_n}))_i = Y_n(i) : i \in \mathbb{N}\}.$$

Finally, $p(x) = \bigcup_n p_n(x)$.

Scott Set Problem without CH

But the problem is still open if CH is not assumed.

Question

If CH fails, does the Scott Set Problem still have a positive answer? What if CH is replaced by some forcing axiom?

A possible approach would be to generalize Ehrenfeucht's Lemma. So it is natural to raise the following question.

Question

Let $M \models \mathrm{PA}$ *be non-standard and* $\mathcal{S}$ *be a Scott set s.t.* $\mathrm{SSy}(M) \subset \mathcal{S}$*, and let* $X \in \mathcal{S} - \mathrm{SSy}(M)$*. Is it always possible to find* $N \succ M$ *s.t.* $X \in \mathrm{SSy}(N) \subseteq \mathcal{S}$*, even if* M *(or* $\mathrm{SSy}(M)$*) is uncountable?*

An Obstacle

One may try to generalize Ehrenfeucht's Lemma (i.e., to answer the previous question) via the following. Given $M \models \mathrm{PA}$ of cardinality $\aleph_1$, write it as an elementary chain of countable submodels M_α's. Then construct N_α's s.t. $X \in \mathrm{SSy}(N_0)$, and the following diagram commutes

$$\begin{array}{ccccccccc} N_0 & \xrightarrow{\prec} & N_1 & \xrightarrow{\prec} & \cdots & \xrightarrow{\prec} & N_\alpha & \xrightarrow{\prec} & \cdots \\ \prec\uparrow & & \prec\uparrow & & & & \prec\uparrow & & \\ M_0 & \xrightarrow{\prec} & M_1 & \xrightarrow{\prec} & \cdots & \xrightarrow{\prec} & M_\alpha & \xrightarrow{\prec} & \cdots \end{array}$$

Although N_0 could be obtained via Ehrenfeucht's Lemma, moving from N_0 to N_1 would encounter an obstacle.

Proposition (Knight, 1982)

For any countable non-standard $M_0 \models \mathrm{PA}$ and any $X \in 2^{\mathbb{N}}$, there are M_1, N_0 s.t. $M_0 \prec M_1$, $M_0 \prec N_1$, $\mathrm{SSy}(M_1) = \mathrm{SSy}(N_0)$, and if N_1 is any model as in the above diagram then $X \in \mathrm{SSy}(N_1)$.

Assuming Proper Forcing Axiom

A Scott set $\mathcal{S}$ is proper, iff $\mathcal{S}/\operatorname{Fin}$ (Fin is the set of finite subsets of $\mathbb{N}$) is a proper forcing poset under almost inclusion.

Theorem (Victoria Gitman)

Assume PFA. If $\mathcal{S}$ is a Scott set which is arithmetically closed (i.e., if $X \in \mathcal{S}$ and $Y \in \Sigma_n^X$ then $Y \in \mathcal{S}$) and proper then it is the standard system of some non-standard model.

However, it is unknown whether there exists a non-trivial ($\neq \mathcal{P}(\mathbb{N})$) uncountable Scott set which is arithmetically closed and proper.

A Weaker Question

Question

(ZFC) Are there non-trivial standard systems of cardinality $2^{\aleph_0}$*?*

Exercise

In ZF, show that there exist non-trivial Scott sets of cardinality $2^{\aleph_0}$*.*

Hint: use Cohen forcing only for arithmetic formulas; and note that every arithmetic closed subset of $2^{\mathbb{N}}$ is a Scott set.

Models with Non-trivial Standard Systems

Theorem

In ZF. For every countable non-standard $M \models \mathrm{PA}$*, there exists a family* $(M_X : X \subset 2^{\mathbb{N}})$ *s.t.* $M = M_\emptyset \preceq_{\mathrm{cof}} M_X$*,* $|M_X| = |\,\mathrm{SSy}(M_X)| = \max\{\aleph_0, |X|\}$ *and*

$$X \subseteq Y \Leftrightarrow M_X \preceq M_Y \Leftrightarrow \mathrm{SSy}(M_X) \subseteq \mathrm{SSy}(M_Y).$$

Two Incomparable Extensions of $\mathrm{SSy}(M)$

Consider a baby case: let $M \models \mathrm{PA}$ be countable and non-standard, construct M_1, M_2 s.t. $M \preceq_{\mathrm{cof}} M_i$ and $\mathrm{SSy}(M_i) \not\subseteq \mathrm{SSy}(M_j)$.

Pick $a \in M - \mathbb{N}$, we construct a type $p(x_1, x_2)$ and then form $M_i = M\langle x_i \rangle$.

We construct p as a union of $\bigcup_{s \in \mathbb{N}} p_s$, where each p_s is a finite type, $p_0 = \{x_1 \in 2^a, x_2 \in 2^a\}$ (2^a identified with the set of binary sequences with length a in M), $p_s \subseteq p_{s+1}$, and in M the size of $p_s(M)$ (the set of realizations of p_s in M) is at least $r(2^a)^2$ for some positive $r \in \mathbb{Q}$.

Given p_s and an M-definable function F, we want to have p_{s+1} implying that $(x_1)_k \neq (F(x_2))_k$ for some $k \in \mathbb{N}$. For $m \in \mathbb{N}$, let

$$q = p_0 \cup \{\forall k < m((x_1)_k = (F(x_2))_k)\}.$$

In M, the size of $q(M)$ is at most $2^{a-m}2^a = 2^{-m}(2^a)^2$. So there exists $k \in \mathbb{N}$, s.t., $p_{s+1} = p_s \cup \{(x_1)_k \neq (F(x_2))_k\}$ can serve our purpose.

Many Extensions of $\mathrm{SSy}(M)$

Given a countable $M \models \mathrm{PA}$, to construct $(M_X : X \subseteq 2^{\mathbb{N}})$ s.t. $\mathrm{SSy}(M_X) \subseteq \mathrm{SSy}(M_Y)$ iff $X \subseteq Y$.

It suffices to find $2^{\aleph_0}$ many $(b_f : f \in 2^{\mathbb{N}})$ s.t. if $X \subseteq 2^{\mathbb{N}}$ then the elementary extension of M generated by $(b_f : f \in X)$ would be M_X as desired. So we construct a type $p(\vec{x})$ of M in $2^{\aleph_0}$ many variables $\vec{x} = (x_f : f \in 2^{\mathbb{N}})$.

To construct $p(\vec{x})$, we construct a sequence of finite approximations $p_n(y_\sigma : \sigma \in 2^n)$, where 2^n denotes the set of binary sequences of length n, s.t.

- if $y_{\sigma_1}, \ldots, y_{\sigma_k} \in 2^{n+1}$, $\rho_i = \sigma_i \upharpoonright n$ (the initial segment of σ_i in 2^n) and $\varphi(y_{\rho_1}, \ldots, y_{\rho_k}) \in p_n$ is a formula with parameters in M, then $\varphi(y_{\sigma_1}, \ldots, y_{\sigma_k}) \in p_{n+1}$ as well (so the first order properties of y_ρ's specified in p_n are inherited by their descendants in p_{n+1});
- if $f_1, \ldots, x_k \in 2^{\mathbb{N}}$, $\sigma_i = f_i \upharpoonright (n+1)$ and $\varphi(y_{\sigma_1}, \ldots, y_{\sigma_k}) \in p_{n+1}$ then $\varphi(x_{f_1}, \ldots, x_{f_k}) \in p$.

Many Extensions of $\mathrm{SSy}(M)$

Pick $a \in M - \mathbb{N}$, let $p_0(y_\emptyset) = \{y_\emptyset \in 2^a\}$.

Suppose that $p_{n-1}(y_\sigma : \sigma \in 2^{n-1})$ and an M-definable function $F(z_1, \ldots, z_k)$ are given. We want to define p_{n+1} s.t. if $\tau, \tau_1, \ldots, \tau_k \in 2^{n+1}$ and $\tau \neq \tau_i$ $(i = 1, \ldots, k)$ then

$$p_n \vdash (y_\tau)_j \neq (F(y_{\tau_1}, \ldots, y_{\tau_k}))_j \text{ for some } j \in \mathbb{N}. \qquad (*)$$

To this end, as in the baby case, assume that in M the following holds

$$|\{\vec{b} \subset M : M \models p_n(\vec{b})\}| > q(2^a)^{2^n} = q|\{(b_\sigma : \sigma \in 2^n) : b_\sigma \in 2^a\}| \quad (**)$$

for some positive standard rational q. Then by the same trick in the baby case, we can get $(*)$ and also $(**)$ for p_{n+1}.

Many Extensions of $\mathrm{SSy}(M)$

By careful arrangement, we can ensure that if $f, f_1, \ldots, f_k \in 2^{\mathbb{N}}$ and F is an M-definable function then for some n and for $\tau = f \restriction (n+1)$ and $\tau_i = f_i \restriction (n+1)$,

$$p_{n+1} \vdash (y_\tau)_j \neq (F(y_{\tau_1}, \ldots, y_{\tau_k}))_j \text{ for some } j \in \mathbb{N}. \qquad (*)$$

As p inherits p_{n+1},

$$p \vdash (x_f)_j \neq (F(x_{f_1}, \ldots, x_{f_k}))_j.$$

Hence, if $(b_f : f \in 2^{\mathbb{N}})$ realizes p then the subset of $\mathbb{N}$ coded by b_f is not in the standard system of the elementary extension of M generated by $(b_g : f \neq g \in 2^{\mathbb{N}})$.

Exercises and Questions

Exercise

1. *Fix a countable* $M \models \mathrm{PA}$ *and* $(f_n \in 2^{\mathbb{N}} : n \in \mathbb{N})$ *s.t.* $f_n \notin \mathrm{SSy}(M)$*, construct* $N \succ M$ *s.t.* $|\mathrm{SSy}(N)| = 2^{\aleph}$ *and* $f_n \notin \mathrm{SSy}(N)$ *for all* n*.*
2. *Assume Martin's Axiom (*MA*), solve the above for* $|M| < 2^{\aleph}$ *and* $(f_\alpha : \alpha < \kappa < 2^{\aleph})$*.*
3. *Assume* MA*, fix a countable* $M \models \mathrm{PA}$*,* $(g_n \in 2^{\mathbb{N}})$*,* $\kappa < 2^{\aleph}$ *and* $(f_\alpha : \alpha < \kappa)$*, s.t., the Turing ideal generated by* $\mathrm{SSy}(M)$ *and* g_n*'s does not contain any of* f_α*, construct* N *s.t.* $M \prec N$*,* $g_n \in \mathrm{SSy}(N)$*,* $f_\alpha \notin \mathrm{SSy}(N)$ *and* $|\mathrm{SSy}(N)| = 2^{\aleph}$*.*

Question

In the last exercise above, can we solve it for uncountable M *and uncountably many* g*'s?*

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Thank you for your attention.