Ramsey Theorem as an intuitionistic property of well-founded relations

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Termination Theorem by Podelski and Rybalchenko

- Transition invariants are used by Podelski and Rybalchenko to prove the termination of a program.
- A transition invariant of a program is a binary relation over program's states which contains the transitive closure of the transition relation of the program; i.e. $T \supseteq R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$.
- A relation is disjunctively well-founded if it is a finite union of well-founded relations.

Theorem (Termination Theorem)

The program P is terminating iff there exists a disjunctively well-founded transition invariant for P.

Example

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while (x > 0 AND y > 0)
        (x,y) = (x-1, x)
          OR
        (x,y) = (y-2, x+1)
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A transition invariant for this program is $T_1 \cup T_2 \cup T_3$, where

$$
\begin{aligned}
T_1 &:= \{(\langle x, y\rangle, \langle x', y'\rangle) \mid x > 0 \wedge x' < x\} \\
T_2 &:= \{(\langle x, y\rangle, \langle x', y'\rangle) \mid x + y > 0 \wedge x' + y' < x + y\}. \\
T_3 &:= \{(\langle x, y\rangle, \langle x', y'\rangle) \mid y > 0 \wedge y' < y\}
\end{aligned}
$$

Since each T_i is well-founded, then the program terminates.

The proof by Podelski and Rybalchenko requires Ramsey Theorem

If you have ω people at a party then either there exists an infinite subset whose members all know each other or an infinite subset none of whose members know each other.

Theorem (Ramsey for pairs)

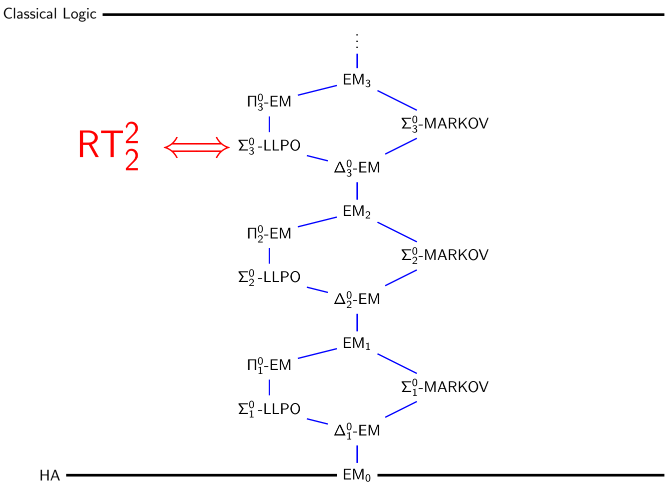
Assume I is a set having some injective enumeration $\sigma = x_0, x_1, \ldots, x_i, \ldots$. Assume $S_1, \ldots, S_n$ are binary relations on I which are a partition of $\{(x_i, x_j) \in I \times I : j < i\}$, that is:

1. $S_1 \cup \cdots \cup S_n = \{(x_i, x_j) \in I \times I : j < i\}$
2. *for all* $1 \leq k < h \leq n$: $S_k \cap S_h = \emptyset$.

Then for some $k \in [1, n]$ there exists some infinite $X \subseteq \mathbb{N}$ such that:

$$\forall i, j \in X.(j < i \implies x_i S_k x_j).$$

How many steps before the program P terminates? Hard to say, because Ramsey Theorem is a purely classical result.



In this work we provide a new intuitionistic version of Ramsey Theorem, which we called H-closure Theorem, and a new intuitionistic proof of the Termination Theorem.

Why do we care?

In order to use our intuitionistic proof to extract some effective bounds for the Termination Theorem.

H-closure Theorem

In order to analyse the bounds implicit in the Termination Theorem we have to replace Ramsey with some intuitionistic result.

Let R be a binary relation on I.

- $H(R)$ is the set of the R-decreasing transitive finite sequences on I:

$$\langle x_1, \ldots, x_n \rangle \in H(R) \iff \forall i, j \in [1, n](i < j \implies x_j R x_i).$$

- R is H-well-founded if $H(R)$ is $\succ$-well-founded.

Theorem (H-closure)

The H-well-founded relations are closed under finite unions:

$$(R_1, \ldots, R_n \text{ } H\text{-well-founded}) \implies ((R_1 \cup \cdots \cup R_n) \text{ } H\text{-well-founded}).$$

H-closure Theorem is classically true, because there exists a simple classical proof of the equivalence between Ramsey Theorem and *H*-closure Theorem.

Where "simple" means:

- it is short;
- it uses only Σ^0_3-LLPO;
- we conjecture it is in RCA_0.

H-closure Theorem is intuitionistically provable.

The proof of H-closure is strictly connected to the one by Erdős and Rado of Ramsey Theorem.

Erdős and Rado build an infinite k-ary tree and they find an infinite branch by using König Lemma. Then, thanks to the Pigeonhole Principle they extract the homogeneous set.

H-closure Theorem is intuitionistically provable.

Since we consider a contrapositive, in our proof we assume that each branch is finite and we show that the tree is finite.

In order to intuitionistically prove it we need to use the inductive definition of well-founded instead of the classical one.
Then instead of using König we use a variant of the Fan Theorem.

It splits the proof of Ramsey Theorem into two parts, one intuitionistic and the other classical but simple (it could be proved using the sub-classical principle Σ^0_3-LLPO).

From H-closure Theorem we may intuitionistically prove the Termination Theorem.

Sketch of the proof of the Termination Theorem by using H-closure. Assume that there exists a disjunctively well-founded transition invariant

$$T = R_1 \cup \cdots \cup R_n \supseteq R^+ \cap (\mathsf{Acc} \times \mathsf{Acc}),$$

- then R_i is H-well-founded for each $i \in [1, n]$;
- hence $R_1 \cup \cdots \cup R_n$ is H-well-founded;
- therefore $R^+ \cap (\mathsf{Acc} \times \mathsf{Acc})$ is H-well-founded and transitive;
- so it is well-founded, and then also $R \cap (\mathsf{Acc} \times \mathsf{Acc})$ is.

Why we need another intuitionistic version of Ramsey?

In 1990 Veldman and Bezem proved, using Choice Axiom and Brouwer thesis, the first intuitionistic negation-free version of Ramsey.

A binary relation R over a set is almost full if for all infinite sequences $x_0, x_1, x_2, \ldots, x_n, \ldots$ on I there are some $i < j$ such that $x_i R x_j$.

Theorem (Almost Full Theorem)

Almost full relations are closed under finite intersections.

Coquand showed that we may bypass the need of Choice Axiom and Brouwer thesis in the Almost Full Theorem, by using the inductive definition of well-foundedness instead of the classical one.

H-closure vs Almost Full

- H-closure and the Almost Full Theorem are classically equivalent.
- We find no easy way to intuitionistically deduce one from the other, due to the use of de' Morgan laws to move from the definition of almost full to the definition of H-well-founded.
- H-closure is in a sense more similar to the original Ramsey theorem, because it was obtained from it with just one classical step, a contrapositive; while almost fullness requires one application of de' Morgan Law, followed by a contrapositive.

What are we doing now?

Theorem

Let P be a program with transition relation R such that there exists a disjunctively well-founded transition invariant whose relations are primitive recursive and have height ω, i.e:

- $T = R_1 \cup \cdots \cup R_n \supseteq R^+ \cap (\mathrm{Acc} \times \mathrm{Acc})$,
- *where R_i is well founded, primitive recursive, and has height ω for any $i \in [1, n]$.*

Then the function computed by P is primitive recursive.

This primitive recursive bound is found by using H-closure Theorem.

What are we doing now?

Vice versa

Theorem

For any primitive recursive function there exists a program P which computes it, for which there is a disjunctively well-founded transition invariant whose relations are primitive recursive and have height ω.

What are we doing now?

Since by Termination Theorem Podelski, Rybalchenko and Cook produced an algorithm, we hope that H-closure Theorem could be useful in order to find bounds also for the algorithm.

Conjecture

A function has at least one implementation in Podelski-Rybalchenko language which the Terminator Algorithm may catch terminating if and only if the function is primitive recursive.

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