

MAPPING BOREL SETS ONTO BALLS AND SELF-SIMILAR SETS BY LIPSCHITZ AND NEARLY LIPSCHITZ MAPS

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ABSTRACT. If X is an analytic metric space satisfying a very mild doubling condition, then for any finite Borel measure μ on X there is a set $N \subseteq X$ such that $\mu(N) > 0$, an ultrametric space Z and a Lipschitz bijection $\phi : N \rightarrow Z$ whose inverse is nearly Lipschitz, i.e., β -Hölder for all $\beta < 1$.

As an application it is shown that a Borel set in a Euclidean space maps onto $[0, 1]^n$ by a nearly Lipschitz map if and only if it cannot be covered by countably many sets of Hausdorff dimension strictly below n .

The argument extends to analytic metric spaces satisfying the mild condition. Further generalization replaces cubes with self-similar sets, nearly Lipschitz maps with nearly Hölder maps and integer dimension with arbitrary finite dimension.

1. INTRODUCTION

It is easy to prove that every compact set $K \subseteq \mathbb{R}$ of real numbers with positive Lebesgue measure can be mapped onto the interval $[0, 1]$ by a Lipschitz map. In [17], Miklós Laczkovich asked if this remains true in higher dimensions. In more detail, if it is true that for every natural number $n > 1$ and every compact set $K \subseteq \mathbb{R}^n$ with positive n -dimensional Lebesgue measure there is a Lipschitz mapping $f : K \rightarrow [0, 1]^n$ onto the n -dimensional cube. This question turned to be very difficult. So far the (affirmative) answer was found only for $n = 2$ (by David Preiss, published years later in [2]).

Vitushkin, Ivanov and Melnikov [27] (see also [15]) constructed an example that shows that no generalization beyond the Laczkovich's question is possible: a compact set $K \subseteq \mathbb{R}^2$ with positive linear measure (i.e., the 1-dimensional Hausdorff measure) that cannot be mapped onto a segment by a Lipschitz map.

Let n be a positive natural number and let us denote the n -dimensional Hausdorff measure by \mathcal{H}^n and the Hausdorff dimension by $\dim_{\mathbb{H}}$. Using recent results on *monotone metric spaces* [28] and ultrametric spaces [19] (see below) Keleti, Máthé and Zindulka [16] proved that if the assumption $\mathcal{H}^n(K) > 0$ is strengthened to $\dim_{\mathbb{H}} K > n$, then K can be mapped by a Lipschitz map onto $[0, 1]^n$ for any analytic metric space K .

From what have been said, it is clear that we do not have a complete understanding of

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- what condition akin to $\mathcal{H}^n(K) > 0$ is equivalent to the existence of a Lipschitz mapping of K onto $[0, 1]^n$,
- what remains true about mapping of K onto $[0, 1]^n$ if we merely suppose that $\mathcal{H}^n(K) > 0$.

The present paper deals with the latter question and provides a partial answer.

We build upon ideas presented in [19] and [16]. In [19] Mendel and Naor prove that a compact metric space contains a large (with respect to Hausdorff dimension) subset with a rather simple metric structure – it is, up to bi-Lipschitz homeomorphism, ultrametric. We also construct a large subset with a simple structure, the “large” and “simple” notions, however, are a bit different.

In more detail, we prove in Section 2 a theorem about spaces satisfying a mild doubling condition henceforth termed *non-exploding* spaces. Roughly speaking, the condition requires that the number of balls of radius r needed to cover X does not increase too fast with decreasing r , cf. Definition 2.1. The following is a simplified version of Theorem 2.6.

Theorem. *Let X be a non-exploding analytic metric space and let μ be a finite Borel measure on X . Then there is a set $N \subseteq X$ such that $\mu(N) > 0$, an ultrametric space Z and a Lipschitz bijection $\phi : N \rightarrow Z$ whose inverse is β -Hölder for all $\beta < 1$.*

Then, based upon this result, we show that if a non-exploding analytic metric space has positive n -dimensional Hausdorff measure, then there is a mapping of X onto $[0, 1]^n$ that is as close to Lipschitz as it gets. (We know from the Vitushkin’s example that it may fail to be Lipschitz.)

Theorem. *Let X be an analytic non-exploding space and $n \in \mathbb{N}$. If $\mathcal{H}^n(X) > 0$, then there is a mapping $f : X \rightarrow [0, 1]^n$ onto $[0, 1]^n$ that is β -Hölder for every $\beta < 1$.*

This is proved in Section 4. The full strength of the mapping theorem is stated in Theorem 4.3 and its corollaries 4.4 and 4.5.

Once we have this mapping result, we may ask if the sufficient condition on the Hausdorff measure is also necessary. And it turns out that it is not, nevertheless there is a simple, natural condition on X involving Hausdorff dimension that is necessary and also sufficient. We discuss this in Section 3.

Further generalization replaces cubes with self-similar sets, nearly Lipschitz maps with nearly Hölder maps and integer dimension with arbitrary finite dimension.

The last Section 5 contains remarks and presents some questions and problems.

The results of this paper already found an application. Namely, Balka, Elekes and Máthé use them in [6] to prove the following theorem on a prevalent behavior of continuous functions. (The dimensions involved are the Hausdorff and packing dimensions of sets and measures.)

Theorem ([6]). *Let K be a non-exploding compact metric space and μ a continuous, finite Borel measure on K . Let n be a positive integer. Then for almost every continuous function on K (in the sense of Christensen’s [8] Haar measure zero) with values in \mathbb{R}^n there is an open set $U_f \subseteq \mathbb{R}^n$ such that $\mu(f^{-1}(U_f)) = \mu(K)$ and for all $y \in U_f$*

$$\dim_{\mathbb{H}} f^{-1}(y) \geq \dim_{\mathbb{H}} \mu \quad \text{and} \quad \dim_{\mathbb{P}} f^{-1}(y) \geq \dim_{\mathbb{P}} \mu.$$

This theorem generalizes a number of previous results and as far as I know, it is the first theorem of its kind proved in such a general context.

All spaces we work with are separable metric spaces. Recall that a metric space is *analytic* if it is a continuous image of a complete metric space (or, equivalently, of the irrational numbers, or equivalently, a Suslin set in a complete metric space). A continuous image of an analytic space is analytic. Every analytic space is separable.

Some of the common notation includes $B(x, r)$ for the closed ball centered at x , with radius r ; $\text{diam } E$ for the diameter of a set E in a metric space; $\text{dist}(A, B)$ the (lower) distance of two sets A, B ; n, m are generic symbols for positive integers. \mathbb{R} and \mathbb{Z} have the usual meaning; ω stands for the set of natural numbers including zero. 2^ω denotes the set of binary sequences; it is also a compact topological space homeomorphic to the standard Cantor ternary set and a topological group. Likewise, ω^ω denotes the set of all sequences of natural numbers, i.e., the maps $f : \omega \rightarrow \omega$.

2. LARGE NEARLY ULTRAMETRIC SETS

In [19], Mendel and Naor proved that every analytic metric space X contains large ultrametric-like subspaces. In more detail, for every $\varepsilon > 0$ there is a subset $Y \subseteq X$ with the following two properties:

- (1) Hausdorff dimension of Y is large: $\dim_{\text{H}} Y \geq \dim_{\text{H}} X - \varepsilon$,
- (2) the metric structure of Y is ultrametric-like: there is a bijection $f : Y \rightarrow U$ onto an ultrametric space U such that both f and its inverse are Lipschitz.

Though it is not clear at first glance how this powerful result is related to mapping metric spaces onto cubes, it is one of two crucial ingredients of the theorems of [16] mentioned in the previous paragraph and its spirit, as we shall see, is also important for the present paper.

In this section we attempt to prove a theorem similar to that of Mendel and Naor: our goal is to find, just like in the theorem, within an analytic metric space X a large ultrametric-like set Y . We want a bit more than (1): given a finite Borel measure on X , our set will have to have positive measure. In order to achieve that, we have to sacrifice some of (2): our set will be still ultrametric-like, but not quite as much as in (2). We term the notion *nearly ultrametric*; it is introduced in Definition 2.4 below.

We do not succeed completely: our proof only works within a framework of analytic spaces that are subject to a growth condition similar to the doubling condition, but much weaker, the so called *non-exploding spaces*. The result is stated in Theorem 2.6.

Non-exploding spaces. We first discuss the doubling-like condition. Recall that a metric space X is *doubling* if there is a number Q such that every ball in the space X can be covered by at most Q many balls of halved radii. This notion and equivalent or similar notions have been defined, investigated and used throughout the literature.

We may generalize the notion as follows. Suppose that the number Q is not fixed but may increase as the radius of the ball in question decreases; but it may not increase very fast. In more detail:

Definition 2.1. Let X be a metric space. If there is a function $Q : (0, \infty) \rightarrow \mathbb{R}$ such that every closed ball in X of radius $r > 0$ is covered by at most $Q(r)$ many closed balls of radius $r/2$ and such that

$$(NE) \quad \lim_{r \rightarrow 0} \frac{\log Q(r)}{\log r} = 0,$$

we call the metric space X *non-exploding*¹.

Needless to say that every doubling metric space and in particular every subset of a Euclidean space is non-exploding.

Nearly Lipschitz maps. We will be frequently making use of the notion of a nearly Lipschitz map that was introduced in [28]. We present two of the several equivalent definitions. Recall that, given $\beta > 0$, a mapping $f : (X, d_X) \rightarrow (Y, d_Y)$ is β -Hölder if there is an $\varepsilon > 0$ and a constant C such that if $d_X(x, y) \leq \varepsilon$, then $d_Y(f(x), f(y)) \leq C d_X(x, y)^\beta$. (Note that we deviate slightly from the usual definition by introducing ε , but if Y is bounded, the definitions are equivalent and, moreover, since we are interested in low scale behavior, we may always suppose that Y is bounded.)

Definition 2.2 ([28]). A mapping $f : (X, d_X) \rightarrow (Y, d_Y)$ between metric spaces is termed *nearly Lipschitz* if f is β -Hölder for all $\beta < 1$.

Proposition 2.3 ([28]). A mapping $f : (X, d_X) \rightarrow (Y, d_Y)$ is *nearly Lipschitz* if and only if there is a function $h : (0, \infty) \rightarrow (0, \infty)$ such that $\underline{\lim}_{r \rightarrow 0} \frac{\log h(r)}{\log r} \geq 1$ and

$$(1) \quad d_Y(f(x), f(y)) \leq h(d_X(x, y)), \quad x, y \in X.$$

Proof. Suppose that $f : X \rightarrow Y$ is nearly Lipschitz. Then there is a decreasing sequence $\delta_n \rightarrow 0$ and a sequence of constants C_n such that

$$d_X(x, y) \leq \delta_n \implies d_Y(f(x), f(y)) \leq C_n d_X(x, y)^{1-1/n}.$$

We may also suppose that $\delta_n \leq C_n^{-n}$ so that $\delta_n^{-1/n}$ is an upper estimate of C_n . Define the function h by

$$h(r) = r^{1-2/n} \text{ if } r \in [\delta_{n+1}, \delta_n).$$

Since $\log h(r)/\log r = 1-2/n$ on the entire interval $[\delta_{n+1}, \delta_n)$, we have $\underline{\lim}_{r \rightarrow 0} \frac{\log h(r)}{\log r} = \underline{\lim}_{n \rightarrow \infty} 1 - \frac{2}{n} = 1$.

If $\delta_{n+1} \leq d_X(x, y) < \delta_n$, then

$$d_Y(f(x), f(y)) \leq \delta_n^{-1/n} d_X(x, y)^{1-1/n} \leq d_X(x, y)^{1-2/n} = h(d_X(x, y))$$

which proves (1).

The reverse implication is straightforward. □

We will say that the metric spaces X, Y are *nearly Lipschitz equivalent* if there is a bijective mapping $f : X \rightarrow Y$ such that both f and its inverse are nearly Lipschitz.

¹The term was coined by Tamás Keleti.

Nearly ultrametric spaces. Recall that a metric space (X, d) is *ultrametric* if any triple $x, y, z \in X$ of points satisfies $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

Definition 2.4. A metric space X is *nearly ultrametric* if it is nearly Lipschitz equivalent to an ultrametric space.

Proposition 2.5. *Let X be a metric space. The following are equivalent.*

- (i) X is nearly ultrametric,
- (ii) there is a nearly Lipschitz bijection $f : X \rightarrow Y$ onto an ultrametric space with Lipschitz inverse,
- (iii) there is a Lipschitz bijection $f : X \rightarrow Y$ onto an ultrametric space with nearly Lipschitz inverse.

Proof. Of course it is enough to prove (i) \Rightarrow (ii) and (i) \Rightarrow (iii). Using Proposition 2.3 there are functions g, h such that $\lim_{r \rightarrow 0} \frac{\log h(r)}{\log r} \geq 1$ and $\lim_{r \rightarrow 0} \frac{\log g(r)}{\log r} \geq 1$ and such that

$$d_X(x, y) \leq g(d_Y(f(x), f(y))) \leq g \circ h(d_X(x, y)).$$

(The latter inequality holds if g is non-decreasing, but we may suppose that.) Define a new metric on Y by $d'_Y = g \circ d_Y$. Since d_Y is an ultrametric, so is d'_Y and it is easy to verify that the mapping $f : (X, d_X) \rightarrow (Y, d'_Y)$ is nearly Lipschitz and has Lipschitz inverse. This proves (i) \Rightarrow (ii) and (i) \Rightarrow (iii) is proved likewise. \square

We have enough to state the first summit of the paper.

Theorem 2.6. *Let X be a non-exploding analytic metric space and let μ be a finite Borel measure on X . Then for every $\varepsilon > 0$ there is a compact nearly ultrametric set $C \subseteq X$ such that $\mu(X \setminus C) < \varepsilon$.*

The rest of this section is devoted to the proof of this theorem. It has two stages: we first prove a very particular case and then reduce the theorem to this particular case.

The infinite dimensional torus. We will prove the theorem first for the infinite dimensional torus, i.e., the compact group $(\mathbb{R}/\mathbb{Z})^\omega$ equipped with a special metric, and its Haar measure.

We will identify \mathbb{R}/\mathbb{Z} with the interval $[0, 1)$. The group operation on $[0, 1)$ is of course addition modulo 1. For $x, y \in [0, 1)$ the distance from x to y is given by $d(x, y) = \|x - y\|$, where $\|z\| = \min\{|z|, |1 - z|\}$, i.e., $[0, 1)$ is a circle and $\|x - y\|$ is the length of the shorter of the two arcs between x and y . We also consider the Lebesgue measure on $[0, 1)$; let us denote it by \mathcal{L} .

Let $\mathbb{T} = [0, 1)^\omega$ and denote the probability Haar measure on \mathbb{T} by $\lambda_{\mathbb{T}}$ (which is clearly obtained as a product measure from \mathcal{L}).

Now let $G \in \omega^\omega$ be a sequence that is not eventually zero. Such a function defines a partition of $\omega = I_0 \cup I_1 \cup I_2 \cup \dots$ into disjoint consecutive intervals such that the length of each I_n is $G(n)$. (Some of the intervals may be empty.) For each n and $k \in I_n$ let $r_k = 2^{-n}$. Clearly $r_k \searrow 0$. Now define the metric on \mathbb{T} as follows:

$$d_G(x, y) = \sup_{k \in \omega} r_k \|x_k - y_k\|.$$

It is easy to check that d_G is an invariant metric on \mathbb{T} . It is clear that if G assumes large values, then the diameters of the coordinate circles decrease slowly, and vice versa. We impose the following growth condition upon G . From now on we define $\log 0 = 0$.

Definition 2.7. We call a sequence $G \in \omega^\omega$ *slow* if $G(n) > 0$ for infinitely many n and $\log G(n)/n \rightarrow 0$.

Lemma 2.8. *If G is slow, then for every $\varepsilon > 0$ there is a compact nearly ultrametric set $C \subseteq (\mathbb{T}, d_G)$ such that $\lambda_{\mathbb{T}}(C) > 1 - \varepsilon$.*

Proof. We first construct the set and then prove that it has the required properties. Begin with defining two sequences of positive real numbers $\langle a_m \rangle, \langle b_k \rangle$: Let

$$a_m = \frac{1}{2(m+1)^2}, \quad b_k = \frac{\varepsilon}{4(n+1)^2 G(n)},$$

where n is the unique number such that $k \in I_n$. Clearly

$$(2) \quad \sum_{m \in \omega} a_m < 1, \quad \sum_{k \in \omega} b_k < \varepsilon/2.$$

We now set up, for each k , a Cantor-like ternary set $C_k \subseteq [0, 1)$. Let $2^{<\omega} = \bigcup_{n \in \omega} 2^n$ be the binary tree consisting of all $\{0, 1\}$ -valued finite sequences. Build recursively a family of closed intervals $\{J_s : s \in 2^{<\omega}\}$ as follows. Let $J_\emptyset = [0, 1 - b_k]$ and if $s \in 2^{<\omega}$ and J_s is constructed, define $J_{s \frown 0}$ and $J_{s \frown 1}$ subject to the following conditions:

- (a) $J_{s \frown 0}$ and $J_{s \frown 1}$ are disjoint equally long subintervals of J_s ,
- (b) the left endpoints of J_s and $J_{s \frown 0}$ coincide,
- (c) the right endpoints of J_s and $J_{s \frown 1}$ coincide,
- (d) the gap between $J_{s \frown 0}$ and $J_{s \frown 1}$ is $2^{-|s|} a_{|s|} b_k$.

Conditions (2) ensure that the construction is possible. Let

$$C_k = \bigcap_{p \in \omega} \bigcup_{s \in 2^p} J_s$$

be the resulting ternary set. Let

$$C = \prod_{k \in \omega} C_k \subseteq \mathbb{T}.$$

The first property we notice is $\lambda_{\mathbb{T}}(C) > 1 - \varepsilon$. Indeed, C_k is omitting a set of Lebesgue measure exactly $b_k(1 + \sum_{m \in \omega} a_m) < 2b_k$. Hence $\mathcal{L}(C_k) > 1 - 2b_k$. Therefore

$$\lambda_{\mathbb{T}}(C) = \prod_k (1 - 2b_k) \geq 1 - 2 \sum_k b_k > 1 - \varepsilon.$$

We claim that C is also nearly ultrametric. To prove it, consider the Cantor cube 2^ω and provide it with the usual least difference metric: $\rho(x, y) = 2^{-|x \wedge y|}$ if $x \neq y$, $\rho(x, x) = 0$. (Here and later, $x \wedge y$ denotes the initial segment common to x and y .) For $x \in 2^\omega$ denote by \hat{x} the unique point of C_k coded by x .

Now consider the power $(2^\omega)^\omega$ of the Cantor set 2^ω and equip it with the metric

$$\rho_G(x, y) = \sup_{k \in \omega} r_k \rho(x_k, y_k).$$

For $x \in (2^\omega)^\omega$ let $\hat{x} = \langle \hat{x}_k : k \in \omega \rangle$. The mapping $x \mapsto \hat{x}$ is obviously bijective. It is also Lipschitz: Let $x, y \in (2^\omega)^\omega$ and let $k \in \omega$. Set $s = x_k \wedge y_k$. Then $\|\hat{x}_k - \hat{y}_k\| \leq \mathcal{L}(J_s) \leq 2^{-|s|} = \rho(x_k, y_k)$. Since this is true for every coordinate, we have $d_G(\hat{x}, \hat{y}) \leq \rho_G(x, y)$.

We now prove that the inverse $\hat{x} \mapsto x$ of the map is nearly Lipschitz. In view of Proposition 2.3 it is thus enough to proceed as follows: Let $j \in \omega$ and suppose

$\rho_G(x, y) = 2^{-j}$. There is n and $k \in I_n$ such that $r_k \rho(x_k, y_k) = 2^{-j}$. Put $s = x_k \wedge y_k$ and $p = |s|$. Since $r_k = 2^{-n}$, we have $2^{-n} 2^{-p} = 2^{-j}$, i.e., $n + p = j$. Estimate $d_G(\hat{x}, \hat{y})$:

$$d_G(\hat{x}, \hat{y}) \geq r_k \|\hat{x}_k - \hat{y}_k\| \geq r_k \text{dist}(J_{s \setminus 0}, J_{s \setminus 1}) = r_k b_k 2^{-p} a_p = 2^{-j} b_k a_p.$$

Therefore (employing definitions of a_m and b_k)

$$\begin{aligned} \frac{\log d_G(\hat{x}, \hat{y})}{\log \rho_G(x, y)} &\leq \frac{j - \log a_p - \log b_k}{j} \\ &= \frac{j + 2 \log(p + 1) + 1 + 2 \log(n + 1) + 2 + \log G(n) - \log \varepsilon}{j}. \end{aligned}$$

Let $\tilde{G}(n) = \max_{i \leq n} G(i)$. It is easy to check that since G is slow, so is \tilde{G} . Since $n + p = j$, we conclude that

$$\begin{aligned} \limsup_{j \rightarrow \infty} \sup_{\rho_G(x, y) = 2^{-j}} \frac{\log d_G(\hat{x}, \hat{y})}{\log \rho_G(x, y)} \\ \leq \limsup_{j \rightarrow \infty} \frac{j + 4 \log(j + 1) + 3 + \log \tilde{G}(j) - \log \varepsilon}{j} = 1, \end{aligned}$$

which is enough. In summary, the mapping $x \mapsto \hat{x}$ is a nearly Lipschitz equivalence of C and $((2^\omega)^\omega, \rho_G)$. Since $((2^\omega)^\omega, \rho_G)$ is an ultrametric space, it follows that C is nearly ultrametric. \square

We now extend the statement to all measures on \mathbb{T} .

Proposition 2.9. *Let μ be a finite Borel measure on \mathbb{T} and $\varepsilon > 0$. If G is slow, then there is a compact nearly ultrametric set $K \subseteq (\mathbb{T}, d_G)$ such that $\mu(\mathbb{T} \setminus K) < \varepsilon$.*

Proof. Assume without loss of generality that $\mu(\mathbb{T}) = 1$. Let $C \subseteq \mathbb{T}$ be the nearly ultrametric set from Lemma 2.8. Let $A = \{(x, y) : x + y \in C\}$. By Fubini Theorem, $(\lambda_{\mathbb{T}} \times \mu)(A) = \int_{\mathbb{T}} \lambda_{\mathbb{T}}(C - x) d\mu = \lambda_{\mathbb{T}}(C) \mu(\mathbb{T})$ and at the same time $(\lambda_{\mathbb{T}} \times \mu)(A) = \int_{\mathbb{T}} \mu(C - x) d\mu$. Thus

$$\int_{\mathbb{T}} \mu(C - x) d\mu = \lambda_{\mathbb{T}}(C) \mu(\mathbb{T}) = \lambda_{\mathbb{T}}(C) > 1 - \varepsilon$$

and consequently there is x such that $\mu(C - x) > 1 - \varepsilon$. Since C is nearly ultrametric and d_G is a shift invariant, it follows that $C - x$ is also nearly ultrametric.

(Observant reader may have noticed that we actually reproved the famous Christensen characterization [8] of Haar measure zero sets in locally compact groups in a slightly stronger setting. We could alternatively use the Christensen's theorem as a black box.) \square

Assouad's embedding revisited. To prove Theorem 2.6, it is enough to reduce it to the case covered by the above proposition. In order to do so we prove the following theorem. It is akin to classical embedding theorems of Aharoni [1] and Assouad [4, 5]. We will reiterate some of the ideas of their proofs, in particular those that appear in [5].

Theorem 2.10. *For every compact non-exploding metric space X there is a slow $G \in \omega^\omega$ and a bi-Lipschitz embedding $f : X \hookrightarrow (\mathbb{T}, d_G)$.*

Construction. Let X be a metric space. The symbol $B(x, r)$ denotes, as usual, a closed ball with center x and radius r .

Fix $\varepsilon > 0$. Suppose $S \subseteq X$ is a maximal ε -separated set in X (i.e., $d(s, s') > \varepsilon$ for distinct $s, s' \in S$). Let I be a finite set and $N \in \omega$ its cardinality. Suppose that for every $x \in X$ we have

$$(3) \quad |S \cap B(x, 8\varepsilon)| \leq N.$$

We employ Assouad's [5, Lemme 2.4] that claims that if (3) holds, then there is a coloring $\chi : S \rightarrow I$ such that if $d(s, s') \leq 8\varepsilon$, then $\chi(s) \neq \chi(s')$. Fix such a coloring χ and define, for each $j \in I$, a function $\phi_j : X \rightarrow \mathbb{R}$ as follows. Let $x \in X$. The set $\chi^{-1}(j) \cap B(x, \frac{3}{2}\varepsilon)$ has at most one point – otherwise there would be two points in S with the same color and within distance 3ε .

- If there is a (unique) $s \in \chi^{-1}(j) \cap B(x, \frac{3}{2}\varepsilon)$, let $\phi_j(x) = d(x, s)$,
- otherwise let $\phi_j(x) = \frac{3}{2}\varepsilon$.

Lemma 2.11. $\forall x, y \in X \forall j \in I |\phi_j(x) - \phi_j(y)| \leq d(x, y)$

Proof. Clearly $\phi_j(x) = \min\{\underline{d}(x, \chi^{-1}(j)), \frac{3}{2}\varepsilon\}$, where \underline{d} denotes the lower distance of a point from a set. The inequality

$$\underline{d}(x, \chi^{-1}(j)) \leq d(x, y) + \underline{d}(y, \chi^{-1}(j))$$

is immediate from the triangle inequality. Hence

$$\phi_j(x) \leq \min\{d(x, y) + \underline{d}(y, \chi^{-1}(j)), \frac{3}{2}\varepsilon\} \leq d(x, y) + \phi_j(y)$$

as required. \square

Lemma 2.12. *If $\frac{5}{2}\varepsilon < d(x, y) \leq 5\varepsilon$, then there is $j \in I$ such that*

$$|\phi_j(y) - \phi_j(x)| \geq \frac{1}{10}d(x, y).$$

Proof. Since S is maximal ε -separated, there is $s \in S \cap B(x, \varepsilon)$. Let $j = \chi(s)$. For every $s' \in S \cap B(y, \frac{3}{2}\varepsilon)$ we have

$$d(s, s') \leq d(s, x) + d(x, y) + d(y, s') \leq \varepsilon + 5\varepsilon + \frac{3}{2}\varepsilon < 8\varepsilon.$$

At the same time

$$d(s, s') \geq d(x, y) - d(s, x) - d(y, s') > \frac{5}{2}\varepsilon - \varepsilon - \frac{3}{2}\varepsilon = 0$$

which proves $s \neq s'$ and thus $\chi(s') \neq \chi(s) = j$. It follows that $\phi_j(y) = \frac{3}{2}\varepsilon$ and consequently

$$|\phi_j(y) - \phi_j(x)| \geq \frac{3}{2}\varepsilon - d(x, s) \geq \frac{1}{2}\varepsilon = \frac{1}{10}5\varepsilon \geq \frac{1}{10}d(x, y). \quad \square$$

Proof of Theorem 2.10. Since X is compact, we may assume that its diameter is bounded by 1. For every n let $\varepsilon_n = 2^{-n}$. Let $S_n \subseteq X$ be a maximal ε_n -separated set and let $G(n) \in \omega$ be a minimal number such that

$$\forall x \in X \quad |S_n \cap B(x, 8\varepsilon_n)| \leq G(n).$$

Claim. G , defined as above, is slow.

Indeed, letting

$$Q(\varepsilon) = \min\{Q \in \omega : \forall x \in X \exists \{x_i : i < Q\} B(x, \varepsilon) \subseteq \bigcup_{i < Q} B(x_i, \varepsilon/2)\}$$

it is easy to check that $G(n) \leq Q(8\varepsilon_n)Q(4\varepsilon_n)Q(2\varepsilon_n)Q(\varepsilon_n)$ and thus

$$\lim_{n \rightarrow \infty} \frac{\log G(n)}{n} \leq \lim_{n \rightarrow \infty} \frac{\log Q(8\varepsilon_n) + \log Q(4\varepsilon_n) + \log Q(2\varepsilon_n) + \log Q(\varepsilon_n)}{\log \varepsilon_n} = 0,$$

because X is non-exploding.

Let $\omega = I_0 \cup I_1 \cup I_2 \cup \dots$ be the partition of ω into disjoint consecutive intervals such that the length of each I_n is $G(n)$, as discussed earlier, and let $\chi_n : S_n \rightarrow I_n$ be the corresponding coloring. For every $j \in I_n$ let $\phi_j : X \rightarrow \mathbb{R}$ be the mapping constructed above for $\chi = \chi_n$, $S = S_n$, $\varepsilon = \varepsilon_n$. It is clear that $\phi_j(X) \subseteq [0, \frac{3}{2}\varepsilon_n]$. Thus the mapping ϕ on X defined by $\phi(x) = \langle \frac{1}{3}\phi_j(x) : j \in \omega \rangle$ maps X into the cube $\prod_{n \in \omega} [0, 2^{-n-1}]^{I_n}$. Equip this cube with the supremum metric. Now Lemma 2.11 proves that ϕ is Lipschitz and Lemma 2.12 proves that ϕ^{-1} is Lipschitz. Overall, $\phi : X \hookrightarrow \prod_{n \in \omega} [0, 2^{-n-1}]^{I_n}$ is a bi-Lipschitz embedding. For each n and $j \in I_n$ let ψ_j be the linear function that maps $[0, 2^{-n-1}]$ onto $[0, \frac{1}{2}]$. The mapping $\psi = \langle \psi_j : j \in \omega \rangle$ is clearly an isometric embedding of $\prod_{n \in \omega} [0, 2^{-n-1}]^{I_n}$ into (\mathbb{T}, d_G) . Thus $f = \psi \circ \phi$ is the required bi-Lipschitz embedding $f : X \hookrightarrow (\mathbb{T}, d_G)$. \square

Proof of Theorem 2.6. The proof of the main theorem is now trivial: Let μ be the finite Borel measure on X ; we may suppose $\mu(X) = 1$. Since X is analytic, there is a compact set $C \subseteq X$ such that $\mu(C) > 1 - \varepsilon/2$. By the above embedding theorem 2.10 there is a slow sequence $G \in \omega^\omega$ and a bi-Lipschitz embedding $C \hookrightarrow (\mathbb{T}, d_G)$. Let ν be the image measure of the restriction of μ to the set C . By Proposition 2.9 there is a nearly ultrametric set $K \subseteq \mathbb{T}$ such that $\nu(K) > \nu(f(C)) - \varepsilon/2 = \mu(C) - \varepsilon/2 > 1 - \varepsilon$. The desired set is $f^{-1}(f(C) \cap K)$. \square

3. INDECOMPOSABILITY

Given $n \in \omega$, if the n -dimensional Hausdorff measure of a metric space X is positive, then its Hausdorff dimension is at least n , but not necessarily vice versa. We will study a condition between the two, whose importance lies in the fact that it is sufficient and also necessary for X to be mapped onto an n -dimensional ball by a nearly Lipschitz mapping. The main result of this section is Theorem 3.3. We have to go through some preliminary material first.

Hausdorff functions. Recall that a right-continuous, non-decreasing function $h : [0, \infty) \rightarrow [0, \infty)$ with $h(0) = 0$ is called a *gauge* or a *Hausdorff function*. Recall that a gauge is *doubling* if there is a constant C such that $h(2r) \leq Ch(r)$ for all $r > 0$. For a gauge h define

$$\text{ord } h = \lim_{r \rightarrow 0} \frac{\log h(r)}{\log r}.$$

Hausdorff measures. Recall that given a gauge g , the *Hausdorff measure* \mathcal{H}^g on a metric space X is defined thus: For each $\delta > 0$ and $E \subseteq X$ set

$$(4) \quad \mathcal{H}_\delta^g(E) = \inf \sum_n g(\text{diam } E_n),$$

where the infimum is taken over all finite or countable covers $\{E_n\}$ of E by sets of diameter at most δ , and put

$$\mathcal{H}^g(E) = \sup_{\delta > 0} \mathcal{H}_\delta^g(E).$$

The basic properties of \mathcal{H}^g are well-known. It is an outer measure and its restriction to Borel sets is a G_δ -regular Borel measure in X . General references: [10, 18, 24]. We shall need the following theorem of Howroyd [12] that generalizes earlier results of Besicovitch [7] and Davies [9].

Theorem 3.1 (Howroyd [12]). *Let X be an analytic metric space and g a doubling gauge. If $\mathcal{H}^g(X) > 0$, then there is a compact set $K \subseteq X$ such that $0 < \mathcal{H}^g(K) < \infty$.*

The particular case of Hausdorff measure when the gauge is given by $g(x) = x^s$ for a fixed $s > 0$ is of major importance. The corresponding Hausdorff measure is called the *s-dimensional Hausdorff measure* and denoted by \mathcal{H}^s .

Hausdorff dimension. The *Hausdorff dimension* of a metric space X is denoted and defined by

$$\dim_{\mathbb{H}} X = \sup\{s : \mathcal{H}^s(X) > 0\}.$$

General references: [10, 18, 24].

Decomposability. The following properties will turn crucial.

Definition 3.2. Let $s > 0$. Say that a metric space X is *s-decomposable* if there is a countable cover $\{X_n\}$ of X such that $\dim_{\mathbb{H}} X_n < s$ for all n , and *s-indecomposable* if it is not *s-decomposable*.

It is easy to show that, for any metric space X

$$(5) \quad \mathcal{H}^s(X) > 0 \Rightarrow X \text{ is } s\text{-indecomposable} \Rightarrow \dim_{\mathbb{H}} X \geq s$$

and also that none of the implications can be reversed. The first implication, though, has a kind of converse in non-exploding spaces. Note that the non-exploding property is an invariant of nearly Lipschitz equivalence, and, as we shall see below in Lemma 4.2, so is *s-indecomposability*.

Theorem 3.3. *If $s > 0$ and X is a non-exploding analytic metric space, then X is *s-indecomposable* if and only if it is nearly Lipschitz equivalent to a metric space Z with $\mathcal{H}^s(Z) > 0$.*

To prove this theorem we prepare two lemmas.

Lemma 3.4. *Let h be a gauge and $0 < \beta \leq \text{ord } h$. There is a gauge \widehat{h} with the following properties:*

- (i) \widehat{h} is strictly increasing,
- (ii) $\widehat{h}(r) \geq h(r) + r^\beta$ on $[0, 1]$,
- (iii) $\text{ord } \widehat{h} = \beta$,
- (iv) the gauge $\widehat{h}^{1/\beta}$ is subadditive and $\text{ord } \widehat{h}^{1/\beta} = 1$,
- (v) \widehat{h} is doubling, and if $\beta \leq 1$, then \widehat{h} is subadditive.

Proof. First set $h^*(r) = h(r) + r^\beta$. It is easy to verify that $\text{ord } h^* = \beta$ and clearly $h^* \geq h$. Now define

$$\psi(r) = \begin{cases} \sup_{r \leq s \leq 1} s^{-\beta} h^*(s), & 0 < r < 1, \\ h^*(1), & r \geq 1. \end{cases}$$

If ψ is bounded, let $\widehat{h}(r) = \sup \psi \cdot r^\beta$. In this case everything is trivial. If ψ is unbounded, put $\widehat{h}(r) = r^\beta \psi(r)$ if > 0 and $\widehat{h}(0) = 0$. Routine calculation proves that \widehat{h} is right-continuous at 0.

First note that obviously $h^* \leq \widehat{h}$. We will often use it. In particular, (ii) holds.

(i) Suppose $0 < r < s \leq 1$. Routine calculation shows that since h^* is right-continuous, the supremum in the definition of ψ is attained. Therefore there is $t \geq r$ such that $\psi(r) = t^{-\beta} h^*(t)$. If $t \geq s$, then $\psi(r) = \psi(s)$ and $\widehat{h}(r) < \widehat{h}(s)$ follows. If $t < s$, then $\widehat{h}(r) = r^\beta t^{-\beta} h^*(t) < h^*(t) \leq h^*(s) \leq \widehat{h}(s)$.

(iii) $\text{ord } \widehat{h} \leq \beta$ follows from (ii). We show that $\text{ord } \widehat{h} \geq \beta$. Suppose for the contrary that there is $\varepsilon > 0$ such that $\text{ord } \widehat{h} < \beta - \varepsilon$. Since $\text{ord } h^* = \beta$, there is $\delta > 0$ such that $h^*(s) < s^{\beta-\varepsilon}$ for all $s \leq \delta$. Since ψ is unbounded, there is $s_0 < \delta$ such that $s_0^{-\beta} h^*(s_0) \geq \psi(\delta)$. Since $\text{ord } \widehat{h} < \beta - \varepsilon$, there is $r < s_0$ such that $\psi(r) > r^{-\varepsilon}$. Therefore there is $s \in [r, 1]$ such that $s^{-\beta} h^*(s) > r^{-\varepsilon}$ and since $r < s_0$, it follows that $s \leq \delta$. Since for every such s we have $h^*(s) < s^{\beta-\varepsilon}$, it follows that $r^{-\varepsilon} < s^{-\beta} h^*(s) < s^{-\beta} s^{\beta-\varepsilon} = s^{-\varepsilon}$. Hence $s < r$, a contradiction.

(iv) Write $g = \widehat{h}^{1/\beta}$. For $r < 1$ we have $g(r) = r \sup_{r \leq s \leq 1} (h^*)^{1/\beta}(s)/s$. Hence $g(r)/r$ is non-increasing. Therefore

$$g(r+s) = r \frac{g(r+s)}{r+s} + s \frac{g(r+s)}{r+s} \leq r \frac{g(r)}{r} + s \frac{g(s)}{s} = g(r) + g(s).$$

Since $\text{ord } \widehat{h} = \beta$, it is clear that $\text{ord } g = 1$.

(v) Both statements can be easily derived from (iv). \square

Lemma 3.5. *Let $s > 0$ and X be a non-exploding analytic space. If X is s -indecomposable, then there is a gauge g with $\text{ord } g \geq s$ and $\mathcal{H}^g(X) > 0$.*

Proof. Since every ball in a non-exploding space X is obviously totally bounded, its closure in the completion of X is compact. It follows that X is contained in a locally compact, subset of its completion. In particular, it is a subset of a σ -compact space.

We may thus employ the following [26, Theorem 6.4] of Sion and Sjerve: if X is an analytic subset of a σ -compact metric space, and $0 < s_0 < s_1 < s_2 < \dots$ is a sequence of reals, then

- (1) either there is a cover $\{X_n\}$ of X such that $\mathcal{H}^{s_n}(X_n) = 0$ for all n ,
- (2) or else there is a gauge g such that $\mathcal{H}^g(X) = \infty$ and $\text{ord } g > s_n$ for all n .

So suppose X is s -indecomposable and pick any sequence $s_n \nearrow s$. Then (1) obviously fails and thus (2) yields a gauge such that $\mathcal{H}^g(X) > 0$ and $\text{ord } g \geq \sup s_n = s$. \square

Proof of Theorem 3.3. The forward implication: Suppose X is s -indecomposable. By the above lemma there is a gauge g with $\text{ord } g \geq s$ and $\mathcal{H}^g(X) > 0$. Using Lemma 3.4 we may suppose that $\text{ord } g = s$, g is strictly increasing and $g(r) \geq r^s$. Define a gauge $h(r) = g(r)^{1/s}$. By Lemma 3.4 we may also suppose that h is strictly increasing, subadditive, $h(r) \geq r$, and $\text{ord } h = 1$.

Let d be the metric of X . Define a new metric on X by $\rho(x, y) = h(d(x, y))$. It is indeed a metric inducing the same topology, because h is subadditive and strictly increasing. The identity mapping $(X, \rho) \rightarrow (X, d)$ is Lipschitz, because $h(r) \geq r$. The identity mapping $(X, d) \rightarrow (X, \rho)$ is nearly Lipschitz, because $\text{ord } h = 1$. Thus (X, d) is nearly Lipschitz equivalent to (X, ρ) . Since $\rho^s(x, y) = g(d(x, y))$, we have $\mathcal{H}^s(X, \rho) = \mathcal{H}^g(X, d) > 0$.

The reverse implication is easy: let Z be nearly Lipschitz equivalent to X and $\mathcal{H}^s(Z) > 0$. Then Z is s -indecomposable by (5) and, by Lemma 4.2 below, so is X . \square

4. MAPPING NON-EXPLODING SPACES ONTO SELF-SIMILAR SETS AND CUBES

We are ready to present and prove the second summit of the paper: an analytic non-exploding space maps onto the cube $[0, 1]^n$ by a nearly Lipschitz map if and only if it is n -indecomposable.

We will actually prove a bit more general statement that involves self-similar sets. The material is taken from [28], where a *self-similar set* is defined as the attractor of an iterated function system with all functions being contracting similarities of \mathbb{R}^n . We also *a priori* impose upon self-similar sets the *Open Set Condition*. All of the relevant definitions can be found in [28]. Interested readers are referred to one of the books [11, 10, 18] for an overview of self-similar sets and related material.

Definition 4.1. A mapping between metric spaces $f : X \rightarrow Y$ is termed *dimension preserving* if $\dim_{\mathbb{H}} f(E) \leq \dim_{\mathbb{H}} E$ for every set $E \subseteq X$. We do not *a priori* impose any continuity on the mapping.

It is well-known and easy to see that Lipschitz mappings are dimension preserving (see, e.g., [10, Lemma 6.1]) and it is also easy to see that nearly Lipschitz mappings are also. It is also worth noticing that dimension preserving maps preserve s -decomposability. Proof is straightforward.

Lemma 4.2. *Let $f : X \rightarrow Y$ be a dimension preserving mapping onto Y . If X is s -decomposable, then so is Y . In particular, the conclusion holds if f is nearly Lipschitz.*

Theorem 4.3. *Let X be a non-exploding analytic metric space and $S \subseteq \mathbb{R}^n$ a self-similar set satisfying the Open Set Condition. Let $s = \dim_{\mathbb{H}} S$. The following are equivalent.*

- (i) X is s -indecomposable.
- (ii) There is a nearly Lipschitz mapping $f : X \rightarrow \mathbb{R}^n$ such that $S \subseteq f[X]$.
- (iii) There is a dimension preserving surjection $g : X \rightarrow S$.

Proof. (ii) \Rightarrow (iii) is easy: start with the nearly Lipschitz map f . Pick a single point $z \in S$ and define $g(x) = f(x)$ if $x \in f^{-1}(S)$ and $g(x) = z$ otherwise. Since, as pointed out above, f is dimension preserving, so is g .

(iii) \Rightarrow (i) is also easy: suppose that (iii) holds and yet X is s -decomposable. By Lemma 4.2 S is s -decomposable as well. But self-similar sets with the Open Set Condition are indecomposable because they have positive Hausdorff measure, see, e.g., [10].

The only remaining implication (i) \Rightarrow (ii) is harder. We postpone its proof until we gather some background material.

Corollary 4.4. *Let X be a non-exploding analytic metric space and $n \in \omega$. The following are equivalent.*

- (i) X is n -indecomposable.
- (ii) There is a nearly Lipschitz surjection $f : X \rightarrow [0, 1]^n$.
- (iii) There is a dimension preserving surjection $g : X \rightarrow [0, 1]^n$.

Proof. This is immediate, because the cube $[0, 1]^n$ is a self-similar set with the Open Set Condition. We only need to take care of values of f that are outside $[0, 1]^n$. But since $[0, 1]^n$ is convex, we may send every such point to the closest point on the boundary of $[0, 1]^n$. This mapping is Lipschitz, which is enough, since a composition of a nearly Lipschitz map and a Lipschitz map is nearly Lipschitz. \square

Let us point out that the above theorem and corollary apply in particular to Borel sets in Euclidean spaces. We present an illustration:

Corollary 4.5. *Let $m \leq n$ be positive integers. For every m -indecomposable Borel set $X \subseteq \mathbb{R}^n$ there is a nearly Lipschitz surjection $f : X \rightarrow [0, 1]^m$. In particular, such a mapping exists whenever $\mathcal{H}^m(X) > 0$.*

So in particular, the Vitushkin example mentioned in the introduction (i.e., a compact set $X \subseteq \mathbb{R}^2$ with positive linear measure that cannot be mapped onto a segment by a Lipschitz map) maps onto $[0, 1]$ by a nearly Lipschitz mapping.

We are aiming towards the proof of the remaining part of Theorem 4.3. We prepare a couple of notions and lemmas.

Monotone spaces. At this point we make use of the notion of monotone metric space introduced in [28], developed in [23] and further investigated in a number of papers, e.g., [13, 14, 20, 21, 22]. By the definition, a metric space (X, d) is *monotone* if there is a linear order $<$ on X and a constant $c > 0$ such that $x < y < z \Rightarrow d(x, y) < cd(x, z)$. The following two facts are crucial for our proof.

Lemma 4.6 ([28, Theorem 4.5, Lemma 3.2]). *Let X be an analytic monotone metric space and $S \subseteq \mathbb{R}^m$ a self-similar set. Let $s = \dim_{\text{H}} S$. If $\mathcal{H}^s(X) > 0$, then there is compact set $K \subseteq X$ such that $\mathcal{H}^s(K) > 0$ and a nearly Lipschitz mapping $g : K \rightarrow S$ onto S .*

Lemma 4.7 ([16, 23]). *Every ultrametric space is monotone.*

We will also need to extend nearly Lipschitz mappings. We prove the extension lemma in a slightly more general setting. Let us call a mapping f between metric spaces *nearly β -Hölder* if it is α -Hölder for all $\alpha < \beta$. This clearly extends the notion of nearly Lipschitz mapping. Proposition 2.3 has a counterpart for nearly Hölder mappings:

Lemma 4.8. *A mapping $f : (X, d_X) \rightarrow (Y, d_Y)$ is nearly β -Hölder if and only if there is a Hausdorff function h such that $\text{ord } h \geq \beta$ and*

$$d_Y(f(x), f(y)) \leq h(d_X(x, y)) \text{ for all } x, y \in X.$$

It is no surprise that nearly Lipschitz and nearly Hölder mappings are extendable:

Lemma 4.9. *Let X be a metric space and $Y \subseteq X$. Let $\beta \leq 1$. Any nearly β -Hölder mapping $f : Y \rightarrow \mathbb{R}^m$ extends to a nearly β -Hölder mapping over X .*

Proof. By Lemma 4.8 and 3.4 there is a subadditive Hausdorff function h such that $\text{ord } h = \beta$ and $|f(x) - f(y)| \leq h(d(x, y))$ for all $x, y \in Y$.

It is obviously enough to prove the statement for the coordinates of f , so assume without loss of generality that $f : Y \rightarrow \mathbb{R}$. Define the extension $f^* : X \rightarrow \mathbb{R}$ by

$$f^*(x) = \inf_{z \in Y} f(z) + h(d(x, z)).$$

Proving that $f^*(x) = f(x)$ for all $x \in X$ is straightforward. We prove that f^* is nearly β -Hölder. Let $x, y \in X$. Since h is subadditive, we have

$$f(z) + h(d(x, z)) \leq f(z) + h(d(y, z)) + h(d(x, y)) \text{ for all } z \in Y.$$

Therefore $f^*(x) \leq f^*(y) + h(d(x, y))$ and thus

$$|f^*(x) - f^*(y)| \leq h(d(x, y)) \text{ for all } x, y \in X.$$

Apply Lemma 4.8 to conclude that f^* is nearly β -Hölder. \square

Proof of Theorem 4.3(i) \Rightarrow (ii). Suppose X is s -indecomposable. By Theorem 3.3 we may suppose that $\mathcal{H}^s(X) > 0$. By the Howroyd theorem (Theorem 3.1) there is a finite Borel measure $\mu \leq \mathcal{H}^s$ such that $\mu(X) > 0$. Apply Theorem 2.6: there is a nearly ultrametric compact set $N \subseteq X$ such that $\mathcal{H}^s(N) \geq \mu(N) > 0$. By Proposition 2.5 there exists an ultrametric space U and a nearly Lipschitz surjection $\phi : N \rightarrow U$, with a Lipschitz inverse. Since U is a Lipschitz preimage of N , we have $\mathcal{H}^s(U) > 0$.

By Lemma 4.7, U is monotone, therefore Lemma 4.6 yields a compact subset $C \subseteq U$ and a nearly Lipschitz mapping $g : C \rightarrow S$ onto S . The composed map $g \circ \phi : N \rightarrow S$ onto S is clearly nearly Lipschitz. Now it is enough to apply Lemma 4.9 to extend $g \circ \phi$ over X . \square

5. COMMENTS AND QUESTIONS

Nearly Hölder mappings. We present a mild generalization of Theorem 4.3. The dimension preserving property can be parameterized as follows: for a mapping $f : (X, d_X) \rightarrow (Y, d_Y)$ between metric spaces define its *Hausdorff dimension* by

$$\dim_{\mathbb{H}} f = \inf\{\alpha : \dim_{\mathbb{H}} f(E) \leq \alpha \dim_{\mathbb{H}} E \text{ for every } E \subseteq X\}.$$

It is routine to show that if f is nearly s -Hölder, then $\dim_{\mathbb{H}} f \leq 1/s$.

Theorem 5.1. *Let X be a non-exploding analytic metric space and $S \subseteq \mathbb{R}^m$ a self-similar set; let $s = \dim_{\mathbb{H}} S \geq t > 0$. The following are equivalent.*

- (i) X is t -indecomposable.
- (ii) There is a nearly $\frac{t}{s}$ -Hölder mapping $f : X \rightarrow \mathbb{R}^m$ such that $S \subseteq f[X]$.
- (iii) There is a surjection $f : X \rightarrow S$ such that $\dim_{\mathbb{H}} f \leq \frac{s}{t}$.

Proof in outline. (ii) \Rightarrow (iii) is obvious and (iii) \Rightarrow (i) follows easily by modification of Lemma 4.2. (i) \Rightarrow (ii) follows the same way as in the proof of Theorem 4.3, one only has to insert between the spaces U and S an ultrametric space $U' = (U, d^{s/t})$ where d the ultrametric of U . \square

Corollary 5.2. *Let X be a non-exploding analytic metric space. Let $t \leq m$. The following are equivalent.*

- (i) X is t -indecomposable.
- (ii) There is a nearly $\frac{t}{m}$ -Hölder surjection $f : X \rightarrow [0, 1]^m$.
- (iii) There is a surjection $f : X \rightarrow [0, 1]^m$ such that $\dim_{\mathbb{H}} f \leq \frac{m}{t}$.

Corollary 5.3 (Peano curves). *For any $m > n > 0$ there is a nearly $\frac{n}{m}$ -Hölder Peano curve, i.e., a surjection $p : [0, 1]^n \rightarrow [0, 1]^m$.*

This seems to have attracted some attention. E.g., Arnold [3, Problem 1988-5] claims that without a proof and asks if “nearly” can be dropped. Semmes [25, 9.1] also discusses this topic.

The condition $s \geq t$ in the above theorem is not necessary, it is only needed for the extension of the nearly $\frac{t}{s}$ -Hölder mapping over the whole space X , guaranteed by Lemma 4.9 for nearly β -Hölder mappings only when $\beta \leq 1$. Inspection of the proofs shows that the following remains true for any $t > 0$. Recall that we *a priori* impose upon self-similar sets the *Open Set Condition*.

Theorem 5.4. *Let X be a non-exploding analytic metric space and $S \subseteq \mathbb{R}^m$ a self-similar set; let $s = \dim_{\mathbb{H}} S$ and $t > 0$. The following are equivalent.*

- (i) X is t -indecomposable.
- (ii) There is a compact set $K \subseteq X$ and a nearly $\frac{t}{s}$ -Hölder surjection $f : K \rightarrow S$.
- (iii) There is a surjection $f : X \rightarrow S$ such that $\dim_{\mathbb{H}} f \leq \frac{s}{t}$.

Lipschitz and Hölder maps. As mentioned in the introduction, in [16] it was proved that any analytic metric space X with $\dim_{\mathbb{H}} X > n$ maps onto the cube $[0, 1]^n$ by a Lipschitz map. The proof builds upon ideas similar to those in the present paper, but deviates in one detail: the constructed mapping factorizes through an interval. Thus it is inevitably short when mappings onto disconnected self-similar sets are under consideration. However, there is, as indicated below, an easy remedy.

Theorem 5.5. *Let X be an analytic metric space and $S \subseteq \mathbb{R}^n$ a self-similar set. If $\dim_{\mathbb{H}} X > \dim_{\mathbb{H}} S$, then there is a compact set $C \subseteq X$ and a Lipschitz surjection $f : C \rightarrow S$.*

Proof. By the theorem of Mendel and Naor [19] quoted above there is a compact set $C \subseteq X$, a compact ultrametric space U and a Lipschitz bijection $g : C \rightarrow U$ such that $\dim_{\mathbb{H}} C = \dim_{\mathbb{H}} U > \dim_{\mathbb{H}} S$. By Lemma 4.7 and [28, Theorem 4.7 and Lemma 3.2] there is a Lipschitz surjection $\phi : U \rightarrow S$. The required mapping is of course $f = \phi \circ g$. \square

Dimension preserving vs. nearly Lipschitz. As to nearly Lipschitz and dimension preserving mappings, we do not really know anything of the relation of the notions except of the following elementary fact.

Proposition 5.6. *A nearly Lipschitz mapping is dimension preserving and continuous.*

However, the theorems we proved indicate that some converse to this proposition might hold. For example, Corollary 4.4 contains the following information: *Let X be an analytic, non-exploding space. If there is a dimension preserving surjection $f : X \rightarrow [0, 1]^m$, then there is a nearly Lipschitz surjection $f : X \rightarrow [0, 1]^m$.* It is thus natural to ask:

Question 5.7. Is there any, however partial, converse to Proposition 5.6?

Non-exploding spaces. The notion of non-exploding space is tailored so that the proof of Theorem 2.6 works. The paramount question is, of course, whether this condition is really needed.

Question 5.8. Is it true that for every analytic (or, equivalently, compact) metric space X and every finite Borel measure μ on X there is a nearly ultrametric set $C \subseteq X$ such that $\mu(C) > 0$?

Note that if the answer were affirmative, the theorem of Balka, Darji and Elekes [6] quoted in the introduction (and its consequences) would hold for all compact metric spaces.

Even a little improvement would be valuable. Maybe there is some room for improvement. First, there may be a more effective way of constructing a bi-Lipschitz embedding of a compact space into ℓ^∞ . And maybe there is another, less regular construction of a nearly ultrametric set in \mathbb{T} with positive Haar measure that does not require the diameters of circles to tend to zero according to a slow function G . The importance of the following question is that all analytic spaces with finite box-counting dimension satisfy the condition and thus Theorem 2.6 would hold for all analytic spaces with finite packing dimension (since they are countable unions of sets with finite box-counting dimension).

Question 5.9. Let X be an analytic metric space. Suppose it satisfies condition

$$\lim_{r \rightarrow 0} \frac{\log Q(r)}{\log r} < \infty$$

in place of condition (NE). Is it true that for every finite Borel measure μ on X there is a nearly ultrametric set $C \subseteq X$ such that $\mu(C) > 0$?

Let us note that an affirmative answer to either of the above questions 5.8 and 5.9 would extend the results of [5].

The other main result, Theorem 4.3, depends mostly on the existence of large nearly ultrametric sets. That is another reason to study Questions 5.8 and 5.9.

Positive Hausdorff measure. We know from [16] that for every analytic metric space X , if $\dim_{\text{H}} X > m$, then there is a Lipschitz surjection of X onto $[0, 1]^m$, and that a non-exploding analytic X is s -indecomposable if and only if there is a nearly Lipschitz surjection of X onto $[0, 1]^m$. The condition $\mathcal{H}^m(X) > 0$ is between the two.

Question 5.10. Is there a condition similar to m -indecomposability that characterizes (non-exploding) compact spaces that map onto $[0, 1]^m$ by a Lipschitz mapping?

Question 5.11. Is there a type of mapping such that the existence of such a mapping of X onto $[0, 1]^m$ or some similar condition characterizes (non-exploding) compact spaces with $\mathcal{H}^m(X) > 0$?

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