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Cyclic Homology of Algebras with One Generator*

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Abstract. We compute the cyclic homology of $A = k[X]/\langle X^n \rangle$ for an arbitrary commutative ring k, and we apply this result to compute the cyclic homology of $k[X]/\langle f \rangle$, when k is a field and f is an arbitrary polynomial.

Key words. Cyclic homology, Hochschild homology.

0. Introduction

The aim of this paper is the computation of the cyclic homology of the algebra $A = k[X]/\langle f \rangle$, where k is a commutative ring with unit and f is a monic polynomial, under suitable hypothesis. In [8], Masuda and Natsume calculate the cyclic homology of this kind of algebra when the characteristic of k is 0. This has also been done by Kassel in [5]. In this work, we generalize these results. The special case $f = X^n - 1$ has been treated in [2] and [3].

We divide the article into three sections. In the first, we build up a simplified resolution using the Taylor series developed in [7] to compute the Hochschild homology of A over k. Both the Hochschild resolution and the simplified resolution are homotopically equivalent, so we compute the maps which give this equivalence. This study was started while one of the authors was at the University of Paris VII working with M. Karoubi [4].

In the second section, we compute the cyclic homology of $A = k[X]/\langle X^n \rangle$ starting with $k = \mathbb{Z}$ and then applying the Künneth formula [1].

We obtain the following results:

(a) $HC_{2r+1}(A) = \bigoplus_{j=0}^{r} \left(\left(\bigoplus_{a=1}^{n-1} \frac{k}{(a+jn)k} \right) \oplus \frac{k}{nk} \right) \quad (r \geqslant 0),$

(b) $HC_{2r}(A) = (\mathrm{Ann}(n)^{(r)}) \oplus k^{(n)} \oplus \left(\bigoplus_{j=0}^{r-1} \left(\bigoplus_{a=1}^{n-1} \mathrm{Ann}(a+nj) \right) \right) \quad (r \geqslant 0),$

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where $\mathrm{Ann}(m)$ is the annihilator of m in k, i.e., the m-torsion of k, and $M^{(h)}$ is the direct sum of h copies of M.

In all computations, we use the normalized double complex $B_{\mathrm{norm}}(A)$ introduced by Loday and Quillen [6].

In the third section, 'Further Results', we study the periodic cyclic homology of $A = k[X]/\langle X^n \rangle$. Furthermore, we compute the cyclic homology of $k[X]/\langle f \rangle$ when k is a field and f is an arbitrary polynomial.

1. Hochschild Homology

In this section, we compute the Hochschild homology of the algebra $A = k[X]/\langle f \rangle$, where k is a commutative ring with 1 and $f = X^n + f_{n-1}X^{n-1} + \cdots + f_0$ is a monic polynomial of degree n.

We start by building a projective resolution of A as a left A^e-module.

We use the Taylor series $T: B \to B \otimes_k B^{\mathrm{Op}}$, where B is an arbitrary k-algebra, defined in [7] as $T(P) = 1 \otimes P - P \otimes 1$. The relation

$$(*) \quad T(PQ) = PT(Q) + QT(P) + T(P)T(Q)$$

holds when $B \otimes_k B^{\mathrm{Op}}$ is considered as a B-module by $a(b \otimes c) = ab \otimes c$. We denote by $\mu: A^e \to A$ the multiplication map; since A is generated as a k-algebra by X, $\ker(\mu)$ is the ideal in A^e generated by $T(X)$.

Most of the computations will be carried out in $k[X]$, and the results seen through their images in A. We will use the fact, that, for every polynomial $P \in k[X]$, $T(P)$ is divisible by $T(X)$.

Since f is a monic polynomial, the division algorithm can be used. For every polynomial P, we denote $\bar{P}$ the quotient and $\dddot{P}$ the remaining, i.e. $P = \bar{P} \cdot f + \dddot{P}$, $\mathrm{dg}(\dddot{P}) < \mathrm{dg}(f)$. The uniqueness of $\bar{P}$ and $\dddot{P}$ is obvious.

Remarks 1.1. (a) We are going to use the maps $\varepsilon_0: A \otimes \bar{A}^s \otimes A \to A \otimes \bar{A}^{s+1} \otimes A$ $(s \geqslant 0)$, defined by $\varepsilon_0(a_0 \otimes \cdots \otimes a_{s+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{s+1}$, where $\bar{A} = A/k$ and $\bar{A}^s$ denotes the s-fold tensor product of $\bar{A}$ over k.

(b) $\gamma: k[X] \otimes k[X] \to A \otimes A$ will be the map $\gamma(P \otimes Q) = \Pi(\dddot{P} \otimes Q)$, where $\Pi: k[X] \otimes k[X] \to A \otimes A$ is the canoncial projection.

(c) The product P_1P_2 in A will be represented in $k[X]$ by $\overset{\dots\dots}{P_1P_2}$, and sometimes explicitly indicated.

PROPOSITION 1.2. *The following result holds*:

$$(1) \quad \frac{T(\overset{\dots\dots}{P_1P_2})}{T(X)} = (P_1 \otimes 1)\frac{T(P_2)}{T(X)} + (1 \otimes P_2)\frac{T(P_1)}{T(X)} - (1 \otimes \overline{P_1P_2})\frac{T(f)}{T(X)} \quad \mathrm{mod}(1 \otimes f, f \otimes 1),$$

$$(2) \quad \frac{T(f)}{T(X)} = \sum_{i>0} f_i \sum_{j=0}^{i-1} X^j \otimes X^{i-j-1},$$

$$(3) \quad (P \otimes 1)\frac{T(f)}{T(X)} = (1 \otimes P)\frac{T(f)}{T(X)} = \frac{T(Pf)}{T(X)} \quad \mathrm{mod}(1 \otimes f, f \otimes 1),$$

$$(4) \quad \gamma\left(X\frac{T(P)}{T(X)}\right) = 0 \otimes 0 \quad \textit{if } \mathrm{dg}(P) < n,$$

$$(5) \quad \gamma\left(X\frac{T(f)}{T(X)}\right) = 1 \otimes 1.$$

Proof: (1) As

$$\overset{\dots\dots}{P_1P_2} = P_1P_2 - \overline{P_1P_2} \cdot f, \qquad T(\overset{\dots\dots}{P_1P_2}) = T(P_1P_2) - T(\overline{P_1P_2} \cdot f),$$

so, by using the formula (*), we obtain in $k[X] \otimes k[X]$,

$$\begin{aligned} T(\overset{\dots\dots}{P_1P_2}) &= (P_1 \otimes 1)T(P_2) + (P_2 \otimes 1)T(P_1) + T(P_1)T(P_2) - \\ &\quad - (\overline{P_1P_2} \otimes 1)T(f) - (f \otimes 1)T(\overline{P_1P_2}) - T(f)T(\overline{P_1P_2}) \\ &= (P_1 \otimes 1)T(P_2) + (1 \otimes P_2)T(P_1) - (1 \otimes \overline{P_1P_2})T(f) \\ &\quad - (f \otimes 1)T(\overline{P_1P_2}), \end{aligned}$$

hence

$$\begin{aligned} \frac{T(\overset{\dots\dots}{P_1P_2})}{T(X)} &= (P_1 \otimes 1)\frac{T(P_2)}{T(X)} + (1 \otimes P_2)\frac{T(P_1)}{T(X)} \\ &\quad - (1 \otimes \overline{P_1P_2})\frac{T(f)}{T(X)} - (f \otimes 1)\frac{T(\overline{P_1P_2})}{T(X)}. \end{aligned}$$

(2) It follows from the linearity of T and the fact that

$$\frac{T(X^i)}{T(X)} = \sum_{j=0}^{i-1} X^j \otimes X^{i-j-1}.$$

(3) As $1 \otimes P - P \otimes 1$ is divisible by $T(X)$, $T(f)$ divides $(1 \otimes P - P \otimes 1)T(f)/T(X)$. Then the last one is zero in $A \otimes A$.

(4) If $\mathrm{dg}(P) < n$, all the monomials in $T(P)/T(X)$ have the form $X^k \otimes X^j$ with $k < n-1$, then $\overset{\dots}{X^{k+1}} \otimes X^j = 0$.

(5) If $f = \Sigma_{i=0}^n f_i X^i$, with $f_n = 1$, then

$$X \cdot \frac{T(f)}{T(X)} = \sum_{i=0}^{n} f_i \sum_{j=0}^{i-1} X^{j+1} \otimes X^{i-j-1}.$$

So,

$$\gamma\left(X\frac{T(f)}{T(X)}\right) = \sum_{i=0}^{n} f_i \sum_{j=0}^{i-1} \overset{\dots}{X^{j+1}} \otimes X^{i-j-1},$$

and since $\overline{X^{j+1}} \neq 0$ only in the case $j = n-1$,

$$\gamma\left(X\frac{T(f)}{T(X)}\right) = f_n \otimes 1 = 1 \otimes 1.$$

PROPOSITION 1.3. *The following sequence is exact*:

$$C_{s*}: \cdots \xrightarrow{d_6} A^2 \xrightarrow{d_5} A^2 \xrightarrow{d_4} A^2 \xrightarrow{d_3} A^2 \xrightarrow{d_2} A^2 \xrightarrow{d_1} A^2 \xrightarrow{\mu} A,$$

where A^2 denotes $A \otimes_k A$ and

$$d_{2i+1}(a \otimes b) = (a \otimes b)T(X), \qquad d_{2i}(a \otimes b) = (a \otimes b)\frac{T(f)}{T(x)}.$$

Proof. To check that the sequence is a complex, it is enough to see that $\mu \cdot T(X)$ and the product $T(X)T(f)/T(X)$ are zero in A^2. Then we build up A-right maps $s_0: A \to A^2$, $s_1: A^2 \to A^2$, and $s_2: A^2 \to A^2$ which are retraction homotopies for this complex, showing its exactness. They are defined as

$$s_0(P) = 1 \otimes P, \qquad s_1(P \otimes 1) = \frac{T(P)}{T(X)}, \qquad s_2(P \otimes 1) = \overline{PX} \otimes 1,$$

$$C_{s*}: \cdots \underset{s_2}{\overset{d_6}{\rightleftarrows}} A^2 \underset{s_1}{\overset{d_s}{\rightleftarrows}} A^2 \underset{s_2}{\overset{d_4}{\rightleftarrows}} A^2 \underset{s_1}{\overset{d_3}{\rightleftarrows}} A^2 \underset{s_2}{\overset{d_2}{\rightleftarrows}} A^2 \underset{s_1}{\overset{d_1}{\rightleftarrows}} A^2 \underset{s_0}{\overset{\mu}{\rightleftarrows}} A.$$

It can be seen by direct computation that

$$s_1 d_{2i-1} + d_{2i}s_2 = \text{id}, \qquad s_2 d_{2i} + d_{2i+1}s_1 = \text{id}, \qquad s_0\mu + d_1 s_1 = \text{id}, \qquad \mu s_0 = \text{id}.$$

Tensoring this sequence by A upon $A \otimes_{A^e}$ and using the identification between $A \otimes_{A^e} A^2$ and A, we obtain the complex

$$\bar{C}_{s*}: \cdots \xrightarrow{0} A \xrightarrow{f'} A \xrightarrow{0} A \xrightarrow{f'} A \xrightarrow{0} A \xrightarrow{f'} A \xrightarrow{0} A,$$

where $A \xrightarrow{f'} A$ is the map $f'(P) = f'P$.

Since A is k-free, its Hochschild homology $H_*(A)$ is $\operatorname{Tor}_*^{A^e}(A, A)$, so it is the homology of the complex $\bar{C}_{s*}$. Hence,

$$H_r(A) = \begin{cases} A & \text{if } r = 0, \\ A/\langle f' \rangle & \text{if } r \text{ is odd}, \\ \operatorname{Ann}(f') & \text{if } r \text{ is even and } r > 0. \end{cases}$$

Let

$$C_*: \cdots \xrightarrow{b'} A \otimes \bar{A}^4 \otimes A \xrightarrow{b'} A \otimes \bar{A}^3 \otimes A \xrightarrow{b'} A \otimes \bar{A}^2 \otimes A$$
$$\xrightarrow{b'} A \otimes \bar{A} \otimes A \xrightarrow{b'} A \otimes A \xrightarrow{b'} A$$

be the canonical reduced Hochschild resolution. We are going to define the maps $g_*: C_* \to C_{s*}$ and $h_*: C_{s*} \to C_*$ which are homotopy equivalences that will be used

later for the computation of the cyclic homology. These maps are defined as follows:

$g_0: A^2 \to A^2$ is the identity map,

$g_1: A \otimes \bar{A} \otimes A \to A^2$ is defined by $g_1(1 \otimes P \otimes 1) = \dfrac{T(P)}{T(X)}$,

$g_2: A \otimes \bar{A}^2 \otimes A \to A^2$ is given by $g_2(1 \otimes P_1 \otimes P_2 \otimes 1) = -1 \otimes \overline{P_1 P_2}$,

and for $s > 2$ $g_s: A \otimes \bar{A}^s \otimes A \to A^s$ will be

$$g_s(1 \otimes P_1 \otimes \cdots \otimes P_s \otimes 1)$$
$$= -g_{s-2}(1 \otimes P_1 \otimes \cdots P_{s-2} \otimes 1)g_2(1 \otimes P_{s-1} \otimes P_s \otimes 1),$$

hence

$$g_{2r}(1 \otimes P_1 \otimes \cdots \otimes P_{2r} \otimes 1) = (-1)^r \prod_{i=1}^{r} (1 \otimes \overline{P_{2i-1}P_{2i}})$$

and

$$g_{2r+1}(1 \otimes P_1 \otimes \cdots \otimes P_{2r+1} \otimes 1) = (-1)^{r+1}\frac{T(P_1)}{T(X)} \prod_{i=1}^{r} (1 \otimes \overline{P_{2i}P_{2i+1}}),$$

which after tensoring by $A \otimes_{A^e}$, become

$$\bar{g}_*: \bar{C}_* \to \bar{C}_{s*},$$

$$\bar{g}_{2r}(1 \otimes P_1 \otimes \cdots \otimes P_{2r}) = (-1)^r \prod_{i=1}^{r} P_{2i-1}P_{2i}$$

and

$$\bar{g}_{2r+1}(1 \otimes P_1 \otimes \cdots \otimes P_{2r+1}) = (-1)^{r+1}P_1' \prod_{i=1}^{r} \overline{P_{2i}P_{2i+1}},$$

where $\bar{C}_*$ denotes $C_* \otimes_{A^e} A$. They are A-maps by the action of A on the first factor.

The A^e-morphisms h_s will be defined by $h_0: A^2 \to A^2$ the identity map, and

$$h_{s+1}: A^2 \to A \otimes \bar{A}^{s+1} \otimes A, \quad h_{s+1}(1 \otimes 1) = \varepsilon_0 h_s d_{s+1}(1 \otimes 1).$$

By direct computations, it follows that

$$h_{2r}(1 \otimes 1) = (-1)^r \sum_{i_1, \ldots, i_r = 1}^{n} (f_{i_1} \cdots f_{i_r}) \sum_{k_1, \ldots, k_r = 1}^{i_1 - 1, \ldots, i_r - 1} 1 \otimes X^{k_1} \otimes X \otimes \cdots$$
$$\otimes X^{k_r} \otimes X \otimes X^{(\sum_{j=1}^r i_j - \sum_{j=1}^r k_j - r)},$$

$$h_{2r+1}(1 \otimes 1) = (-1)^{r+1} \sum_{i_1, \ldots, i_r = 1}^{n} (f_{i_1} \cdots f_{i_r}) \sum_{k_1, \ldots, k_r = 1}^{i_1 - 1, \ldots, i_r - 1} 1 \otimes X \otimes X^{k_1} \otimes$$
$$\cdots \otimes X^{k_r} \otimes X \otimes X^{(\sum_{j=1}^r i_j - \sum_{j=1}^r k_j - r)}$$

which, after tensoring, become

$$\bar{h}_{2r}(1) = (-1)^r \sum_{i_1,\ldots,i_r=1}^{n} (f_{i_1}\cdots f_{i_r}) \sum_{k_1,\ldots,k_r=1}^{i_1-1,\ldots,i_r-1} X^{(\sum_{j=1}^r i_j - \sum_{j=1}^r k_j - r)} \otimes X^{k_1}$$
$$\otimes X \otimes \cdots \otimes X^{k_r} \otimes X,$$

$$\bar{h}_{2r+1}(1) = (-1)^{r+1} \sum_{i_1,\ldots,i_r=1}^{n} (f_{i_1}\cdots f_{i_r}) \sum_{k_1,\ldots,k_r=1}^{i_1-1,\ldots,i_r-1} X^{(\sum_{j=1}^r i_j - \sum_{j=1}^r k_j - r)}$$
$$\otimes X \otimes X^{k_1} \otimes \cdots \otimes X^{k_r} \otimes X.$$

We shall need the following auxiliary result:

LEMMA 1.4. *For all* $r > 0$,

$$g_{r+1}(1 \otimes P_1' \otimes \cdots \otimes P_{r+1} \otimes P_{r+2})$$
$$= s_{r+1} g_r b'(1 \otimes P_1 \otimes \cdots \otimes P_{r+1} \otimes P_{r+2}).$$

Proof. Let us first show that $s_{r+1} g_r (1 \otimes P_1 \otimes \cdots \otimes P_{r+1}) = 0$. In fact, for $r = 2t$

$$g_r(1 \otimes P_1 \otimes \cdots \otimes P_{r+1}) = (-1)^t \prod_{i=1}^{t} (1 \otimes \overline{P_{2i-1}P_{2i}})(1 \otimes P_{r+1})$$

and

$$s_{r+1}(1 \otimes 1) = 0,$$

so the result follows because s_{r+1} is a right A-map; and, for $r = 2t+1$

$$g_r(1 \otimes P_1 \otimes \cdots \otimes P_{r+1}) = (-1)^{t+1} \frac{T(P_1)}{T(X)} \prod_{i=1}^{t} (1 \otimes \overline{P_{2i}P_{2i+1}})(1 \otimes P_{r+1}).$$

Since $\mathrm{dg}(P_1) < n$, all the monomials in the above product are of the type $X^i \otimes \alpha$ with $1 < n-1$, so

$$s_{r+1} g_r (1 \otimes P_1 \otimes \cdots \otimes P_{r+1}) = \sum \overline{X^{i+1}} \otimes \alpha = 0.$$

Using this, we have

$$s_{r+1} g_r b'(1 \otimes P_1 \otimes \cdots \otimes P_{r+1} \otimes P_{r+2}) = s_{r+1} g_r (P_1 \otimes \cdots \otimes P_{r+1} \otimes P_{r+2}).$$

Thus, we only need to show that

$$g_{r+1}(P_1 \otimes \cdots \otimes P_{r+1} \otimes P_{r+2})$$
$$= s_{r+1} g_r (P_1 \otimes \cdots \otimes P_{r+1} \otimes P_{r+2}).$$

Let $r = 2t$, then

$$g_{2t}(P_1 \otimes \cdots \otimes P_{2t+2}) = (-1)^t \prod_{i=1}^{t} (1 \otimes \overline{P_{2i}P_{2i+1}})(P_1 \otimes P_{2t+2}),$$

so

$$s_{2t+1} g_{2t} b'(1 \otimes P_1 \otimes \cdots \otimes P_{2t+2})$$
$$= s_{2t+1} g_{2t} (P_1 \otimes \cdots \otimes P_{2t+2})$$
$$= (-1)^{t+1} \frac{T(P_1)}{T(X)} \prod_{i=1}^{t} (1 \otimes \overline{P_{2i}P_{2i+1}})(1 \otimes P_{2t+2})$$
$$= g_{2t+1}(1 \otimes P_1 \otimes \cdots \otimes P_{2t+2}).$$

Finally, let $r = 2t+1$. Since

$$g_{2t+1}(P_1 \otimes \cdots \otimes P_{2t+3})$$
$$= (-1)^{t+1} \frac{T(P_2)}{T(X)} \prod_{i=1}^{t} (1 \otimes \overline{P_{2i+1}P_{2i+2}})(P_1 \otimes P_{2t+3})$$

and s_{2t+2} is an A-module map with the A-structure of $A \otimes A$ by multiplication on the right.

$$s_{2t+2} g_{2t+1} (P_1 \otimes \cdots \otimes P_{2t+3})$$
$$= (-1)^{t+1} \prod_{i=1}^{t} (1 \otimes \overline{P_{2i+1}P_{2i+2}})(1 \otimes P_{2t+3}) s_{2t+2}\left((P_1 \otimes 1)\frac{T(P_2)}{T(X)}\right).$$

Using

$$\frac{T(\overset{\cdots}{P_1P_2})}{T(X)} = (P_1 \otimes 1)\frac{T(P_2)}{T(X)} + (1 \otimes P_2)\frac{T(P_1)}{T(X)} -$$
$$-(1 \otimes \overline{P_1P_2})\frac{T(f)}{T(X)} \quad \text{(Proposition 1.2)},$$

we obtain

$$s_{2t+2}\left((P_1 \otimes 1)\frac{T(P_2)}{T(X)}\right) = s_{2t+2}\left(\frac{T(\overset{\cdots}{P_1P_2})}{T(X)} - (1 \otimes P_2)\frac{T(P_1)}{T(X)} + (1 \otimes \overline{P_1P_2})\frac{T(f)}{T(X)}\right)$$
$$= s_{2t+2}\left(\frac{T(\overset{\cdots}{P_1P_2})}{T(X)}\right) - (1 \otimes P_2) s_{2t+2}\left(\frac{T(P_1)}{T(X)}\right) +$$
$$+ (1 \otimes \overline{P_1P_2}) s_{2t+2}\left(\frac{T(f)}{T(X)}\right)$$
$$= 1 \otimes \overline{P_1P_2}.$$

Hence,

$$
\begin{aligned}
& s_{2t+2}g_{2t+1}(P_1 \otimes \cdots \otimes P_{2t+3}) \\
&= (-1)^{t+1}(1 \otimes \overline{P_1 P_2}) \prod_{i=1}^{t} (1 \otimes \overline{P_{2i+1}P_{2i+2}})(1 \otimes P_{2t+3}) \\
&= (-1)^{t+1} \prod_{i=0}^{t} (1 \otimes \overline{P_{2i+1}P_{2i+2}})(1 \otimes P_{2t+3}) \\
&= g_{2t+2}(1 \otimes P_1 \otimes \cdots \otimes P_{2t+3}).
\end{aligned}
$$

PROPOSITION 1.5. *g_* and h_* are maps of complexes.*

Proof. First, we shall prove that the diagrams

$$
\begin{tikzcd}
A \otimes \bar{A}^{r+1} \otimes A \arrow[r, "b'"] \arrow[d, "g_{r+1}"] & A \otimes \bar{A}^{r} \otimes A \arrow[d, "g_r"] \\
A^2 \arrow[r, "d_{r+1}"] & A^2
\end{tikzcd}
$$

commute. For $r = 0$, we have

$$
\begin{aligned}
d_1 g_1(1 \otimes P \otimes 1) = d_1\left(-\frac{T(P)}{T(X)}\right) &= -T(P) = P \otimes 1 - 1 \otimes P \\
&= g_0(P \otimes 1 - 1 \otimes P) \\
&= g_0 b'(1 \otimes P \otimes 1).
\end{aligned}
$$

Now, let $r > 0$. We know that $d_{r+1}s_{r+1} + s_r d_r = \mathrm{id}$ and our last lemma shows that $d_{r+1}g_{r+1} = d_{r+1}s_{r+1}g_r b'$.

So,

$$d_{r+1}g_{r+1} = d_{r+1}s_{r+1}g_r b' = g_r b' - s_r d_r g_r b'.$$

By inductive hypothesis $d_r g_r = g_{r-1}b'$, so

$$d_{r+1}g_{r+1} = g_r b' - s_r g_{r-1} b' b' = g_r b'.$$

Now we shall prove that

$$
\begin{tikzcd}
A^2 \arrow[r, "d_{r+1}"] \arrow[d, "h_{r+1}"] & A^2 \arrow[d, "h_r"] \\
A \otimes \bar{A}^{r+1} \otimes A \arrow[r, "b'"] & A \otimes \bar{A}^{r} \otimes A.
\end{tikzcd}
$$

commutes. For $r = 0$, $h_0 = \mathrm{id}$. So

$$
\begin{aligned}
h_0 d_1(P \otimes Q) = (P \otimes Q)T(X) = P \otimes QX - PX \otimes Q &= -b'(P \otimes X \otimes Q) \\
&= b'h_1(P \otimes Q).
\end{aligned}
$$

For $r > 0$,

$$h_{r+1}(1 \otimes 1) = \varepsilon_0 h_r d_{r-1}(1 \otimes 1) \quad \text{and} \quad b'\varepsilon_0 = \mathrm{id} - \varepsilon_0 b',$$

so

$$b'h_{r+1}(1 \otimes 1) = b'\varepsilon_0 h_r d_{r+1}(1 \otimes 1) = h_r d_{r+1}(1 \otimes 1) - \varepsilon_0 b' h_r d_{r+1}(1 \otimes 1).$$

Since by inductive hypothesis,

$$\varepsilon_0 b' h_r d_{r+1}(1 \otimes 1) = \varepsilon_0 h_{r-1} d_r d_{r+1}(1 \otimes 1) = 0,$$

then

$$b'h_{r+1}(1 \otimes 1) = h_r d_{r+1}(1 \otimes 1).$$

2. Cyclic Homology of $k[X]/\langle X^n \rangle$

In this section, we compute the cyclic homology of $A = k[X]/\langle X^n \rangle$ for k an arbitrary commutative ring with unit. We first study the case $k = Z$ and then apply the Künneth formula for general k. The final result is stated in Theorem 2.6. For the computations, we use the SBi sequence.

PROPOSITION 2.1. *Let k be a commutative ring with* 1 *and $A = k[X]/\langle f \rangle$, where $f = X^n + f_{n-1}X^{n-1} + \cdots + f_0$ is a monic polynomial of degree n. Then the morphism*

$$B_{s_m} = A \xrightarrow{\bar{h}_m} A \otimes \bar{A}^m \xrightarrow{B_m} A \otimes \bar{A}^{m+1} \xrightarrow{\bar{g}_{m+1}} A$$

is given by

$$
\begin{aligned}
& B_{s_{2r}}(X^a) = -a \cdot X^{a-1} - r \cdot \overline{f' \cdot X^a} \quad \textit{if } a \geqslant 0 \ (\textit{in particular, } B_{s_{2r}}(1) = 0) \\
& B_{s_{2r+1}} = 0.
\end{aligned}
$$

Proof. We shall first prove that $B_{s_{2r+1}} = 0$. As A is free over k with basis $(X^a)_{0 \leqslant a < n}$ and $\bar{h}_{2r+1}(X^a)$ is a linear combination over k of expressions of the type $X^{k_0} \otimes X \otimes X^{k_1} \otimes \cdots \otimes X^{k_r} \otimes X$, it will be enough to prove that

$$\bar{g}_{2r+2} B_{2r+1}(X^{k_0} \otimes X \otimes X^{k_1} \otimes \cdots \otimes X^{k_r} \otimes X) = 0.$$

Since

$$
\begin{aligned}
& \bar{g}_{2r+2}\varepsilon_0 t^s(X^{k_0} \otimes X \otimes X^{k_1} \otimes \cdots \otimes X^{k_r} \otimes X) \\
&\quad = (-1)^s \bar{g}_{2r+2}\varepsilon_0(X^{k_0} \otimes X \otimes X^{k_1} \otimes \cdots \otimes X^{k_r} \otimes X),
\end{aligned}
$$

where $t: A\otimes \bar A^i \to A\otimes \bar A^i$ is the map induced by the permutation $t': A^{i+1}\to A^{i+1}$ given by

$$t'(a_0\otimes\cdots\otimes a_i) = (-1)^i a_i\otimes a_0\otimes\cdots\otimes a_{i-1} \quad\text{and}\quad B_{2r+1}=\varepsilon_0\left(\sum_{s=0}^{2r+1} t^s\right),$$

the result follows.

To prove that $B_{s_{2r}}(X^a) = -a\cdot X^{a-1} - r\cdot\overline{f'X^a}$ we shall use

$$\bar h_{2r}(X^a) = (-1)^r \sum_{i_1,\dots,i_r=1}^{n} (f_{i_1}\cdots f_{i_r}) \times$$
$$\times \sum_{k_1,\dots,k_r=1}^{i_1-1,\dots,i_r-1} X^{k_0}\otimes X^{k_1}\otimes\cdots\otimes X^{k_r}\otimes X,$$

where $k_0=\Sigma_{j=1}^r (i_j-k_j) - r + a$, hence $B_{2r}\bar h_{2r}(X^a)$ is a sum of monomials of one of the following types

$$A = (-1)^{r+1} f_{i_1}\cdots f_{i_r}(1\otimes X^{k_0}\otimes X^{k_1}\otimes\cdots\otimes X^{k_r}\otimes X),$$
$$B = (-1)^{r+1} f_{i_1}\cdots f_{i_r}(1\otimes X\otimes X^{k_j}\otimes\cdots\otimes X\otimes X^{k_r}\otimes X\otimes X^{k_0}\otimes X^{k_1}\otimes$$
$$\otimes X\otimes\cdots\otimes X^{k_{j-2}}\otimes X\otimes X^{k_{j-1}}),$$

and

$$C = (-1)^{r+1} f_{i_1}\cdots f_{i_r}(1\otimes X^{k_j}\otimes X\otimes\cdots\otimes X^{k_r}\otimes X\otimes X^{k_0}\otimes X^{k_1}\otimes X\otimes$$
$$\cdots\otimes X^{k_{j-2}}\otimes X\otimes X^{k_{j-1}}\otimes X).$$

Now, $\bar g_{2r+1}(A)$ is not zero only if $k_1=\cdots=k_r=n-1$ (so $i_1=\cdots=i_r=n$ and $k_0=a$). In this case, $\bar g_{2r+1}(A) = -aX^{a-1}$. Similarly $\bar g_{2r+1}(B)\neq 0$ only if $k_2=\cdots=k_r=n-1$ (so $i_2=\cdots=i_r=n$ and $k_0=i_1-k_1+a$). Then

$$\bar g_{2r+1}(B) = -\overline{X^{k_0}X^{k_1}} = \overline{X^{k_0}}X^{k_1} - \overline{X^{k_0+k_1}}.$$

Finally $\bar g_{2r+1}(C)\neq 0$ only if $k_1=\cdots=\hat k_j=\cdots=k_r=n-1$ (and consequently $i_1=\cdots=\hat i_j=\cdots=i_r=n$). Hence

$$k_0=i_j-k_j-1+a$$

and

$$\bar g_{2r+1}(C) = -k_jX^{k_j-1}\overline{X^{k_0}X} = k_jX^{k_j-1}(\overline{X^{k_0}}X - \overline{X^{k_0+1}}).$$

Combining these results we obtain

$$\bar g_{2r+1}B_{2r}\bar h_{2r}(X^a) = -aX^{a-1} - r\sum_{i=1}^n f_i\sum_{k=1}^{i-1}(\overline{X^{i+a-1}} - \overline{X^{i-k+a-1}}\cdot X^k) -$$
$$- r\sum_{i=1}^n f_i \sum_{k=1}^{i-1} kX^{k-1}(\overline{X^{i-k+a}} - \overline{X^{i-k+a-1}}\cdot X).$$

By computing

$$\sum_{k=1}^{i-1} kX^{k-1}(\overline{X^{i-k+a}} - \overline{X^{i-k+a-1}}\cdot X)$$
$$= \sum_{k=1}^{i-1} kX^{k-1}\cdot\overline{X^{i-k+a}} - \sum_{k=2}^{i} kX^{k-1}\cdot\overline{X^{i-k+a}} + \sum_{k=2}^{i} X^{k-1}\cdot\overline{X^{i-k+a}}$$
$$= \overline{X^{i-k+a}} - iX^{i-1}\cdot\overline{X^a} + \sum_{k=2}^i X^{k-1}\cdot\overline{X^{i-k+a}}$$
$$= \sum_{k=1}^i X^{k-1}\cdot\overline{X^{i-k+a}} = \sum_{k=0}^{i-1} X^k\cdot X^{i-k-1+a},$$

and replacing this value in the previous formula, it follows that

$$\bar g_{2r+1}B_{2r}\bar h_{2r}(X^a)$$
$$= -aX^{a-1} - r\sum_{i=1}^n f_i\sum_{k=1}^{i-1}(\overline{X^{i+a-1}} - \overline{X^{i-k+a-1}}\cdot X^k) -$$
$$- r\sum_{i=1}^n f_i\sum_{k=0}^{i-1} X^k\cdot\overline{X^{i-k-1+a}}$$
$$= -aX^{a-1} - r\sum_{i=1}^n f_i(i-1)\overline{X^{i+a-1}} + r\sum_{i=1}^n f_i\sum_{k=1}^{i-1}\overline{X^{i-k+a-1}}\cdot X^k -$$
$$- r\sum_{i=1}^n f_i \sum_{k=0}^{i-1} X^k\cdot\overline{X^{i-k-1+a}}$$
$$= -aX^{a-1} - r\sum_{i=1}^n f_i(i-1)\overline{X^{i+a-1}} - r\sum_{i=1}^n f_i\overline{X^{i+a-1}}$$
$$= -aX^{a-1} - r\sum_{i=1}^n f_i i\overline{X^{i+a-1}} = -a\cdot X^{a-1} - r\cdot\overline{f'X^a}.$$

LEMMA 2.2. *If $f=X^n$, the sequence*

$$0\longrightarrow \operatorname{Coker} B_{2r}\longrightarrow HC_{2r+1}(A)\xrightarrow{S} HC_{2r-1}(A)\longrightarrow 0$$

splits ($r\geqslant 1$). Hence,

$$HC_{2r+1}(A)\cong HC_{2r-1}(A)\oplus \operatorname{Coker} B_{2r}\cong \bigoplus_{j=0}^r \operatorname{Coker} B_{2j}$$

and

$$B_{2r-1}: HC_{2r-1}(A)\to H_{2r}(A)$$

is zero.

Proof. It is convenient to remark that, for $f=X^n$, A is graded and both b and B are homogeneous maps. We shall give a map $R: HC_{2r-1}(A)\to HC_{2r+1}(A)$ such that $S\cdot R=\mathrm{id}$. We first define a k-free submodule of the cycles of dimension $2n-1$

whose images generate the cyclic homology, and a map $\bar{R}$ of this submodule into the $(2r+1)$-dimensional cycles. We prove later that $\bar{R}$ maps boundaries into boundaries, hence, inducing the desired map.

Let $X^a_{0,t} = \bar{h}_{2t+1}(X^a)$. We shall see by induction on r that for each a $(0 \leqslant a < n)$ there is a sequence $(X^a_{s,t})$ $(s \geqslant 0)$

$$X^a_{s,t} = \sum \lambda X^{\alpha_0} \otimes X^{\alpha_1} \otimes \cdots \otimes X^{\alpha_{2s+2t+1}} \in A \otimes \bar{A}^{2s+2t+1}$$

(where λ depends on $\alpha_0, \ldots, \alpha_{2s+2t+1}$), statisfying:

(i) $\alpha_0 + \cdots + \alpha_{2s+2t+1} = tn + a + 1$,
(ii) $(X^a_{r-t,t}; X^a_{r-t-1,t}; \ldots; X^a_{0,t}; 0; \ldots; 0)$ $(0 \leqslant t \leqslant r, 0 \leqslant a \leqslant n)$ generate $HC_{2r+1}(A)$,
(iii) $B_{2r+1}(X^a_{r-t,t}) = 0$.

In the case $r = 0$, conditions (i) to (iii) are immediate consequences of the equality $X^a_{0,0} = \bar{h}_1(X^a) = -X^a \otimes X$. For $r > 0$, we consider the exact sequence

$$H_{2r-1}(A) \xrightarrow{i} HC_{2r+1}(A) \xrightarrow{S} HC_{2r-1}(A).$$

By the inductive hypothesis, there exists $X^a_{s,t}$ $(0 \leqslant s, 0 \leqslant t, s+t < r$ and $0 \leqslant a < n)$ verifying (i), (ii) and (iii). Since $B_{2r-1}(X^a_{r-t-1,t})$ is zero in $H_{2r}(A)$ there is a $X^a_{r-t,t} \in A \otimes \bar{A}^{2r+1}$ such that

$$b_{2r+1}(X^a_{r-t,t}) = -B_{2r-1}(X^a_{r-t-1,t}).$$

As b and B are homogeneous we can take $X^a_{r-t,t}$ such that $\mathrm{dg}(X^a_{r-t,t}) = tn + a + 1$, where dg means degree. It is clear that the conditions (i) and (ii) are verified taking into account that the image of i gives $(X^a_{0,r}; 0; \ldots; 0)$. To check condition (iii), we apply $\bar{g}_{2r+2}B_{2r+1}$ and condition (i) to the elements of the type $(X^a_{r-t,t}; X^a_{r-t-1,t}; \ldots; X^a_{0,t}; 0; \ldots; 0)$ $(t \neq r)$ and use Proposition 2.1 for the elements $(X^a_{0,r}; 0; \ldots; 0)$.

The set $(X^a_{r-t-1,t}; \ldots; X^a_{0,t}; 0; \ldots; 0)$ is linearly independent in $(A \otimes \bar{A}^{2r-1}) \oplus (A \otimes \bar{A}^{2r-3}) \oplus \cdots \oplus (A \otimes \bar{A})$. This can be proved componentwise by showing that the set $(X^a_{0,t})$ is linearly independent in $A \otimes \bar{A}^{2t+1}$. So, the application $\bar{R}$ defined by

$$\begin{aligned}\bar{R}(X^a_{r-t-1,t}; \ldots; X^a_{0,t}; 0; \ldots; 0) \\ = (X^a_{r-t,t}; X^a_{r-t-1,t}; \ldots; X^a_{0,t}; 0; \ldots; 0)\end{aligned}$$

induces a map of a submodule of the cycles of dimension $2r+1$, which generates the $HC_{2r-1}(A)$, into those of dimension $2r+1$. To prove that this morphism induces $R: HC_{2r-1}(A) \to HC_{2r+1}(A)$ it will be enough to check that it maps boundaries into boundaries. To prove this fact, we may assume that

$$M = \sum \lambda_{a,t}(X^a_{r-t-1,t}; \ldots; X^a_{0,t}; 0; \ldots; 0)$$

is a boundary, i.e. there exists a $(y_r, \ldots, y_0)$ with $y_t \in A \otimes \bar{A}^{2t}$, such that

$$M = (b(y_r) + B(y_{r-1}); \ldots; b(y_1) + B(y_0); b(y_0)).$$

As b and B are homogeneous, then we may take the y's with the same total degree of the elements X's. Let $T = \Sigma\, \lambda_{a,t} X^a_{r-t,t} - B(y_r)$. So,

$$b(T) = b\Big(\sum \lambda_{a,t} X^a_{r-t,t}\Big) - bB(y_r) = b\Big(\sum \lambda_{a,t} X^a_{r-t,t}\Big) + Bb(y_r)$$

and since

$$b(y_r) + B(y_{r-1}) = \sum \lambda_{a,t} X^a_{r-t-1,t} \quad \text{and} \quad \sum \lambda_{a,t}(X^a_{r-t,t}; \ldots; X^a_{0,t}; 0; \ldots; 0)$$

is a cycle,

$$b(T) = b\Big(\sum \lambda_{a,t} X^a_{r-t,t}\Big) + B\Big(\sum \lambda_{a,t} X^a_{r-t-1,t}\Big) = 0.$$

Hence, T is a b-cycle and, using the fact that the total degree of T is at most rn, $\bar{g}_{2r+1}(T) = 0$ according to the definition of $\bar{g}_{2r+1}$. So it is a boundary, and there exists $y_{r+1} \in A \otimes \bar{A}^{2r+2}$ such that $b(y_{r+1}) + B(y_r) = \Sigma\, \lambda_{a,t} X^a_{r-t,t}$ as we wished to prove.

LEMMA 2.3. *If* $A = \mathbb{Z}[X]/\langle X^n \rangle$, Ker $B_{2r} = \mathbb{Z}$ *(as a consequence, the sequence*

$$0 \longrightarrow H_{2r+2}(A) \longrightarrow HC_{2r+2}(A) \longrightarrow \operatorname{Ker} B_{2r} \to 0$$

splits). Moreover, we can define a map $\bar{R}: \operatorname{Ker} B_{2r} \to HC^{2r+2}(A)$, *which actually maps into* Ker B_{2r+2}, *such that* $S \cdot \bar{R} = \mathrm{id}$.

Proof. According to Proposition 1.1, $H_{2r}(A) = \operatorname{Ann}(f')$ in A for $r > 0$. In this case, $f' = nX^{n-1}$, so $\operatorname{Ann}(f')$ is the submodule generated by $\{X, X^2, \ldots, X^{n-1}\}$; $H_0(A) = A$ is free with basis $\{1, X, \ldots, X^{n-1}\}$; and

$$H_{2r+1}(A) = A/\langle f' \rangle = \mathbb{Z}[X]/\langle X^n, nX^{n-1} \rangle$$

is the free module generated by $\{1, X, \ldots, X^{n-2}\}$, plus $(\mathbb{Z}/n\mathbb{Z})X^{n-1}$.

As

$$B_{2r} i(X^a) = -(a + rn)X^{a-1}, \qquad \operatorname{Ker}(B_0 i) = \langle 1 \rangle = \mathbb{Z}$$

and

$$\operatorname{Ker}(B_{2r} i) = 0 \quad \text{for} \quad r > 0$$

We shall prove by induction on r that the class of the element $(0, \ldots, 0, 1) \in (A \otimes \bar{A}^{2r}) \oplus \cdots \oplus A$ generates Ker B_{2r}. If $r = 0$, the result follows from the fact that Ker $B_0 = \operatorname{Ker}(B_0 i) = \langle 1 \rangle$ and is, hence, equal to $\mathbb{Z}$. If $r > 0$, let $\beta \in \operatorname{Ker} B_{2r}$. We know that $S(\beta) \in \operatorname{Ker} B_{2r-2}$, hence $S(\beta) = c(0, \ldots, 0, 1)$, where $(0, \ldots, 0, 1) \in (A \otimes \bar{A}^{2r-2}) \oplus \cdots \oplus A$, and $c \in \mathbb{Z}$. So, for $(0, \ldots, 0, 1) \in (A \otimes \bar{A}^{2r}) \oplus \cdots \oplus$

A, $S(\beta - c(0, \ldots, 0, 1)) = 0$. Because of the exactness of the SBi-sequence, there is a $\gamma \in H_{2r}(A)$ such that $\beta = c(0, \ldots, 0, 1) + i(\gamma)$. As β and $c(0, \ldots, 0, 1)$ belong to Ker B_{2r}, and $\mathrm{Ker}(B_{2r} i) = 0$, $\gamma = 0$. This proves that β is a multiple of $(0, \ldots, 0, 1)$. Since all even-dimensional boundaries in the double complex end in zero, then the element $(0, \ldots, 0, 1)$ generates a free submodule of $HC_{2r}(A)$ and the proof is achieved.

As $(0, \ldots, 0, 1) \in (A \otimes \bar{A}^{2r}) \oplus \cdots \oplus A$ freely generates Ker B_{2r} and $(0, \ldots, 0, 1) \in (A \otimes \bar{A}^{2r+2}) \oplus \cdots \oplus A$ generates Ker B_{2r+2}, we can define $\bar{R}$ mapping the first one into the second one.

PROPOSITION 2.4. *If* $A = \mathbb{Z}[X]/\langle X^n \rangle$, *then*

$$HC_{2r+1}(A) \cong \bigoplus_{j=0}^{r} \mathrm{Coker}(B_{2j} i)$$

Proof. $HC_1(A) \oplus \mathrm{Coker}\, B_1$ and Lemma 2.2 shows that

$$HC_{2r+1}(A) \cong HC_{2r-1}(A) \oplus \mathrm{Coker}\, B_{2r}.$$

So, we can conclude by induction that $HC_{2r+1}(A) \cong \bigoplus_{j=0}^{r} \mathrm{Coker}\, B_{2j}$. Finally, $\mathrm{Coker}\, B_{2j} = \mathrm{Coker}(B_{2j} \circ i)$ because

$$HC_{2j}(A) = i(H_{2j}(A)) \oplus \bar{R}(\mathrm{Ker}\, B_{2j-2}) \quad \text{and} \quad \bar{R}(\mathrm{Ker}\, B_{2j-2}) = \mathrm{Ker}\, B_{2j}$$

by Lemma 2.3.

COROLLARY 2.5. *If* $A = \mathbb{Z}[X]/\langle X^n \rangle$,

$$HC_{2r+1}(A) = \bigoplus_{j=0}^{r} \left(\left(\bigoplus_{a=1}^{n-1} \frac{\mathbb{Z}}{(a+jn)\mathbb{Z}} \right) \oplus \frac{\mathbb{Z}}{n\mathbb{Z}} \right) \quad (r \geqslant 0)$$

and

$$HC_{2r}(A) = \mathbb{Z}^{(n)} \quad (r \geqslant 0),$$

where $M^{(h)}$ *means a direct sum of h copies of M.*

Proof. According to the proof of Lemma 2.3, we have

$$B_{2r} \circ i(X^a) = -(a+rn)X^{a-1}, \quad H_0(A) = \bigoplus_{a=0}^{n-1} \mathbb{Z} \cdot X^a,$$

$$H_{2j}(A) = \bigoplus_{a=1}^{n-1} \mathbb{Z} \cdot X^a \quad (j > 0)$$

and

$$H_{2j+1}(A) = \left(\bigoplus_{a=0}^{n-2} \mathbb{Z} \cdot X^a \right) \oplus \frac{\mathbb{Z}}{n\mathbb{Z}} \cdot X^{n-1}.$$

So,

$$\mathrm{Coker}(B_{2j} i) = \left(\bigoplus_{a=1}^{n-1} \frac{\mathbb{Z}}{(a+jn)\mathbb{Z}} \cdot X^{a-1} \right) \oplus \frac{\mathbb{Z}}{n\mathbb{Z}} \cdot X^{n-1}.$$

Hence, from Lemma 2.3 and Proposition 2.4, we obtain

$$HC_{2r}(A) = H_{2r}(A) \oplus \mathbb{Z} = \mathbb{Z}^{(n)} \quad (r > 0)$$

and

$$HC_{2r+1}(A) \cong \bigoplus_{j=0}^{r} \mathrm{Coker}(B_{2j} i) \cong \bigoplus_{j=0}^{r} \left(\left(\bigoplus_{a=1}^{n-1} \frac{\mathbb{Z}}{(a+jn)\mathbb{Z}} \right) \oplus \frac{\mathbb{Z}}{n\mathbb{Z}} \right) \quad (r \geqslant 0),$$

respectively.

THEOREM 2.6. *If* $A = k[X]/\langle X^n \rangle$, *with k an arbitrary commutative ring with* 1,

$$HC_{2r+1}(A) = \bigoplus_{j=0}^{r} \left(\left(\bigoplus_{a=1}^{n-1} \frac{k}{(a+jn)k} \right) \oplus \frac{k}{nk} \right) \quad (r \geqslant 0).$$

and

$$HC_{2r}(A) = (\mathrm{Ann}(n)^{(r)}) \oplus k^{(n)} \oplus \left(\bigoplus_{j=0}^{r-1} \left(\bigoplus_{a=1}^{n-1} \mathrm{Ann}(a+nj) \right) \right) \quad (r \geqslant 0),$$

where $\mathrm{Ann}(m)$ *means the annihilator of m in k, i.e., the m-torsion of k.*

Proof. Since the complex $B(\mathbb{Z}[X]/\langle X^n \rangle)_{\mathrm{norm}}$ is free and finitely generated, then so are the cycles and boundaries, hence Künneth formula applies ([1], Ch. VI, Th. 3.3, p. 113) and the theorem follows immediately.

3. Further Results

In this section, we shall compute the periodic cyclic homology of the k-algebra $A = k[X]/\langle X^n \rangle$. Moreover, we shall apply the previous results to the case of a field k and an arbitrary polynomial f. As a corollary, we obtain that the map B_{2r-1}: $HC_{2r-1}(A) \to H_{2r}(A)$ is zero if k is an integral domain and f a monic polynomial.

PROPOSITION 3.1. *Let k be a commutative ring with* 1 *and* $A = k[X]/\langle X^n \rangle$. *Then we have the equalities*

$$HC_{2r+1}^{\mathrm{per}}(A) = \prod_{j=0}^{\infty} \left(\left(\bigoplus_{a=1}^{n-1} \frac{k}{(a+jn)k} \right) \oplus \frac{k}{nk} \right)$$

and

$$HC_{2r}^{\mathrm{per}}(A) = k \oplus \prod_{j=0}^{\infty} \left(\left(\bigoplus_{a=1}^{n-1} \mathrm{Ann}(a+nj) \right) \oplus \mathrm{Ann}(n) \right).$$

Proof. The periodic cyclic homology $HC_*^{\mathrm{per}}(A)$ and the cyclic homology $HC_*(A)$ are related by the following exact sequence,

$$0 \to \varprojlim_i{}^1 (HC_{m+1+2i}(A)) \to HC_m^{\mathrm{per}}(A) \to \varprojlim_i (HC_{m+2i}(A)) \to 0$$

(see, for instance, [9] (1.4)).

We are going to prove that

$$\varprojlim_i{}^1 (HC_{m+1+2i}(A)) = 0$$

and, hence, the periodic cyclic homology will be computed as $\varprojlim_i (HC_{m+2i}(A))$ for both m even and odd.

Let $m = 2r+1$. Using Lemma 2.2 and Proposition 2.4, we can see that the following square

$$\begin{tikzcd} HC_{2r+1}(A) \arrow[r] \arrow[d] & HC_{2r-1}(A) \arrow[d] \\ \bigoplus_{j=0}^{r}\left(\left(\bigoplus_{a=1}^{n-1} \frac{k}{(a+jn)k}\right) \oplus \frac{k}{nk}\right) \arrow[r, "\tau"] & \bigoplus_{j=0}^{r-1}\left(\left(\bigoplus_{a=1}^{n-1} \frac{k}{(a+jn)k}\right) \oplus \frac{k}{nk}\right) \end{tikzcd}$$

commutes, when τ is the canonical projection. This implies that the sequence $\{HC_{2i+1}(A), S\}$ satisfies the Mittag–Leffler condition, hence $\varprojlim_i{}^1 (HC_{2i+1}(A)) = 0$. Thus

$$HC_m^{\mathrm{per}}(A) = \varprojlim_i (HC_{m+2i}(A)) = \prod_{j=0}^{\infty}\left(\left(\bigoplus_{a=1}^{n-1} \frac{k}{(a+jn)k}\right) \oplus \frac{k}{nk}\right).$$

If $m = 2r$, we shall use the following short exact sequence of diagrams,

$$\begin{tikzcd}[column sep=small] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 \arrow[r] & HC_{2r+2}(\mathbb{Z}[X]/\langle X^n\rangle)\otimes k \arrow[r] \arrow[d, "S\otimes 1_k"] & HC_{2r+2}(A) \arrow[r] \arrow[d, "S"] & \mathrm{Tor}_1(HC_{2r+1}(\mathbb{Z}[X]/\langle X^n\rangle), k) \arrow[r] \arrow[d, "S_1"] & 0 \\ 0 \arrow[r] & HC_{2r}(\mathbb{Z}[X]/\langle X^n\rangle)\otimes k \arrow[r] \arrow[d] & HC_{2r}(A) \arrow[r] \arrow[d] & \mathrm{Tor}_1(HC_{2r-1}(\mathbb{Z}[X]/\langle X^n\rangle), k) \arrow[r] \arrow[d] & 0 \\ & \vdots & \vdots & \vdots & \end{tikzcd}$$

from which we obtain

$$\begin{tikzcd}[column sep=small] & & 0 \arrow[d] & & 0 \arrow[d] & \\ \text{(a)} \arrow[rr] & & k^{(n-1)}\oplus k \arrow[rr, "0\otimes 1_k"] \arrow[d] & & k^{(n-1)}\oplus k \arrow[d] \arrow[r] & {} \\ \text{(b)} \arrow[rr] & & k^{(n-1)}\oplus k \oplus\left(\bigoplus_{j=0}^{r}\left(\bigoplus_{a=1}^{n-1}\mathrm{Ann}(a+nj)\oplus \mathrm{Ann}(n)\right)\right) \arrow[rr, "S"] \arrow[d] & & k^{(n-1)}\oplus k \oplus\left(\bigoplus_{j=0}^{r-1}\left(\bigoplus_{a=1}^{n-1}\mathrm{Ann}(a+nj)\left(\oplus \mathrm{Ann}(n)\right)\right)\right) \arrow[d] \arrow[r] & {} \\ \text{(c)} \arrow[rr] & & \left(\bigoplus_{j=0}^{r}\left(\bigoplus_{a=1}^{n-1}\mathrm{Ann}(a+nj)\oplus \mathrm{Ann}(n)\right)\right) \arrow[rr, "S_1"] \arrow[d] & & \left(\bigoplus_{j=0}^{r-1}\left(\bigoplus_{a=1}^{n-1}\mathrm{Ann}(a+nj)\oplus \mathrm{Ann}(n)\right)\right) \arrow[d] \arrow[r] & {} \\ & & 0 & & 0 & \end{tikzcd}$$

where S_1 is the canonical projection from $\oplus_{j=0}^{r}$ to $\oplus_{j=0}^{r-1}$. Since the diagrams (a)

and (c) satisfy the Mittag–Leffler condition, we have that

$$\varprojlim_i{}^1 (HC_{2i}(\mathbb{Z}[X]/\langle X^n\rangle)\otimes k) \text{ is } 0$$

and

$$\varprojlim_i{}^1 (\mathrm{Tor}_1(HC_{2i+1}(\mathbb{Z}[X]/\langle X^n\rangle), k)) \text{ is also } 0.$$

So, by using the long exact sequence we have $\varprojlim_i{}^1 (HC_{2i}(A)) = 0$, and the limit sequence

$$\text{(d)}\quad 0 \to k \xrightarrow{\alpha} \varprojlim_i (HC_{2i}(A)) \to \prod_{j=0}^{\infty}\left(\left(\bigoplus_{a=1}^{n-1}\mathrm{Ann}(a+nj)\right)\oplus \mathrm{Ann}(n)\right) \to 0$$

is exact. Now, the canonical map

$$\begin{aligned} p_{2r} : k = \varprojlim_i (HC_{2r+2i}(\mathbb{Z}[X]/\langle X^n\rangle)\otimes k) &\longrightarrow HC_{2r}(\mathbb{Z}[X]/\langle X^n\rangle)\otimes k \\ &= k^{(n-1)}\oplus k \end{aligned}$$

composed with the projection π on the second factor $k^{(n-1)}\oplus k \to k$ is an isomorphism. So, if β is the composite

$$\varprojlim_i (HC_{2r+2i}(A)) \xrightarrow{p_{2r}} HC_{2r}(A) \xrightarrow{\pi_{2r}} k^{(n-1)}\oplus k \xrightarrow{\pi} k \xrightarrow{(\pi p_{2r})^{-1}} k,$$

where p_{2r} is the natural map and π_{2r} is the canonical projection of

$$HC_{2r}(A) = (k^{(n-1)}\oplus k)\oplus\left(\bigoplus_{j=0}^{r}\left(\left(\bigoplus_{a=1}^{n-1}\mathrm{Ann}(a+nj)\right)\oplus \mathrm{Ann}(n)\right)\right)$$

into its first component, then $\beta\circ\alpha$ is the identity. Hence, the sequence (d) splits and

$$HC_{2r}^{\mathrm{per}}(A) = \varprojlim_i (HC_{2i}(A)) = k \oplus \prod_{j=0}^{\infty}\left(\left(\bigoplus_{a=1}^{n-1}\mathrm{Ann}(a+nj)\right)\oplus \mathrm{Ann}(n)\right).$$

LEMMA 3.2. *$HC_*(A\times B)$ is isomorphic to $HC_*(A)\oplus HC_*(B)$.*

Proof. We are going to consider the Hochschild resolution of $A\times B$ and the direct sum of the resolutions of A and B

$$A\times B \xleftarrow{b'} (A\times B)^{\otimes 2} \xleftarrow{b'} (A\times B)^{\otimes 3} \xleftarrow{b'} (A\times B)^{\otimes 4} \xleftarrow{b'} (A\times B)^{\otimes 5} \xleftarrow{b'} \cdots$$

and

$$A\times B \xleftarrow{b'\times b'} A^{\otimes 2}\times B^{\otimes 2} \xleftarrow{b'\times b'} A^{\otimes 3}\times B^{\otimes 3} \xleftarrow{b'\times b'} A^{\otimes 4}\times B^{\otimes 4} \xleftarrow{b'\times b'} \cdots$$

As both are $(A\times B)^e$-resolutions which are relatively projective, the maps

$$\lambda_n((a_1, b_1)\otimes\cdots\otimes(a_n, b_n)) = (a_1\otimes\cdots\otimes a_n, b_1\otimes\cdots\otimes b_n)$$

induce an isomorphism in homology. Since the maps λ_n commute with t, they induce a map in the L-Q complexes, hence a map in the cyclic homology. This map is an isomorphism in Hochschild homology, so it is also an isomorphism in cyclic homology.

PROPOSITION 3.3. *If k is algebraically closed, then for $f = (X - a_1)^{n_1} \cdots (X - a_m)^{n_m}$,*

$$HC_*(k[X]/\langle f \rangle) \cong \bigoplus_{j=1}^{m} HC_*\left(\frac{k[X]}{(X - a_j)^{n_j}}\right) \cong \bigoplus_{j=1}^{m} HC_*\left(\frac{k[X]}{X^{n_j}}\right),$$

so we may compute the cyclic homology as in Section 2.

Proof. It is clear by Lemma 3.2.

Remark 3.4. If $A = k[X]/\langle f \rangle$, and $\bar{k}$ is a field containing k as a subfield, then $A \otimes_k \bar{k} = \bar{k}[X]/\langle f \rangle$. So, the L-Q complex, for $A \otimes_k \bar{k}$, is the same as that one of A tensorized by $\bar{k}$ over k. Since $\bar{k}$ is faithfully flat over k, $HC_*(A \otimes_k \bar{k}) = HC_*(A) \otimes_k \bar{k}$. This implies that, for an arbitrary field k, letting $\bar{k}$ be its algebraic closure, we can also compute the cyclic homology of A.

PROPOSITION 3.5. *For $A = k[X]/\langle f \rangle$, where k is an integral domain and f monic,*

$$B_{2r} : HC_{2r-2}(A) \longrightarrow H_{2r}(A) \text{ is zero.}$$

Proof. Let F be the field of quotients of k and let $i : k \to F$ be the canonical inclusion. i induces an injective map $A \to A' = F[X]/\langle f \rangle$, which we shall also denote i. There is a commutative diagram

$$\begin{array}{ccccc}
HC_{2r-1}(A) & \xrightarrow{B_{2r-1}} & H_{2r}(A) \cong \mathrm{Ann}(f') & \hookrightarrow & A \\
\downarrow & & \downarrow j & & \downarrow i \\
HC_{2r-1}(A') & \xrightarrow{B'_{2r-1}} & H_{2r}(A') \cong \mathrm{Ann}'(f') & \hookrightarrow & A'
\end{array}$$

where all the vertical arrows are induced by the inclusion i. Since $B'_{2r-1} = 0$ and j is injective, the B_{2r-1} is zero.

COROLLARY 3.6. *If k is an hereditary integral domain and f monic, then $HC_{2r}(k[X]/\langle f \rangle)$ is projective and its rank equals the dimension of $HC_{2r}(F[X]/\langle f \rangle)$, where F is the field of quotients of k and $F[X]/\langle f \rangle$ is considered as an F-algebra.*

Proof. Let $A = k[X]/\langle f \rangle$. By Proposition 3.5, we have the short exact sequence

$$0 \longrightarrow H_{2r}(A) \longrightarrow HC_{2r}(A) \longrightarrow \mathrm{Ker}\, B_{2r-2} \longrightarrow 0.$$

In order to prove that $HC_{2r}(A)$ is projective, we shall show that both $\mathrm{Ker}\, B_{2r-2}$ and $H_{2r}(A)$ are so. In fact, by the inductive hypothesis, we may suppose that $HC_{2r-2}(A)$ is projective. Then

$$\mathrm{Ker}\, B_{2r-2} \subseteq HC_{2r-2}(A) \quad \text{and} \quad H_{2r}(A) = \mathrm{Ann}(f') \subseteq A$$

are projective, because they are submodules of projective modules and k is hereditary.

The rank of $HC_{2r}(A)$ may be computed by tensoring by the field of quotients F or k and using the previous results (we recall that F is flat over k).

References

1. Cartan, H. and Eilenberg, S.: *Homological Algebra*, Princeton Univ. Press, 1956.
2. Cortiñas, G. and Villamayor, O. E.: Cyclic homology of $k[\mathbb{Z}/2\mathbb{Z}]$. *Rev. Un. Mat. Argentina* **33**, 1987.
3. Cortiñas, G., Guccione, J. and Villamayor, O. E.: Cyclic homology of $k[\mathbb{Z}/p\mathbb{Z}]$. *K-Theory* **2** (1989), 603–616.
4. Karoubi, M. and Villamayor, O. E.: Homologie cyclique d'algebres de groupe, *CRAS* (*serie I*) **311**(1) (1990), 1–3.
5. Kassel, C.: Cyclic homology, comodules, and mixed complexes, *J. Algebra* **107** (1987), 195–216.
6. Loday, J. L. and Quillen, D.: Cyclic homology and the Lie algebra homology of matrices, *Coment. Math. Helv.* **59** (1984), 565–591.
7. Mount, K. and Villamayor, O. E.: Taylor series and higher derivations, Pub. del Dept. de Matemätiça, Universidad de Buenos Aires No. 18, 1969.
8. Masuda, T. and Natsume, T.: Cyclic cohomology of certain affine schemes, *Publ. Res. Inst. Math. Sci.* **21** (1985), 1261–1279.
9. Weibel, C. A.: Nil *K*-Theory maps to cyclic homology, *Trans. Amer. Math. Soc.* **303** (1987), 541–558.