Filters for Square-bracket Partition Relations

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It is a classical theorem of infinitary combinatorics that any filter for the relation: $\kappa\to(\kappa)^2$ must be a κ-additive ultrafilter. In this paper we explore the properties of filters for the generally weaker square-bracket partition relations. In §1, we prove that a filter for $\kappa\to[\kappa]^2_3$ can be extended to a κ-additive ultrafilter, and that the existence of a filter for $\kappa\to[\kappa]^2_4$ implies κ is weakly compact. In §2, we prove that a Jonsson filter is, in at least one natural case, a Rowbottom filter. In §3, we establish that most of our hypotheses imply the existence of an inner model of ZFC with a measurable cardinal. In §4, we apply these techniques to the filter of ω-closed sets and the Axiom of Determinateness.

§0 Preliminaries

Throughout this paper, κ will represent an infinite cardinal. $[\kappa]^\alpha$ will represent the set of all increasing sequences of length α from κ, and $[\kappa]^{<\omega}$ will denote the collection of all finite subsets of κ.

κ satisfies $\kappa\to[\kappa]^\alpha_\beta$ ($\kappa\to[\kappa]^\alpha_{\beta,\gamma}$) iff for all partitions $F:[\kappa]^\alpha\to\beta$ there is a set $X\subseteq\kappa$, $\|X\|=\kappa$, such that $F''[X]^\alpha\neq\beta$ ($\|F''[X]^\alpha\|\leq\gamma$). The set X is called <u>homogeneous</u> for F. The relation $\kappa\to[\kappa]^\alpha_{\beta,<\gamma}$ requires that $\|F''[X]^\alpha\|<\gamma$. In all these definitions, $<\omega$ may be substituted for α. $\kappa\to(\kappa)^\alpha$ is the relation $\kappa\to[\kappa]^\alpha_2$.

$\mathcal{F}$ is a filter for a given partition relation if it is a uniform filter, and if homogeneous sets in the filter may be found for all partitions. This is written by placing $\mathcal{F}$ inside the bracket, for example $\kappa\to[\mathcal{F}]^\alpha_\beta$ with the understanding that $\mathcal{F}$ is a uniform filter over κ.

$\mathcal{F}$ is a Jonsson filter if κ satisfies $\kappa \to [\mathcal{F}]^{<\omega}_{\kappa}$. $\mathcal{F}$ is a δ Rowbottom filter if $\mathcal{F}$ satisfies $\kappa \to [\mathcal{F}]^{<\omega}_{\lambda,<\delta}$ for all λ.

Ultrafilters U which satisfy $\kappa \to [U]^2_2$ are called Ramsey, and as mentioned at the beginning such ultrafilters are κ-additive over κ, and hence κ is either ω or a measurable cardinal. Ultrafilters which satisfy $\kappa \to [U]^2_3$ were first discussed by A. Blass [1] who called them weakly Ramsey, and since then, several results have appeared. For example, A. Blass [2] shows that if $\omega \to [U]^2_5$, then U is a p-point, and A. Kanamori [7] provides constructions for measurable cardinals.

We will have occasion to call upon some further concepts involving ultrafilters. The Rudin-Keisler partial order is defined as follows: If U is an ultrafilter over a set I, and V is over J, say $V \le_{RK} U$ iff there is a function f: $I \to J$ such that $V = f_*(U)$, where $f_*(U) = \{X \subseteq J \mid f^{-1}(X) \in U\}$. We write $U =_{RK} V$ iff $U \le_{RK} V$ and $V \nleq_{RK} U$. It is well-known that $U =_{RK} V$ iff $V = f_*(U)$ for some f which is one-to-one on a set in U. U is said to be RK-minimal iff there is no (non-principal) $V <_{RK} U$. It is well-known that U is RK-minimal iff U is Ramsey.

An ultrafilter U is called (μ,ν)-regular iff there are ν sets in U any μ of which have empty intersection. Such a family of sets is called a (μ,ν)-regularizing family for U. An ultrafilter U over κ is λ-decomposable iff there is a uniform ultrafilter V over λ such that $V \le_{RK} U$. This is equivalent to saying that there is a partition $\bigcup_{\alpha<\lambda} A_\alpha = \kappa$ such that for any $X \subset \lambda$ with $|X| < \lambda$, $\bigcup_{\alpha \in X} A_\alpha \notin U$. An ultrafilter U over κ is weakly normal iff whenever $\{\alpha<\kappa \mid f(\alpha)<\alpha\} \in U$, there is a $\beta < \kappa$ so that $\{\alpha<\kappa \mid f(\alpha) \le \beta\} \in U$.

The regularity and decomposability of ultrafilters were first considered by H. J. Keisler (see [3] 4.3); K. Prikry [17] is the definitive paper on decomposability, and A. Kanamori [8] connects irregularity with weak normality. W. W. Comfort and S. Negrepontis [4] is a good general reference for ultrafilters.

With the exception of §4, we assume the Axiom of Choice throughout.

§1 Filters implying measurability and weak compactness

Our chief tool in this section will be a technique for extending filters to ultrafilters.

<u>Theorem 1.1</u> Let κ, $\mathcal{F}$ satisfy $\kappa\to[\mathcal{F}]^{\alpha}_{n}$, for some $n<\omega$, α, κ regular. Then $\mathcal{F}$ can be extended to a uniform ultrafilter $\mathcal{F}^*$ with additivity at least that of $\mathcal{F}$.

Proof: Let n be least such that $\kappa\not\to[\mathcal{F}]^{\alpha}_{n}$, and let $F\colon\ [\kappa]^{\alpha}\to n$ be a partition with no homogeneous set. For any $X\subseteq\kappa$, let F_X be the partition on $[\kappa]^{\alpha}$:

$$F_X(p) = \begin{cases} n-1 & \text{if } F(p) = n-1 \text{ and } p(0)\notin X \\ n & \text{if } F(p) = n-1 \text{ and } p(0)\in X \\ F(p) & \text{otherwise.} \end{cases}$$

Since $\kappa\to[\mathcal{F}]^{\alpha}_{n+1}$, there is a set $Y\subseteq\kappa$ homogeneous for F_X. Since $\|F''[Y]^{\alpha}\|=n$, exactly one of $\{n-1,n\}$ fails to be in $F_X''[Y]^{\alpha}$. Furthermore, if Y' is any set in $\mathcal{F}$, $Y\cap Y'$ is also homogeneous, so that $n\in F_X''[Y]^{\alpha}$ iff $n\in F_X''[Y']^{\alpha}$. Thus we may define: $\mathcal{F}^* = \{X\,|$ for any or all $Y\in\mathcal{F}$, $n\in F''_X[Y]^{\alpha}\}$.
By construction, this is an ultrafilter. That $\mathcal{F}^*$ extends $\mathcal{F}$ follows from the observation that $n\in F_Y''[Y]^{\alpha}$. Since $\mathcal{F}$ is uniform, it is easy to see that $\kappa-\delta$ $\mathcal{F}^*$ for all $\delta<\kappa$. Thus sets in $\mathcal{F}^*$ are unbounded and hence size κ by regularity.

Finally, suppose δ is the additivity of $\mathcal{F}$, and $\{X_\alpha\}_{\alpha<\eta}$ are members of $\mathcal{F}^*$, $\eta<\delta$. For each α, let $Y_\alpha \in \mathcal{F}$ be homogeneous for F_{X_α}, and let $Y = \bigcap_{\alpha<\eta} Y_\alpha$. It follows that Y is homogeneous for F_X, $X = \bigcap_{\alpha<\eta} X_\alpha$ and $n \in F_X''[Y]^\alpha$, so that $X \in \mathcal{F}^*$. □

For specific n and α, we can say more:

<u>Theorem 1.2</u> Let κ be a regular uncountable cardinal and $\mathcal{F}$ a filter satisfying $\kappa \to [\mathcal{F}]^2_3$. Then κ is a measurable cardinal.

Proof: If $\kappa \to [\mathcal{F}]^2_2$ then $\mathcal{F}$ is a κ-additive ultrafilter, as we remarked earlier. If $\kappa \not\to [\mathcal{F}]^2_2$, let $F: [\kappa]^2 \to 2$ be a partition without a homogeneous set in $\mathcal{F}$. Form the ultrafilter $\mathcal{F}^*$ as in 1.1, and suppose its additivity is $\delta<\kappa$. Let $g: \kappa \to \delta$ be a function such that for all $\alpha<\delta$, $g^{-1}(\alpha) \notin \mathcal{F}^*$. We define a partition G by

$$G(\alpha,\beta) = \begin{cases} 0 & \text{if } g(\alpha) = g(\beta) \\ 1 & \text{if } g(\alpha) < g(\beta) \\ 2 & \text{if } g(\alpha) > g(\beta) \end{cases}$$

Let $X \in \mathcal{F}$ be homogeneous for G, i.e., $F''[X]^2 \neq 3$. Since $\mathcal{F}$ is uniform, 0 must be in $G''[X]^2$. 1 must also be in $G''[X]^2$, since otherwise

$$X \subset \bigcup_{\alpha\leq\beta} g^{-1}(\alpha) \quad \text{where } \beta = g(\cap X).$$

By δ-additivity, this would imply $g^{-1}(\alpha) \in \mathcal{F}^*$. for some α.

These facts together compel $2 \notin G''[X]^2$, but then by regularity, there must be η,γ such that for $\alpha,\beta \in X-\eta$, $\gamma = g(\alpha) = g(\beta)$. Since $X-\eta \in \mathcal{F}^*$, and $X-\eta \subseteq g^{-1}(\gamma)$, $g^{-1}(\gamma) \in \mathcal{F}^*$, a contradiction. □

That κ be regular in 1.2 is essential, as the following example attests: Let $\langle \kappa_n | n \in \omega \rangle$ be an increasing sequence of measurable cardinals, with U_n a normal κ_n-additive ultrafilter over κ_n. Let U be a Ramsey ultrafilter over ω, i.e. $\omega \to [U]^2_2$. Such U exist under a variety of assumptions, e.g. Martin's Axiom. Let $\lambda = \sup \kappa_n$; and define a uniform ultrafilter V over λ by:

$$X \in V \text{ \underline{iff} } X \subseteq \lambda \ \& \ \{n \mid X \cap \kappa_n \in U_n\} \in U.$$

λ is singular, yet we can show that $\lambda \to [V]^2_3$: Suppose that $F: [\lambda]^2 \to 3$. Using the κ_n-additivity of U_n and the fact that $\kappa_n \to [U_n]^2_2$ by a well-known result of Rowbottom about normal ultrafilters, it is not difficult to find sets $X_n \in U_n$ and $i_n < 3$ for every $n \in \omega$, and a function $f: [\omega]^2 \to 3$ such that:

(a) $F''[X_n] = \{i_n\}$,

(b) $X_{n+1} \subseteq (\kappa_{n+1} - \kappa_n)$, and

(c) if $\alpha \in X_m$ and $\beta \in X_n$ and $m < n$, then $F(\alpha,\beta) = f(m,n)$.

Finally, we can find a $Y \in U$ and a fixed $i < 3$ such that $n \in Y$ implies $i_n = i$, and using the Ramseyness of U, a fixed $j < 3$ such that $f''[Y] = \{j\}$. Thus, if $Z = \bigcup_{n \in Y} X_n$, then $Z \in V$ and $F''[Z]^2 = \{i,j\}$.

Theorem 1.3 Let $\mathcal{F}, \kappa$ regular satisfy $\kappa \to [\mathcal{F}]^2_n$. Then κ satisfies $\kappa \to [\kappa]^2_{n-2}$.

Proof: Let $H: [\kappa]^2 \to n-2$ be any partition. We can assume that κ is not measurable. Thus, if we expand $\mathcal{F}$ to $\mathcal{F}^*$, then we can suppose that its additivity is some $\delta < \kappa$, and define g and G as in the proof of 1.2. Let G_1 map $[\kappa]^2$ into $3 \times (n-2)$ by:

$$G_1(\alpha,\beta) = (G(\alpha,\beta),\ H(\alpha,\beta))$$

Since $\kappa \to [\mathcal{F}]^2_n$ implies $\kappa \to [\mathcal{F}]^2_{3(n-2),n}$, let $X \in \mathcal{F}$ be such that $\| G''[X]^2 \| \le n-1$. It follows that $\| \{k \mid (0,k) \in G_1''[X]^2\} \| \le n-3$, since for some a, b, $(1,a)$ and $(2,b)$ are in $G_1''[X]^2$ as above. Thus if we choose α such that $Y = X \cap g^{-1}(\alpha)$ is size κ, then $\| H''[Y]^2 \| \le n-3$. $\square$

Corollary 1.4 If κ is a regular uncountable cardinal with a filter $\mathcal{F}$ satisfying $\kappa \to [\mathcal{F}]^2_4$, then κ is a weakly compact cardinal.

In a similar manner to the above, one may prove for κ regular that the following imply measurability:

$\kappa\to[\not{F}]^3_{13}$ $\kappa\to[\not{F}]^4_{75}$ ... $\kappa\to[\not{F}]^n_{h(n)}$, ...

and in general:

$$\kappa\to[\not{F}]^n_k \quad \text{implies} \quad \kappa\to[\kappa]_{\kappa+1-h(n)}$$

where $h(n)$ is the number of different ways n objects may be prewellordered. The chief open question here is: does $\kappa\to[\not{F}]^2_4$ imply measurability for regular κ? An example can be found of a κ, and an ultrafilter such that $\kappa\to[\not{F}]^2_4$, but $\kappa\not\to[\not{F}]^2_3$. In the example, however, κ is measurable, and $\not{F}$ is formed by gluing together a quantity of normal measures. In §3, we shall establish that $\kappa\to[\not{F}]^2_n$ for an $n\leq\omega$ implies that there is an inner model with a measurable cardinal.

We now consider infinite subscripts.

<u>Theorem 1.5</u> Given $\kappa,\alpha,\not{F}$ satisfying $\kappa\to[\not{F}]^\alpha_\omega$, then $\kappa\to[\not{F}]^\alpha_n$ for some n.

<u>Proof</u>: Suppose the theorem is false, and for each n, F_n is a partition with no homogeneous set in $\not{F}$. Call a set $T\subseteq[\kappa]^\alpha$ <u>unavoidable</u> if $T\cap[X]^\alpha\neq\emptyset$ for all $X\in\not{F}$. Call $S\subseteq[\kappa]^\alpha$ <u>good</u> if for all $k<\omega$, there is $n<\omega$ such that at least k of the sets

$$S\cap F_n^{-1}(0),\ S\cap F_n^{-1}(1),\dots,\ S\cap F_n^{-1}(n-1)$$

are unavoidable. Note that if S is good and S is partitioned into m pieces, $S=\bigcup_{j<m}S_j$, one of these pieces must be good, for if not, let k be such that for all $n<\omega$ and $j<m$, fewer than k of the sets $\{S_j\cap F_n^{-1}(i)\}_{j<m,\ i<n}$ are unavoidable. Choose n so that at least mk of the sets $\{S\cap F_n^{-1}(i)\}_{i<n}$ are unavoidable. Then mk of the sets $\{S_j\cap F_n^{-1}(i)\}_{j<m,i<n}$ are unavoidable, hence for some $j<m$, k of the sets $\{S_j\cap F_n^{-1}(i)\}_{i<n}$ are unavoidable.

We now construct disjoint unavoidable sets $\{T_i\}_{i<\omega}$ and good sets

$S_0 \subseteq S_1 \subseteq S_2 \ldots$ such that $S_i \cap T_j = \emptyset$ for $j \leq i$ as follows: The sets $F_2^{-1}(0)$ and $F_2^{-1}(1)$ are both unavoidable and at least one is good. Let S_0 be the one which is good, T_0 the other.

Given T_i, S_i choose n so that at least two of the sets:

$\{S_i \cap F_n^{-1}(j)\}_{j<n}$ are unavoidable.

Let S_{i+1} be one which is good, and T_{i+1} another which is unavoidable. Finally, we define that partition $F:[\kappa]^\alpha \to \omega$ by:

$$F(p) = \begin{cases} n & \text{if } p \in T_n \\ 0 & \text{otherwise} \end{cases}$$

By construction, $F^{-1}(n) \supseteq T_n$ is unavoidable for each $n<\omega$, violating $\kappa \to [\mathcal{F}]^\alpha_\omega$. □

<u>Theorem 1.6</u> If there exist $\kappa, \alpha, \mathcal{F}$ which satisfy $\kappa \to [\mathcal{F}]^\alpha_{\omega,<\omega}$, then there exists a measurable cardinal.

Proof: By 1.5 and 1.1, $\mathcal{F}$ can be extended to an ultrafilter $\mathcal{F}^*$. The stronger relation here shows that $\mathcal{F}^*$ is $\aleph_1$-additive since if

$\bigcup_{n \in \omega} A_n = \kappa$ is a partition of κ into ω parts, let $X \in \mathcal{F}^*$ be homogeneous for the partition:

$F(p) =$ that n such that $p(0) \in A_n$.

Since $F''[X]^\alpha$ is finite, there is some n_0 such that $X \subseteq \bigcup_{n \leq n_0} A_n$, and so some $A_n \in \mathcal{F}^*$

Thus the additivity of $\mathcal{F}^*$ is greater than ω, and by the usual arguments there is a measurable cardinal. □

The last result of this section is applicable only to situations with limited amounts of Choice.

<u>Theorem 1.7</u> (DC) If $\kappa, \alpha, \mathcal{F}$ satisfy $\kappa \to [\mathcal{F}]^\alpha_\omega$, then either $\kappa \to [\mathcal{F}]^\alpha_{\omega,<\omega}$ or there is a non-principal ultrafilter on ω.

Proof: Suppose $\kappa \not\to [\mathcal{F}]^\alpha_{\omega,<\omega}$ and let $F:[\kappa]^\alpha \to \omega$ witness this fact.

Altering our notation somewhat, call a set $B \subseteq \omega$ avoidable if $F''[X]^\alpha B=\emptyset$ for some $X \in \mathcal{F}$, unavoidable otherwise.

Let $B = \{n | \{n\}$ is avoidable$\}$. The set $\omega - B$ cannot be finite, for then the partition:

$$G(p) = \begin{cases} n_0 & \text{if } F(p) \in B \\ F(p) & \text{otherwise} \end{cases}$$

(where $n_0 \in \omega - B$) would contradict $\kappa \to [\mathcal{F}]^\alpha_\omega$. Similarly, B itself is not avoidable, for if $F''[X]^\alpha \cap B=\emptyset$ for some $X \in \mathcal{F}$, then $F:[X]^\alpha \to \omega - B$ would again contradict $\kappa \to [\mathcal{F}]^\alpha_\omega$.

Claim: There is an infinite subset $D \subseteq B$, unavoidable, such that D cannot be split into two unavoidable subsets. If not, split B itself into two unavoidable sets D_1 and A_1. Next split D_1 into two unavoidable sets D_2 and A_2 and continue this procedure which results in an infinite disjoint collection $\{A_n\}_{n<\omega}$ of unavoidable sets. Then the partition:

$$G(p) = \begin{cases} n \text{ if } F(p) \in A_n \\ 0 \text{ otherwise} \end{cases}$$

contradicts $\kappa \to [\mathcal{F}]^\alpha_\omega$, establishing the claim.

Finally, let $D \subseteq B$ be as claimed. Let U be the collection of unavoidable subsets of D. That U is a non-principal ultrafilter on D is straight-forward. We show for example, that D is ω-additive. If $S, T \in U$ then $S \cap T$ must be unavoidable, otherwise $S-T$ and $T-S$ would be disjoint unavoidable subsets of D, a contradiction. $\square$

§2 Jonsson and Rowbottom Filters

In [14] E. M. Kleinberg proved that Jonsson and Rowbottom cardinals are almost the same. We prove here a similar result for filters.

It is stated without proof in [7] (see 6.10, 6.11), where connections with Prikry forcing and the structure theory of ultrafilters over a measurable cardinal are made. Specifically,

Theorem 2.1 Let κ be the least cardinal with a Jonsson filter. Then any Jonsson filter on κ is a δ-Rowbottom filter for some $\delta<\kappa$.

Proof Let κ be as stated, $\mathcal{F}$ a Jonsson filter.

We first claim that κ, $\mathcal{F}$ satisfy $\kappa\to[\mathcal{F}]^{<\omega}_{\delta}$ for some $\delta<\kappa$. If not, then for each $\delta<\kappa$, let $F_\delta:[\kappa]^{<\omega}\to\delta$ be a partition such that for all $X\in$, $F''_\delta[X]^{<\omega}=\delta$. Define $F:[\kappa]^{<\omega}\to\kappa$ by:

$$F(\alpha_1,\alpha_2,\ \ldots,\alpha_n) = \begin{cases} F_{\alpha_1}(\alpha_2,\ \ldots,\ \alpha_n) & \text{if } n>1, \\ 0 & \text{otherwise} \end{cases}$$

No $X\in\mathcal{F}$ can be homogeneous for F, since for $\gamma<\kappa$ we can choose $\delta>\gamma$, $\delta\in X$ and ordinals $\alpha_1,\ldots,\alpha_j\in X$, $\delta<\alpha_1$ such that $F_\delta(\alpha_1,\ldots,\alpha_j) = \gamma$. This contradicts $\kappa\to[\mathcal{F}]^{<\omega}_{\kappa}$.

Let δ be least such that $\kappa\to[\mathcal{F}]^{<\omega}_{\delta}$ We next claim that $\kappa\to[\mathcal{F}]^{<\omega}_{\delta\ <\delta}$. Let $G:[\kappa]^{<\omega}\to\delta$ be any partition. Since δ is least, again choose $F_\alpha:[\kappa]^{<\omega}\to\alpha$ for each $\alpha<\delta$ such that $F_\alpha''[X]^{<\omega}\ \alpha$, for all $X\in\mathcal{F}$. Define $H:[\kappa]^{<\omega}\to\delta$ by:

$$H(\alpha_1,\ \ldots,\ \alpha_n) = \begin{cases} F_{G(\alpha_1,\ \ldots,\ \alpha_k)}(\alpha_{k+1},\ \ldots,\ \alpha_{k+m}) \\ \qquad\qquad \text{if } n = 2^k3^m > 6 \\ 0 \quad \text{otherwise} \end{cases}$$

Let X be homogeneous for H. Then $\|G''[X]^{<\omega}\|<\delta$, otherwise, if $\eta<\delta$ there would be $\beta_1,\ldots,\beta_k\in X$ such that $G(\beta_1,\ldots,\beta_k)>\delta$ and then $\gamma_1,\ldots,\gamma_m\in X$ such that $F_{G(\beta_1,\ldots,\beta_k)}(\gamma_1,\ldots,\gamma_m) = \eta$ and by expanding the set $\beta_1,\ldots\beta_k,\gamma_1,\ldots\gamma_m$ one could show that $\eta\in H''[X]^{<\omega}$, contradicting $\kappa\to[\mathcal{F}]^{<\omega}_{\delta}$.

Lastly, we claim that $\mathcal{F}$ is a δ-Rowbottom filter, i.e., $\kappa\to[\mathcal{F}]^{<\omega}_{\lambda,<\delta}$ for all $\lambda,\delta\le\lambda<\kappa$. For this it is sufficient to show

$\kappa\to[\mathcal{F}]^{<\omega}_{\lambda,<\lambda}$ for all such λ, since given $F:[\kappa]^{<\omega}\to\lambda$, we can find successive sets $X_1\supseteq X_2\supseteq\ldots\supseteq X_k$ with $\|F''[X_i]^{<\omega}\|$ a decreasing sequence of ordinals until $\|F''X_k]^{<\omega}\|<\delta$

Given, $\delta<\lambda<\kappa$ suppose that $F:[\kappa]^{<\omega}\to\lambda$ is a partition such that $\|F''[X]^{<\omega}\|=\lambda$ for all $X\in\mathcal{F}$. Let $\mathcal{F}_\lambda\subseteq 2^\lambda$ be the filter defined by: $A\in\mathcal{F}_\lambda$ if for some $X\in\mathcal{F}$, $F''[X]^{<\omega}\subseteq A$. $\mathcal{F}_\lambda$ is easily seen to be a filter on λ. We will show that $\mathcal{F}_\lambda$ is actually a Jonsson filter, contradicting the leastness of κ. This will prove the theorem. Suppose $G:[\kappa]^{<\omega}\to\lambda$. Since $\kappa\to[\mathcal{F}]^{<\omega}_\delta$ and hence $\kappa\to[\mathcal{F}]^{<\omega}_\lambda$, we define a partition $H:[\kappa]^{<\omega}\to\lambda$ such that whenever $X\in\mathcal{F}$ and $H''[X]^{<\omega}\neq\lambda$, then $G''[F''[X]^{<\omega}]^{<\omega}\neq\lambda$. Since $F''[X]^{<\omega}\in\mathcal{F}_\lambda$, $\mathcal{F}_\lambda$ will be a Jonsson filter. Specifically,

$$H(\alpha_1,\ldots,{}_n)=\begin{cases} G(F(\alpha_{i_{1,1}},\ldots,\alpha_{i_{1,k}}),\ F(\alpha_{i_{2,1}},\ldots,\alpha_{i_{2,k_2}}),\ \ldots,F(\alpha_{i_{m,1}},\ldots,{}_{i_{m,k_m}})) \\ \quad \text{if } n=2^m p_1^{n} p_2^{n}\ldots p_m^{n_m} \text{ and} \\ \quad \text{for } j\le m,\ n_j=2^{k_j}p_1^{i_{j,1}}p_2^{i_{j,2}}\ldots p_{k_j}^{i_{j,k_j}} \text{ (prime factorizations)} \\ 0 \text{ otherwise} \end{cases}$$

H has the required property. □

§3 Inner Models of Measurability

Turning from direct combinatorial consequences to consistency strength, in this section we shall establish that, in most cases having $\kappa\to[\mathcal{F}]^2_n$ for some $n\le\omega$ implies the existence of an inner model of ZFC with a measurable cardinal. We also establish that the same conclusion can be drawn from the existence of a Jonsson filter. The following simple observation has its own intrinsic interest.

<u>Theorem 3.1</u> If U is an ultrafilter over κ such that $\kappa\to[U]^2_{n+1}$ and V is a uniform ultrafilter over λ such that $V<_{RK}U$, then $\lambda\to[V]^2_n$. Hence, there is no RK descending chain of length n starting with any U satisfying $\kappa\to[U]^2_n$.

Proof: Let $f_*(U) = V$, and suppose that $F:[\lambda]^2 \to n$.

Define $G: [\kappa]^2 \to n+1$ by:

$$G(\alpha, \beta) = \begin{cases} F(f(\alpha), f(\beta)) & \text{if } f(\alpha) \neq f(\beta) \\ n & \text{otherwise.} \end{cases}$$

By hypothesis, there is an $X \in U$ and an $i < n+1$ such that $i \notin G''[X]^2$.

But $i \neq n$, else f would be one-to-one on X, contradicting $V <_{RK} U$.

Hence, if $Y = f''X \in V$, we have $F''[Y]^2 \neq n$.

The second sentence of the theorem follows from the fact that $\kappa \to [U]^2_2$ iff U is RK-minimal.

Next, we compile several facts from which our main result will follow. We shall only provide a proof for the first, which is an instance of the model-theoretic universality of regular ultrapowers (see [3] p. 207), and references for the rest.

Theorem 3.2. If U is an $(\omega, 2^\lambda)$-regular ultrafilter over κ and V is any ultrafilter over λ, then $V \leq_{RK} U$.

Proof: Let $\{A_X | X \subseteq \lambda\}$ be a $(\omega, 2^\lambda)$-regularizing family for U, indexed by subsets of λ. Define $f:\kappa \to \lambda$ by choosing

$$f(\alpha) \in \cap\{X | X \in V \ \& \ \alpha \in A_X\}.$$

This is possible, since this last set is in V. It is not difficult to see that $f_*(U) = V$. □

Theorem 3.3 (Kunen-Prikry [15]) If an ultrafilter is λ^+-decomposable and λ is regular, then the ultrafilter is λ-decomposable.

Theorem 3.4

(a) (Kanamori [8]) If U is a uniform non-(κ, κ^+)-regular ultrafilter over κ^+, then there is such an ultrafilter $\leq_{RK}$ U which is also weakly normal.

(b) (Kanamori [8] and Ketonen [9] independently)

If U is a uniform ultrafilter over a regular κ which is not $(\omega\ \lambda)$-regular for some $\lambda<\kappa$, then there is such an ultrafilter $\leq_{RK}U$ which is also weakly normal.

Theorem 3.5

(a) (Jensen [5]) Suppose that $\kappa^{<\kappa}=\kappa$ and there is a uniform weakly normal ultrafilter over κ. Then there is an inner model with a measurable cardinal.

(b) (Koppelberg for regular κ[5]; Donder (unpublished) for singular κ). Suppose that there is a uniform ultrafilter over κ which is λ-indecomposable for some regular $\lambda<\kappa$. Then there is an inner model with a measurable cardinal.

The main result is at hand:

Theorem 3.6: Suppose that there are κ and $\mathcal{F}$ satisfying $\kappa\to[\mathcal{F}]^2_n$ for some $n\leq\omega$ where either

(i) $\kappa\geq\omega_{n-3}$ if $n\geq 4$, or else

(ii) The GCH holds and $\kappa>\omega$.

Then there is an inner model with a measurable cardinal.

Proof: By 1.2 and 1.5 we can suppose that $3<n<\omega$ and by 1.1 we can replace $\mathcal{F}$ by an ultrafilter U. For case (i), it suffices by 3.5(b) to establish that U is not ω_{n-3}-decomposable. So, suppose it were. By repeated applications of 3.1 and 3.3 there would be an RK-descending chain $U\geq_{RK}V_1>_{RK}V_2>_{RK}\ldots$, where V_i is uniform over ω_{n-2-i} and $\omega_{n-2-i}\to[V_i]^2_{n+1-i}$. But when $i=n-3$, we would be confronted with $\omega_1\to[V_{n-3}]^2_4$, contradicting 1.4.

For case (ii), note first that U cannot be $(\omega,2^\omega)$-regular, else by 3.2 there would be too many ultrafilters below U in the RK order.

For example, if V and W are ultrafilters over I and J respectively, define $V \times W$ over $I \times J$ by

$$X \in V \times W \text{ iff } \{i \mid \{j \mid \langle i, j \rangle \in X\} \in W\} \in V.$$

Then set $V^2 = V \times V$ and $V^{k+1} = V^k \times V$. When V is a non-principal ultrafilter over ω, the projection to the first coordinate verifies that $V^\kappa <_{RK} V^{\kappa+1}$, and thus $V^n \leq_{RK} U$ would contradict 3.1

Next, by CH if $\kappa = \omega_1$ we can assume by 3.4(a) that U is weakly normal. If $\kappa > \omega_1$, we can first assume that $\kappa < \omega_\omega$ by case (i) and thus that κ is regular. By CH and 3.4(b), we can again assume that U is weakly normal.

Finally, by 3.5(a) we can conclude that there is an inner model with a measurable cardinal for case (ii) as well. $\square$

We can draw the same conclusion for Jonsson filters:

<u>Theorem 3.7</u>: If there is a Jonsson filter, then there is an inner model with a measurable cardinal.

<u>Proof</u>: Let $\kappa \to [\mathcal{F}]^{<\omega}_{\delta}$ be as in the second paragraph of the proof of 2.1. It is not difficult to see that δ must be regular: If not, there is a $G:[\kappa]^{<\omega} \to \operatorname{cf}(\delta)$ such that $G''[X]^{<\omega} = \operatorname{cf}(\delta)$ for any $X \in \mathcal{F}$, and proceed just as in the proof of $\kappa \not\to [\mathcal{F}]^{<\omega}_{\delta,<\delta}$ to get a contradiction.

Secondly, by the argument of 1.6 using a partition $\bigcup_{\alpha<\delta} A_\alpha = \kappa$, we can show that any ultrafilter U extending $\mathcal{F}$ must be δ-indecomposable.

These facts together with 3.5(b) imply that there is an inner model with a measurable cardinal. $\square$

This result contrasts with a result of Mitchell [16] that a Jonsson cardinal is Jonsson (in fact, Ramsey) in K, the core model of Dodd and Jensen. Since there are no inner models of K with a measurable cardinal, the consistency strength of having a Jonsson filter is strictly stronger than that of merely having a Jonsson cardinal.

§4 Applications

In [10], E. M. Kleinberg proved that the partition relation $\kappa\to(\kappa)^{\omega}$ implies that the filter U_{ω} generated by the ω-closed, unbounded sets is an ultrafilter on κ. If AC_{ω} is also assumed, the additivity of U_{ω} is at least $\aleph_1$, and is a measurable cardinal. With our techniques, we can improve this.

__Theorem 4.1__ AC_{ω} + $\kappa\to[\kappa]^{\omega}_{\omega}$ imply there is a measurable cardinal.

Proof: The relation $\kappa\to[\kappa]^{\omega}_{\omega}$ in effect implies $\kappa\to[U_{\omega}]^{1}_{\omega}$ since if $F:[\kappa]^1\to\omega$, let $F^*:[\kappa]^{\omega}\to\omega$ be the partition: $F^*(p)=F(\cup p)$, then if $X\subseteq\kappa$ is homogeneous for F^*, then $(X)_{\omega}$, the collection of ω-sups from X is closed and unbounded, and is homogeneous for F as well.

By 1.5, $\kappa\to[U_{\omega}]^{1}_{n}$ for some n, and by 1.1, U_{ω} can be extended to an ultrafilter U^*_{ω} (AC_{ω} is sufficient for these purposes.) Finally, AC_{ω} implies this additivity is at least $\aleph_1$, and hence a measurable cardinal. $\square$

For any regular cardinal λ, let U_{λ} be the filter generated by the λ-closed, unbounded sets, Well-ordered choice of length κ implies that U_{λ} is not an ultrafilter (see [11]).

In applying the theorems of §1 here, our advantage is the natural additivity of U_{λ}.

__Theorem 4.2__ (AC_{β}) $\kappa\to[U_{\lambda}]^{\alpha}_{\beta}$ α, $\lambda,\beta<\kappa$ implies $\kappa\to[U_{\lambda}]^{\alpha}_{\beta,<\beta}$

Proof: Let $F:[\kappa]^{\alpha}\to\beta$ be any partition. In the terminology of 1.5, let $B = \{\delta|\{\delta\}$ is avoidable$\}$. For each $\delta\in B$, choose $X_{\delta}\in U_{\lambda}$ such that $\delta\notin F''[X_{\delta}]^{\alpha}$. Since each X_{δ} contains a λ-closed, unbounded subset, we may assume X_δ is itself λ-closed and unbounded. By usual arguments, $X = \bigcap_{\delta\in B} X_\delta$ is λ-closed and unbounded, and $F''[X]^\alpha \cap B=\emptyset$. It follows that $\|\beta-B\|<\beta$(otherwise $F\restriction[X]^\alpha$ would contradict $\kappa\to[U_\lambda]^\alpha_\beta$) and so $\|F''[X]^\alpha\|<\beta$. □

<u>Theorem 4.3</u> $(AC_\omega)\kappa\to[U_\lambda]^\alpha_\omega$,α λ<κ implies $\kappa\to[U_\lambda]^\alpha_{\omega,k}$ for some $k<\omega$.

Proof: By 1.1 and 4.2, $\kappa\to[U_\lambda]^\alpha_{\omega,<\omega}$ and $\kappa\to[U_\lambda]^\alpha_n$ for some n. Together these imply the theorem, for given F, we can find $X\in U_\lambda$ such that $\|F''[X]^\alpha\|<\omega$, and hence, by repeated applications of $\kappa\to[U_\lambda]^\alpha_k$ to suitable partitions can reduce the size of the range further to a set of size less than n. □

In [12], E. M. Kleinberg proved as a consequence of the Axiom of Determinateness (AD) and Dependent Choice (DC) that $\aleph_n$ is a Jonsson cardinal for all $n<\omega$, and that $\aleph_\omega$ is Rowbottom. In [6], J. M. Henle proved that the $\{\aleph_n\}_{n<\omega}$ each had a Jonsson filter. The question remained open whether or not $\aleph_\omega$ had a Rowbottom filter. This question will not be answered here, but we do exhibit a partition relation satisfied by $\aleph_\omega$ for which $\aleph_\omega$ has no filter.

<u>Theorem 4.4</u> AD+DC+ZF $\vdash$"$\aleph_\omega\to[\aleph_\omega]^2_\omega$ but there is no filter $\mathcal{F}$ on $\aleph_\omega$ satisfying $\aleph_\omega\to[\mathcal{F}]^2_\omega$".

Proof: The second part of this theorem follows from the work of §1. By 1.7 either $\aleph_\omega\to[\mathcal{F}]^{<\omega}_{\omega,<\omega}$ or there is a non-principal ultrafilter on ω. The first relation is demonstrated false by the partition, $F:[\aleph_\omega]^2\to\omega$ defined by $F(\alpha,\beta)$ = the least n such that $\alpha<\aleph_n$. That the second is false is a well-known consequence of AD.

To prove the first part of the theorem, we use Kleinberg's methods of [13]. Let $F:[\aleph_\omega]^2\to\omega$ be any partition. Theorem 6.4 of [5] proves that if κ satisfies $\kappa\to(\kappa)^\kappa$, then a related cardinal κ_ω is Rowbottom. In the case of AD, κ is $\aleph_1$ and κ_ω is $\aleph_\omega$. The proof deals with partitions of $[\aleph_\omega]^{<\omega}$ into γ pieces, $\gamma < \aleph_\omega$. We consider here only the simpler problem of the partition F. The proof proceeds by finding subsets D_n of κ_n (for AD, $\aleph_n$) each of size κ_{n-1} such that (for our case) $\| F:[D_n]^2 \| = 1$ for all $n<\omega$ and $\| F''[D_n x D_m] \| = 1$ for all $n,m<\omega$.

Let $g:\omega\to\omega$, $h:[\omega]^2\to 2$ be defined by:

$$g(n) = \begin{cases} 0 \text{ if } F''[D_n]^2 \text{ is a multiple of } 3 \\ 1 \text{ otherwise} \end{cases}$$

$$\text{and} \quad h(n,m) = \begin{cases} 0 \text{ if } F''[D_n x D_m] \text{ is even} \\ 1 \text{ otherwise} \end{cases}$$

By Ramsey's theorem, let A be an infinite subset of ω such that h is constant on $[A]^2$. Let B be an infinite subset of A such that g is constant on B. It follows that $E = \bigcup_{n\in B} D_n$ is our desired homogeneous set for F. It has cardinality $\aleph_\omega$, and $F:[E]^2$ will be disjoint from one of the four sets: even multiples of 3, odd multiples of 3, even non-multiples of 3, and odd non-multiples of 3. □

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