From replica Bethe ansatz solution of KPZ growth to non-crossing directed polymers

P. Le Doussal (LPTENS)

with : Pasquale Calabrese (Univ. Pise, SISSA)

Alberto Rosso (LPTMS Orsay)

Thomas Gueudre (LPTENS,Torino)

most recent: Andrea de Luca (LPTENS,Orsay)

- many discrete models in “KPZ class” exhibit universality related to random matrix theory: Tracy Widom distributions: of largest eigenvalue of GUE,GOE..

RBA: method integrable systems (Bethe Ansatz) +disordered systems(replica)

- provides solution directly continuum KPZ eq./DP (at all times)

KPZ eq. is in KPZ class !

- also to discrete models => allowed rigorous replica

Outline:

Part I

- KPZ equation, KPZ class, random matrices,Tracy Widom distributions.
- solving KPZ at any time by mapping to directed paths
  then using (imaginary time) quantum mechanics
  attractive bose gas (integrable) => large time TW distrib. for KPZ height

- droplet initial condition

P. Calabrese, PLD, A. Rosso EPL 90 20002 (2010)

P. Calabrese, M. Kormos, PLD, EPL 10011 (2014)

- flat initial condition

P. Calabrese, PLD, PRL 106 250603 (2011)
J. Stat. Mech. P06001 (2012)

- KPZ in half space

T. Gueudre, PLD, EPL 100 26006 (2012).

Part II

- Non-crossing probability of directed polymers

Andrea De Luca, PLD, arXiv1505.04802.

Phys. Rev. E 92, 040102 (2015)

Generalized Bethe-Ansatz

Macdonald process (Borodin-Corwin)

Kardar Parisi Zhang equation

Phys Rev Lett 56 889 (1986)

growth of an interface of height h(x,t)

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2}(\partial_x h)^2 + \eta(x,t)$$

diffusion

noise

$$\overline{\eta(x,t)\eta(x',t')} = D\delta(x-x')\delta(t-t')$$

- 1D scaling exponents $h \sim t^{1/3} \sim x^{1/2}$ $x \sim t^{2/3}$

- P(h=h(x,t)) non gaussian

even at large time PDF depends on some details of initial condition

related to RMT

flat h(x,0) =0

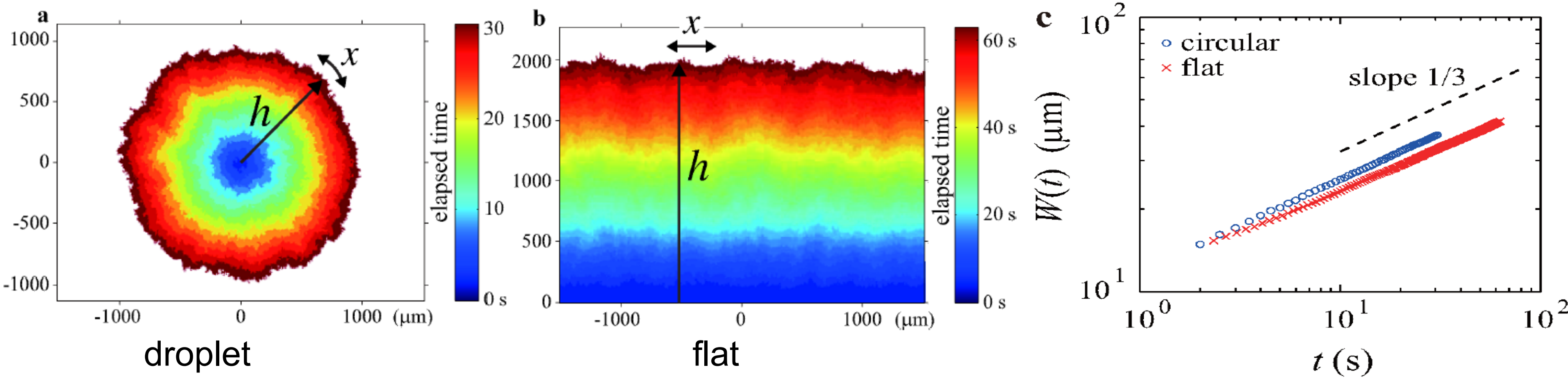
wedge h(x,0) = - w |x|

(droplet)

$\lambda_0 = 0$ Edwards Wilkinson P(h) gaussian

- Turbulent liquid crystals

Takeuchi, Sano PRL 104 230601 (2010)



droplet

flat

$$W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$$

$$h(x,t) \simeq_{t \to +\infty} v_\infty t + \chi \, t^{1/3}$$

χ is a random variable

$$h \sim t^{1/3} \sim x^{1/2}$$

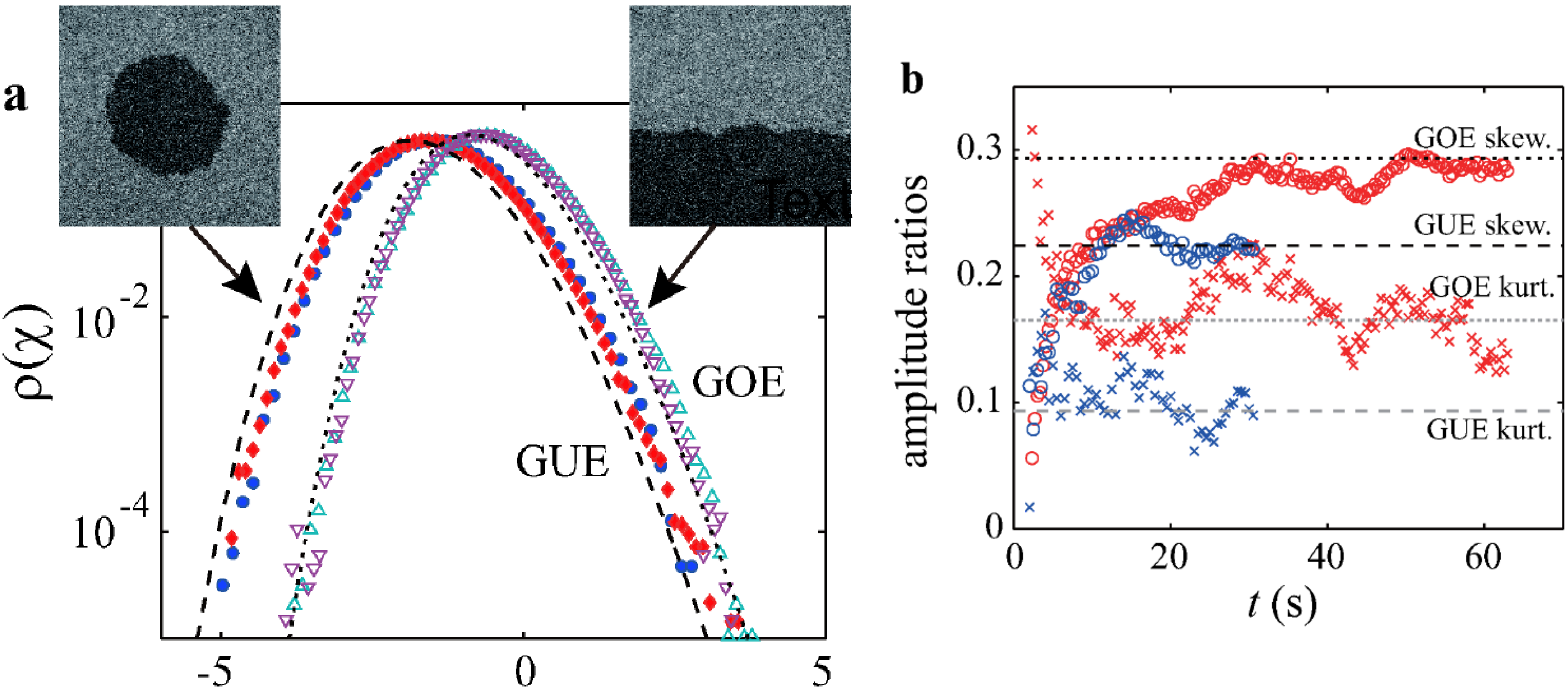
also reported in:

- slow combustion of paper — J. Maunuksela et al. PRL 79 1515 (1997)
- bacterial colony growth — Wakita et al. J. Phys. Soc. Japan. 66, 67 (1996)
- fronts of chemical reactions — S. Atis (2012)
- formation of coffee rings via evaporation — Yunker et al. PRL (2012)

$$h \simeq v_\infty t + (\Gamma t)^{1/3} \chi,$$

skewness =

$$\frac{< (h- < h >)^3 >}{< (h- < h >)^2 >^{3/2}}$$



Large N by N random matrices H, with Gaussian independent entries

eigenvalues λ_i $i = 1, ..N$

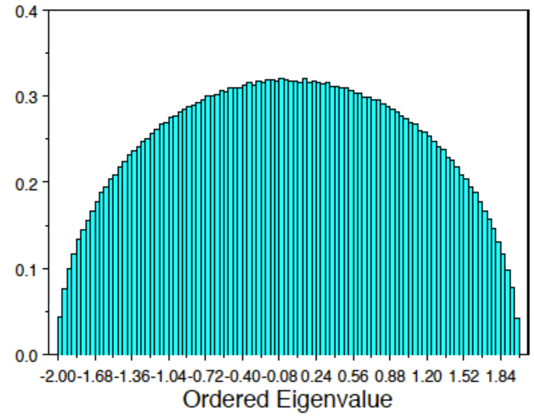
$$P[\lambda] = c_{N,\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta N}{4} \sum_{k=1}^N \lambda_k^2}$$

H is:

$\beta =$
- 1 (GOE) real symmetric
- 2 (GUE) hermitian
- 4 (GSE) symplectic

Universality large N :

- DOS: semi-circle law



histogram of eigenvalues N=25000

- distribution of the largest eigenvalue

$H \to NH$

$$\lambda_{max} = 2N + \chi N^{1/3}$$

$$Prob(\chi < s) = F_\beta(s)$$

Tracy Widom (1994)

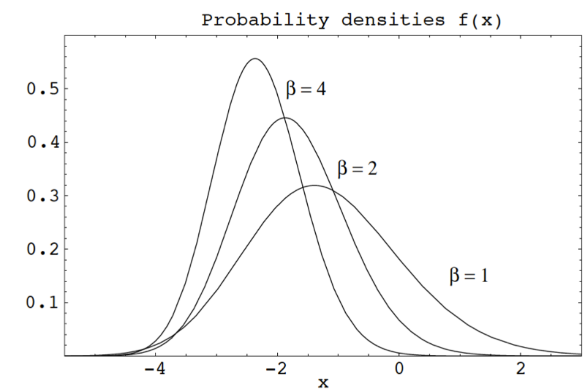
Tracy-Widom distributions (largest eigenvalue of RM)

Fredholm determinants

$$(I-K)\phi(x) = \phi(x) - \int_y K(x,y)\phi(y)$$

GOE $$F_1(s) = Det[I - K_1]$$

$$K_1(x,y) = \theta(x) Ai(x+y+s)\theta(y)$$



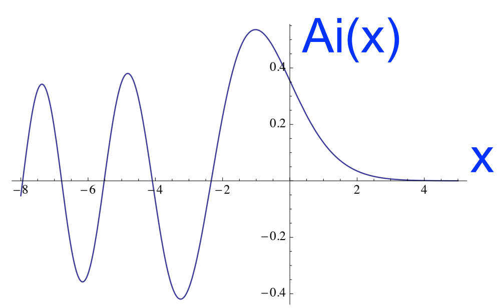
GUE $$F_2(s) = Det[I - K_2]$$

$$K_2(x,y) = K_{Ai}(x+s, y+s)$$

$$K_{Ai}(x,y) = \int_{v>0} Ai(x+v) Ai(y+v)$$

Ai(x-E)

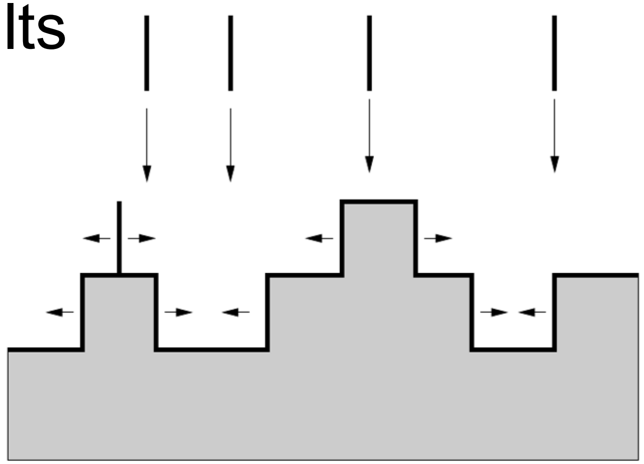
is eigenfunction E
particle linear potential



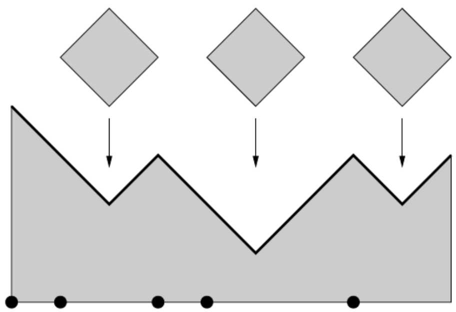
discrete models in KPZ class/exact results

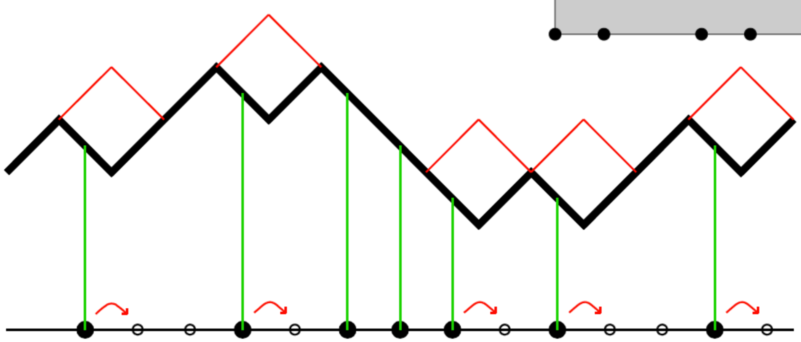
- polynuclear growth model (PNG)

  Prahofer, Spohn, Baik, Rains (2000)

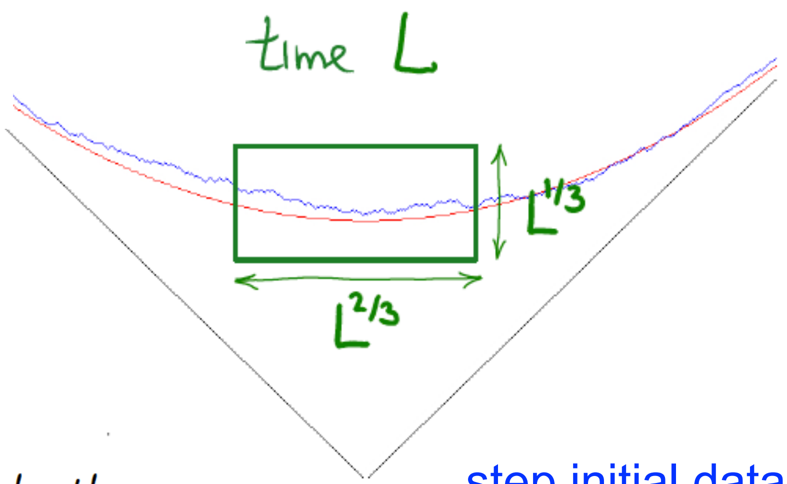


- totally asymmetric exclusion process (TASEP)





Red boxes are added independently at rate 1. Equivalently, particles with no neighbour on the right jump independently with waiting time distributed as exp(−x)dx.



step initial data

Johansson (1999)

Exact results for height distributions for some discrete models in KPZ class

- PNG model

Baik, Deft, Johansson (1999)

$h(0,t) \simeq_{t\to\infty} 2t + t^{1/3}\chi$

droplet IC
GUE

Prahofer, Spohn, Ferrari, Sasamoto,..
(2000+)

flat IC $\chi = \chi_1$ GOE

multi-point correlations
Airy processes

$A_2(y)$ GUE

$A_1(y)$ GOE

$$h(yt^{2/3},t) \simeq_{t\to\infty} 2t - \frac{y^2}{2t} + t^{1/3} A_n(y)$$

- similar results for TASEP

Johansson (1999), ...

Exact results for height distributions for some discrete models in KPZ class

- PNG model

droplet IC

Baik, Deft, Johansson (1999) $h(0,t) \simeq_{t\to\infty} 2t + t^{1/3}\chi$ GUE

Prahofer, Spohn, Ferrari, Sasamoto,..
(2000+)

flat IC $\chi = \chi_1$ GOE

multi-point correlations
Airy processes

$A_2(y)$ GUE

$A_1(y)$ GOE

$$h(yt^{2/3},t) \simeq_{t\to\infty} 2t - \frac{y^2}{2t} + t^{1/3} A_n(y)$$

- similar results for TASEP Johansson (1999), ...

question: is KPZ equation in KPZ class ?

Cole Hopf mapping

KPZ equation ⟷ Continuum
Directed paths (polymers)
in a random potential

↕

Quantum mechanics
of bosons
(imaginary time)

Kardar 87

- **Droplet** (Narrow wedge) KPZ/Continuum DP fixed endpoints

Replica Bethe Ansatz (RBA)

- P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)
- V. Dotsenko, EPL 90 20003 (2010) J Stat Mech P07010
  Dotsenko Klumov P03022 (2010).

Weakly ASEP

- T Sasamoto and H. Spohn PRL 104 230602 (2010)
  Nucl Phys B 834 523 (2010) J Stat Phys 140 209 (2010).
- G.Amir, I.Corwin, J.Quastel Comm.Pure.Appl.Math. 64 466 (2011)

- **Droplet** (Narrow wedge) KPZ/Continuum DP fixed endpoints

Replica Bethe Ansatz (RBA)

- P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)
- V. Dotsenko, EPL 90 20003 (2010) J Stat Mech P07010 Dotsenko Klumov P03022 (2010).

Weakly ASEP

- T Sasamoto and H. Spohn PRL 104 230602 (2010) Nucl Phys B 834 523 (2010) J Stat Phys 140 209 (2010).
- G.Amir, I.Corwin, J.Quastel Comm.Pure.Appl.Math. 64 466 (2011)

- **Flat** KPZ/Continuum DP one free endpoint (RBA)

P. Calabrese, P. Le Doussal, PRL 106 250603 (2011) and J. Stat. Mech. P06001 (2012)

ASEP J. Ortmann, J. Quastel and D. Remenik arXiv1407.8484 and arXiv 1503.05626

- **Droplet** (Narrow wedge) KPZ/Continuum DP fixed endpoints

Replica Bethe Ansatz (RBA)

- P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)
- V. Dotsenko, EPL 90 20003 (2010) J Stat Mech P07010 Dotsenko Klumov P03022 (2010).

Weakly ASEP

- T Sasamoto and H. Spohn PRL 104 230602 (2010) Nucl Phys B 834 523 (2010) J Stat Phys 140 209 (2010).
- G.Amir, I.Corwin, J.Quastel Comm.Pure.Appl.Math. 64 466 (2011)

- **Flat** KPZ/Continuum DP one free endpoint (RBA)

P. Calabrese, P. Le Doussal, PRL 106 250603 (2011) and J. Stat. Mech. P06001 (2012)

ASEP J. Ortmann, J. Quastel and D. Remenik arXiv1407.8484 and arXiv 1503.05626

- **Stationary** KPZ

Cole Hopf mapping

KPZ equation:

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2} (\partial_x h)^2 + \eta(x,t)$$

define:

$$Z(x,t) = e^{\frac{\lambda_0}{2\nu} h(x,t)}$$

$$\lambda_0 h(x,t) = T \ln Z(x,t)$$

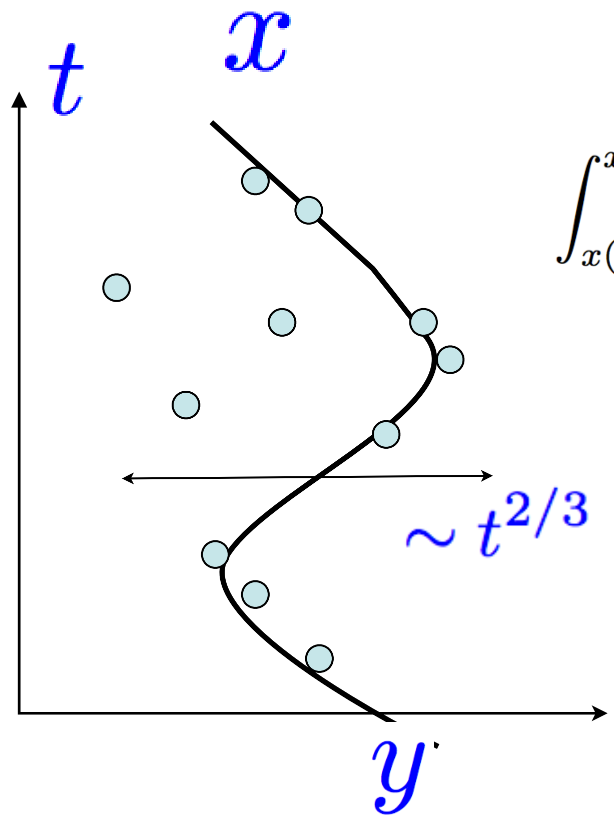
$$T = 2\nu$$

it satisfies:

$$\partial_t Z = \frac{T}{2} \partial_x^2 Z - \frac{V(x,t)}{T} Z$$

$$\lambda_0 \eta(x,t) = -V(x,t)$$

describes directed paths in random potential V(x,t)



$$Z(x,t|y,0) =$$

$$\int_{x(0)=y}^{x(t)=x} Dx(\tau)\, e^{-\frac{1}{T}\int_0^t d\tau \frac{\kappa}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau),\tau)}$$

$$\overline{V(x,t)V(x',t')} = \bar{c}\ \ \delta(t-t')\delta(x-x')$$

Feynman Kac

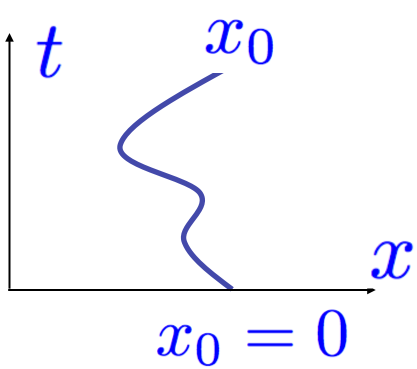
$$Z(x,y,t=0) = \delta(x-y)$$

$$\partial_t Z = \frac{T}{2\kappa}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

initial conditions

$$e^{\frac{\lambda_0}{2\nu}h(x,t)} = \int dy Z(x,t|y,0) e^{\frac{\lambda_0}{2\nu}h(y,t=0)}$$

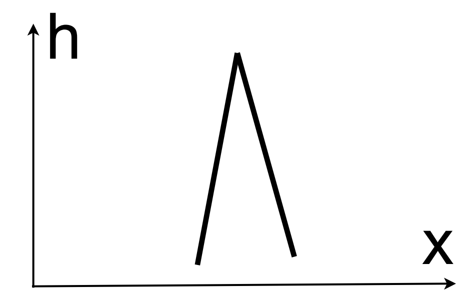
1) DP both fixed endpoints $Z(x_0,t|x_0,0)$



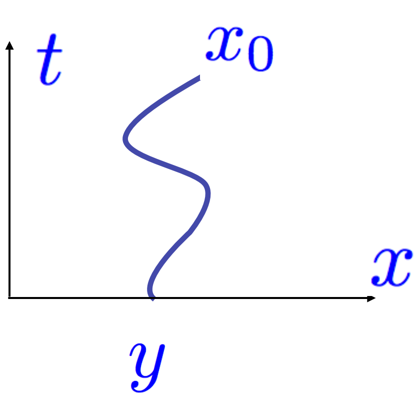
KPZ: narrow wedge <=> droplet initial condition

$$h(x,t=0) = -w|x|$$

$$w \to \infty$$



2) DP one fixed one free endpoint $\int dy Z(x_0,t|y,0)$



KPZ: flat initial condition

$$h(x,t=0) = 0$$

Schematically

$$Z = e^{\frac{\lambda_0 h}{2\nu}}$$

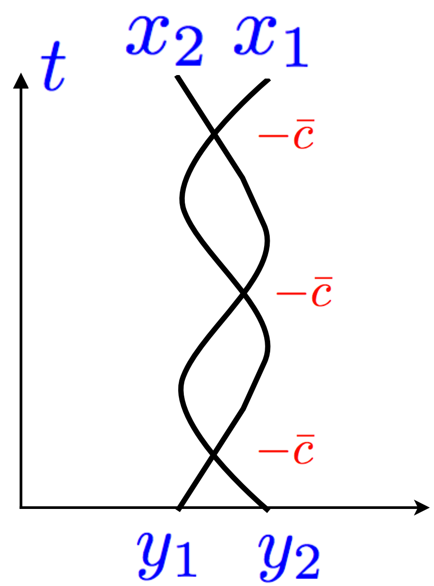
calculate $$\overline{Z^n} = \int dZ\, Z^n P(Z) \qquad n \in \mathbb{N}$$

“guess” the probability distribution from its integer moments:

$$P(Z) \to P(\ln Z) \to P(h)$$

Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1, t|y_1, 0)..Z(x_n, t|y_n 0)} = \langle x_1, ..x_n | e^{-tH_n} | y_1, ..y_n \rangle$$



$$\partial_t \mathcal{Z}_n = -H_n \mathcal{Z}_n$$

$$x = T^3 \kappa^{-1} \tilde{x} \quad , \quad t = 2T^5 \kappa^{-1} \tilde{t}$$

drop the tilde..

$$H_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1 \le i < j \le n} \delta(x_i - x_j)$$

Attractive Lieb-Lineger (LL) model (1963) what do we need from quantum mechanics ?

- KPZ with droplet initial condition

  = fixed endpoint DP partition sum

μ eigenstates

E_μ eigen-energies

$$e^{-tH} = \sum_\mu |\mu > e^{-E_\mu t} < \mu|$$

$$\overline{Z(x_0 t|x_0 0)^n} = < x_0 ... x_0 | e^{-tH_n} | x_0, .. x_0 >$$

symmetric states = bosons

$$= \sum_\mu \Psi^*_\mu(x_0..x_0) \Psi_\mu(x_0..x_0) \frac{1}{||\mu||^2} e^{-E_\mu t}$$

what do we need from quantum mechanics ?

- KPZ with droplet initial condition

  = fixed endpoint DP partition sum

μ eigenstates

E_μ eigen-energies

$$e^{-tH} = \sum_\mu |\mu > e^{-E_\mu t} < \mu|$$

$$\overline{Z(x_0 t|x_0 0)^n} =< x_0...x_0|e^{-tH_n}|x_0, ..x_0 >$$

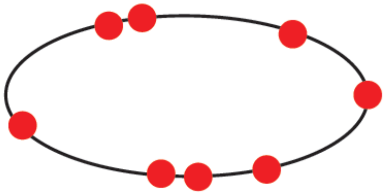
symmetric states = bosons

$$= \sum_\mu \Psi_\mu^*(x_0..x_0)\Psi_\mu(x_0..x_0)\frac{1}{||\mu||^2}e^{-E_\mu t}$$

- flat initial condition

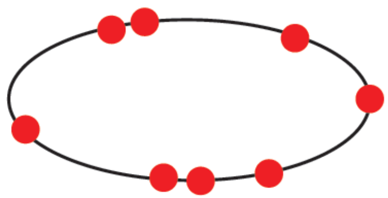
$$\overline{(\int_y Z(x_0 t|y0))^n} = \sum_\mu \Psi_\mu^*(x_0, .x_0)\int_{y_1,.y_n}\Psi_\mu(y_1, .y_n)\frac{1}{||\mu||^2}e^{-E_\mu t}$$

LL model: n bosons on a ring with local delta attraction



$$H_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1\leq i<j\leq n} \delta(x_i - x_j)$$

LL model: n bosons on a ring with local delta attraction



$$H_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1 \le i < j \le n} \delta(x_i - x_j)$$

Bethe Ansatz:

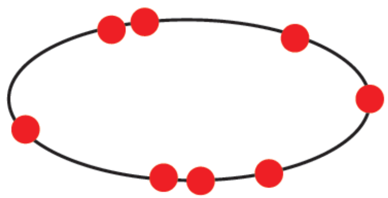
all (un-normalized) eigenstates are of the form (plane waves + sum over permutations)

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^{n} e^{i\lambda_{P_\ell} x_\ell}$$

$$E_\mu = \sum_{j=1}^{n} \lambda_j^2 \qquad A_P = \prod_{n \ge \ell > k \ge 1} \left(1 - \frac{ic\ \mathrm{sgn}(x_\ell - x_k))}{\lambda_{P_\ell} - \lambda_{P_k}}\right)$$

They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

LL model: n bosons on a ring with local delta attraction



$$H_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1 \le i < j \le n} \delta(x_i - x_j)$$

Bethe Ansatz:

all (un-normalized) eigenstates are of the form (plane waves + sum over permutations)

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P_\ell} x_\ell}$$

$$E_\mu = \sum_{j=1}^{n} \lambda_j^2$$

$$A_P = \prod_{n \ge \ell > k \ge 1} \left(1 - \frac{ic\ \mathrm{sgn}(x_\ell - x_k))}{\lambda_{P_\ell} - \lambda_{P_k}}\right)$$

They are indexed by a set of rapidities $\lambda_1, ..\lambda_n$

which are determined by solving the N coupled Bethe equations (periodic BC)

$$e^{i\lambda_j L} = \prod_{\ell \ne j} \frac{\lambda_j - \lambda_\ell - i\bar{c}}{\lambda_j - \lambda_\ell + i\bar{c}}$$

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

Derrida Brunet 2000

- ground state = a single bound state of n particules Kardar 87

$$\psi_0(x_1, ..x_n) \sim \exp(-\frac{\bar{c}}{2}\sum_{i<j}|x_i - x_j|) \qquad E_0(n) = -\frac{\bar{c}^2}{12}n(n^2-1)$$

$$\overline{Z^n} = \overline{e^{n \ln Z}} \quad \sim_{t\to\infty} e^{-tE_0(n)} \sim e^{\frac{\bar{c}^2}{12}n^3 t}$$

exponent 1/3

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

Derrida Brunet 2000

- ground state = a single bound state of n particules Kardar 87

$$\psi_0(x_1,..x_n) \sim \exp(-\frac{\bar{c}}{2}\sum_{i<j}|x_i - x_j|) \qquad E_0(n) = -\frac{\bar{c}^2}{12}n(n^2-1)$$

$$\overline{Z^n} = \overline{e^{n\ln Z}} \quad \sim_{t\to\infty} e^{-tE_0(n)} \sim e^{\frac{\bar{c}^2}{12}n^3 t}$$

exponent 1/3

can it be continued in n ? NO !

information about the tail
of the distribution of "free energy" f = - ln Z = - h

$$P(f) \sim_{f\to-\infty} \exp(-\frac{2}{3}(-f)^{3/2})$$

n bosons+attraction => bound states

Bethe equations + large L => rapidities have imaginary parts

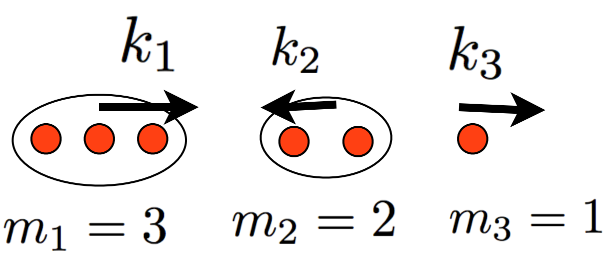
Derrida Brunet 2000

- ground state = a single bound state of n particules Kardar 87

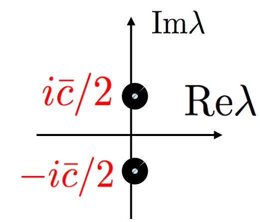
$$E_0(n) = -\frac{\bar{c}^2}{12} n(n^2-1)$$

need to sum over all eigenstates !

- all eigenstates are: All possible partitions of n into ns "strings" each with mj particles and momentum kj



$$n = \sum_{j=1}^{n_s} m_j$$



$$\lambda_{j,a_j} = k_j + \frac{i\bar{c}}{2}(m_j + 1 - 2a_j) \qquad a_j = 1, ..m_j$$
$$j = 1, ..n_s$$

=> $$E_\mu = \sum_{j=1}^{n_s} (m_j k_j^2 - \frac{\bar{c}^2}{12} m_j (m_j^2 - 1))$$

Integer moments of partition sum: fixed endpoints (droplet IC)

$$\overline{Z^n} = \sum_\mu \frac{|\Psi_\mu(0..0)|^2}{||\mu||^2} e^{-E_\mu t}$$

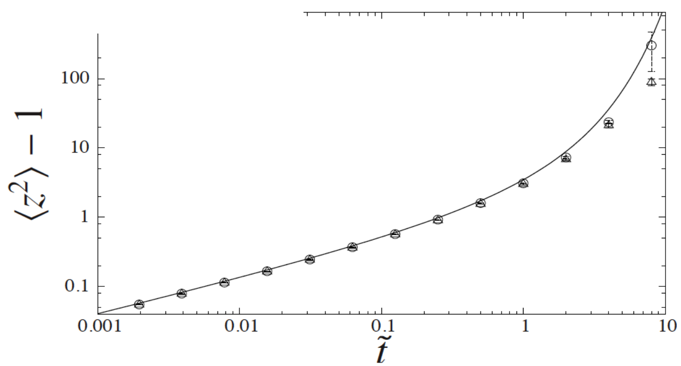
$$\Psi_\mu(0..0) = n!$$

norm of states: Calabrese-Caux (2007)

$$\overline{\hat{Z}^n} = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,\ldots m_{n_s})_n} \qquad n = \textstyle\sum_{j=1}^{n_s} m_j$$

$$\int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi[k,m] \prod_{j=1}^{n_s} e^{m_j^3 \frac{\bar{c}^2 t}{12} - m_j k_j^2 t},$$

$$\Phi[k,m] = \prod_{1\leq i<j\leq n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 c^2/4}{(k_i - k_j)^2 + (m_i + m_j)^2 c^2/4}$$



how to get P(ln Z) i.e. P(h) ?

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$\ln Z = -\lambda f$$

f = - ln Z = - h random variable expected O(1)

introduce generating function of moments g(x):

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

how to get P(ln Z) i.e. P(h) ?

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$\ln Z = -\lambda f$$

f = - ln Z = h

random variable
expected O(1)

introduce generating function of moments g(x):

what we aim
to calculate=
Laplace transform
of P(Z)

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

what we actually study

so that at large time:

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f-x)} = Prob(f > x)$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

$$Z(n_s, x) = \sum_{m_1, \ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

reorganize sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

Airy trick

$$\int_{-\infty}^{\infty} dy Ai(y) e^{yw} = e^{w^3/3}$$

$$Z(n_s, x) = \sum_{m_1, \ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

double Cauchy formula

$$det[\frac{1}{i(k_i - k_j)\lambda^{-3/2} + (m_i + m_j)}]$$

$$= \prod_{i<j} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{i=1}^{n_s} \frac{1}{2m_i}$$

$$\frac{1}{X} = \int_0^{\infty} dv e^{-vX}$$

Results: 1) g(x) is a Fredholm determinant at any time t

$$Z(n_s, x) = \prod_{j=1}^{n_s} \int_{v_j > 0} dv_j \; det[K(v_j, v_\ell)]$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$K(v_1, v_2) = -\int \frac{dk}{2\pi} dy Ai(y + k^2 - x + v_1 + v_2) e^{-ik(v_1 - v_2)} \frac{e^{\lambda y}}{1 + e^{\lambda y}}$$

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x) \quad = Det[I + K]$$

by an equivalent definition of a Fredholm determinant

$$K(v_1, v_2) \equiv \theta(v_1) K(v_1, v_2) \theta(v_2)$$

Results: 1) g(x) is a Fredholm determinant at any time t

$$Z(n_s,x) = \prod_{j=1}^{n_s} \int_{v_j>0} dv_j \; det[K(v_j,v_\ell)] \qquad \lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$K(v_1,v_2) = -\int \frac{dk}{2\pi} dy Ai(y+k^2-x+v_1+v_2) e^{-ik(v_1-v_2)} \frac{e^{\lambda y}}{1+e^{\lambda y}}$$

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s,x) = Det[I+K]$$

by an equivalent definition of a Fredholm determinant

$$K(v_1,v_2) \equiv \theta(v_1) K(v_1,v_2) \theta(v_2)$$

2) large time limit $\lambda = +\infty$ $\quad \frac{e^{\lambda y}}{1+e^{\lambda y}} \longrightarrow \theta(y)$

Airy function identity

$$\int dk Ai(k^2+v+v') e^{ik(v-v')} = 2^{2/3} \pi Ai(2^{1/3} v) Ai(2^{1/3} v')$$

g(x)= $Prob(f > x = -2^{2/3} s) = Det(1 - P_s K_{Ai} P_s) = F_2(s)$

$$K_{Ai}(v,v') = \int_{y>0} Ai(v+y) Ai(v'+y)$$

GUE-Tracy-Widom distribution

An exact solution for the KPZ equation with flat initial conditions

P. Calabrese, P. Le Doussal, (2011)

needed:

$$\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$$

1) g(s=-x) is a Fredholm Pfaffian at any time t

$$Z(n_s) = \sum_{m_i \geq 1} \prod_{j=1}^{n_s} \int_{k_j} \prod_{q=1}^{m_j} \frac{-2}{2ik_j+q} e^{\frac{\lambda^3}{3}m_j^3 - 4m_j k_j^2 \lambda^3 - \lambda m_j s}$$

$$\times \mathrm{Pf}\left[\begin{pmatrix} \frac{2\pi}{2ik_i}\delta(k_i+k_j)(-1)^{m_i}\delta_{m_i,m_j} + \frac{1}{4}(2\pi)^2\delta(k_i)\delta(k_j)(-1)^{\min(m_i,m_j)}\mathrm{sgn}(m_i-m_j) & \frac{1}{2}(2\pi)\delta(k_i) \\ -\frac{1}{2}(2\pi)\delta(k_j) & \frac{2ik_i+m_i-2ik_j-m_j}{2ik_i+m_i+2ik_j+m_j} \end{pmatrix} \right]$$

$$Z(n_s) = \prod_{j=1}^{n_s} \int_{v_j>0} \mathrm{Pf}[\mathbf{K}(v_i,v_j)]_{2n_s,2n_s}$$

$$g_\lambda(s) = \mathrm{Pf}[\mathbf{J}+\mathbf{K}] = \sum_{n_s=0}^{\infty} \frac{1}{n_s!} Z(n_s)$$

$$\mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

An exact solution for the KPZ equation with flat initial conditions

P. Calabrese, P. Le Doussal, (2011)

needed:

$$\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$$

1) g(s=-x) is a Fredholm Pfaffian at any time t

$$Z(n_s) = \sum_{m_i \geq 1} \prod_{j=1}^{n_s} \int_{k_j} \prod_{q=1}^{m_j} \frac{-2}{2ik_j+q} e^{\frac{\lambda^3}{3} m_j^3 - 4 m_j k_j^2 \lambda^3 - \lambda m_j s}$$

$$\times \mathrm{Pf}\left[\begin{pmatrix} \frac{2\pi}{2ik_i}\delta(k_i+k_j)(-1)^{m_i}\delta_{m_i,m_j} + \frac{1}{4}(2\pi)^2\delta(k_i)\delta(k_j)(-1)^{\min(m_i,m_j)}\mathrm{sgn}(m_i-m_j) & \frac{1}{2}(2\pi)\delta(k_i) \\ -\frac{1}{2}(2\pi)\delta(k_j) & \frac{2ik_i+m_i-2ik_j-m_j}{2ik_i+m_i+2ik_j+m_j} \end{pmatrix} \right]$$

$$Z(n_s) = \prod_{j=1}^{n_s} \int_{v_j>0} \mathrm{Pf}[\mathbf{K}(v_i,v_j)]_{2n_s,2n_s}$$

$$g_\lambda(s) = \mathrm{Pf}[\mathbf{J}+\mathbf{K}] = \sum_{n_s=0}^{\infty} \frac{1}{n_s!} Z(n_s)$$

$$\mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

2) large time limit $\lambda = +\infty$

$$g_\infty(s) = F_1(s) = \det[I - \mathcal{B}_s]$$

GOE Tracy Widom

$$\mathcal{B}_s = \theta(x) Ai(x+y+s)\theta(y)$$

Fredholm Pfaffian Kernel at any time t

$$K_{11} = \int_{y_1,y_2,k} Ai(y_1+v_i+s+4k^2)Ai(y_2+v_j+s+4k^2)[\frac{e^{-2i(v_i-v_j)k}}{2ik} f_{k/\lambda}(e^{\lambda(y_1+y_2)})$$

$$+ \frac{\pi\delta(k)}{2} F(2e^{\lambda y_1}, 2e^{\lambda y_2})]$$

$$K_{12} = \frac{1}{2}\int_y Ai(y+s+v_i)(e^{-2e^{\lambda y}}-1)\ \delta(v_j)$$

$$K_{22} = 2\delta'(v_i-v_j)\,,$$

$$f_k(z) = \frac{-2\pi k z\,{}_1F_2\left(1;2-2ik,2+2ik;-z\right)}{\sinh\left(2\pi k\right)\Gamma\left(2-2ik\right)\Gamma\left(2+2ik\right)}\,, \qquad (19)$$

$$F(z_i,z_j) = \sinh(z_2-z_1)+e^{-z_2}-e^{-z_1}+\int_0^1 du$$

$$\times J_0(2\sqrt{z_1z_2(1-u)})[z_1\sinh(z_1u)-z_2\sinh(z_2u)]\,.$$

$$g_\lambda(s) = \sqrt{Det(1-2K_{10})}(1+\langle\tilde{K}|(1-2K_{10})^{-1}|\delta\rangle)$$

$$K_{10}(v_1,v_2) = \partial_{v_1}K_{11}(v_1,v_2) \qquad\qquad K_{12}(v_1,v_2) = \tilde{K}(v_1)\delta(v_2)$$

Fredholm Pfaffian Kernel at any time t

$$K_{11} = \int_{y_1,y_2,k} Ai(y_1+v_i+s+4k^2)Ai(y_2+v_j+s+4k^2)\Big[\frac{e^{-2i(v_i-v_j)k}}{2ik} f_{k/\lambda}(e^{\lambda(y_1+y_2)})$$

$$+\frac{\pi\delta(k)}{2}F(2e^{\lambda y_1},2e^{\lambda y_2})\Big]$$

$$K_{12} = \frac{1}{2}\int_y Ai(y+s+v_i)(e^{-2e^{\lambda y}}-1)\ \delta(v_j)$$

$$K_{22} = 2\delta'(v_i-v_j)\,,$$

$$f_k(z) = \frac{-2\pi k z\,{}_1F_2\left(1;2-2ik,2+2ik;-z\right)}{\sinh\left(2\pi k\right)\Gamma\left(2-2ik\right)\Gamma\left(2+2ik\right)}\,, \qquad (19)$$

$$F(z_i,z_j) = \sinh(z_2-z_1)+e^{-z_2}-e^{-z_1}+\int_0^1 du$$

$$\times J_0(2\sqrt{z_1z_2(1-u)})[z_1\sinh(z_1u)-z_2\sinh(z_2u)]\,.$$

large time limit

$$\lim_{\lambda\to+\infty} f_{k/\lambda}(e^{\lambda y}) = -\theta(y)$$

$$\lim_{\lambda\to+\infty} F(2e^{\lambda y_1},2e^{\lambda y_2}) =$$

$$\theta(y_1+y_2)(\theta(y_1)\theta(-y_2)-\theta(y_2)\theta(-y_1))$$

$$g_\lambda(s) = \sqrt{Det(1-2K_{10})}(1+\langle\tilde{K}|(1-2K_{10})^{-1}|\delta\rangle)$$

$$K_{10}(v_1,v_2) = \partial_{v_1}K_{11}(v_1,v_2)$$

$$K_{12}(v_1,v_2) = \tilde{K}(v_1)\delta(v_2)$$

Summary: we found

for droplet initial conditions

$$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + 2^{2/3}(\frac{t}{t^*})^{1/3}\chi$$

χ at large time has the same distribution as the largest eigenvalue of the GUE

for flat initial conditions

$$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + (\frac{t}{t^*})^{1/3}\chi$$

similar (more involved)

χ at large time has the same distribution as the largest eigenvalue of the GOE

$$t^* = \frac{8(2\nu)^5}{D^2\lambda_0^4}$$

in addition: g(x) for all times
=> P(h) at all t (inverse LT)

decribes full crossover from Edwards Wilkinson to KPZ

t^* is crossover time scale

large for weak noise, large diffusivity

GSE ?

Summary:

for droplet initial conditions $$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + 2^{2/3}(\frac{t}{t^*})^{1/3}\chi$$

χ at large time has the same distribution as the largest eigenvalue of the GUE

for flat initial conditions $$\frac{\lambda_0 h}{2\nu} \equiv \ln Z = v_\infty t + (\frac{t}{t^*})^{1/3}\chi$$

similar (more involved)

χ at large time has the same distribution as the largest eigenvalue of the GOE

$$t^* = \frac{8(2\nu)^5}{D^2\lambda_0^4}$$

in addition: g(x) for all times
=> P(h) at all t (inverse LT)

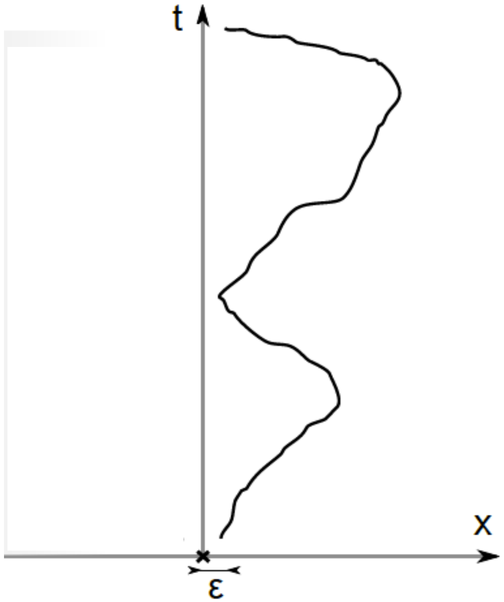
decribes full crossover from Edwards Wilkinson to KPZ

t^* is crossover time scale

GSE ? KPZ in half-space

DP near a wall = KPZ equation in half space

T. Gueudre, P. Le Doussal, EPL 100 26006 (2012)



$$g(s) = \sqrt{\mathrm{Det}[I + \mathcal{K}]}$$

$$\mathcal{K}(v_1, v_2) = -2\theta(v_1)\theta(v_2)\partial_{v_1} f(v_1, v_2)$$

$$f(v_1, v_2) = \int \frac{dk}{2\pi} \int_y Ai(y + s + v_1 + v_2 + 4k^2)\, f_{k/\lambda}(e^{\lambda y}) \frac{e^{-2ik(v_1 - v_2)}}{2ik}$$

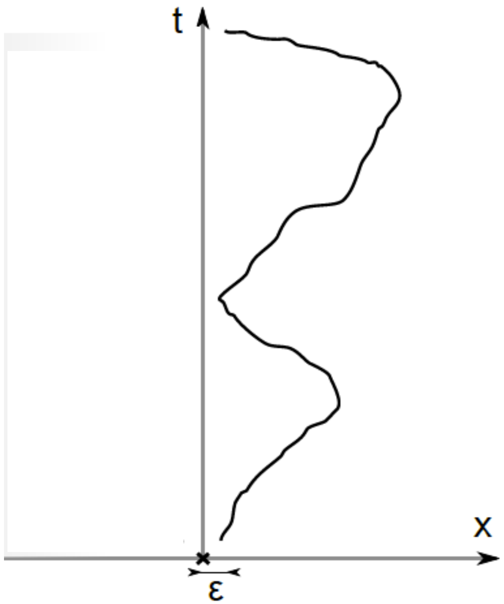
$$f_k[z] = \frac{2\pi k}{\sinh(4\pi k)} \left(J_{-4ik}(\frac{2}{\sqrt{z}}) + J_{4ik}(\frac{2}{\sqrt{z}}) \right)$$
$$- {}_1F_2\,(1; 1 - 2ik, 1 + 2ik; -1/z)$$

$$Z(x, 0, t) = Z(0, y, t) = 0$$

$\nabla h(0, t)$ fixed

DP near a wall = KPZ equation in half space

T. Gueudre, P. Le Doussal,
EPL 100 26006 (2012)



$$g(s) = \sqrt{\mathrm{Det}[I + \mathcal{K}]}$$

$$\mathcal{K}(v_1, v_2) = -2\theta(v_1)\theta(v_2)\partial_{v_1} f(v_1, v_2)$$

$$f(v_1, v_2) = \int \frac{dk}{2\pi} \int_y Ai(y + s + v_1 + v_2 + 4k^2)\, f_{k/\lambda}(e^{\lambda y}) \frac{e^{-2ik(v_1 - v_2)}}{2ik}$$

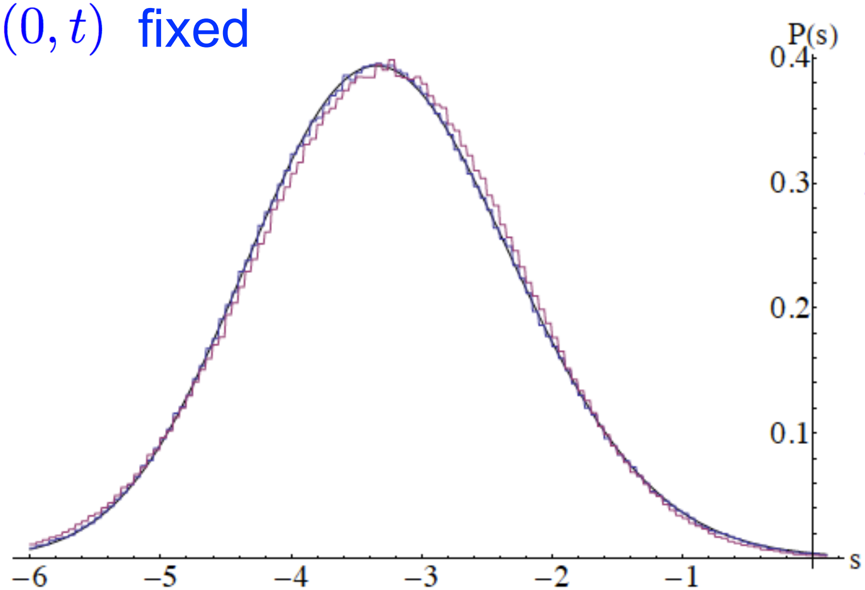
$$f_k[z] = \frac{2\pi k}{\sinh(4\pi k)} \left(J_{-4ik}(\frac{2}{\sqrt{z}}) + J_{4ik}(\frac{2}{\sqrt{z}}) \right)$$
$$- {}_1F_2\,(1; 1 - 2ik, 1 + 2ik; -1/z)$$

$$\lim_{\lambda \to \infty} f_{k/\lambda}[e^{\lambda y}] = -\theta(y)(1 - \cos(2ky))$$

$Z(x, 0, t) = Z(0, y, t) = 0$

$\nabla h(0, t)$ fixed

$$\lambda = (\frac{\bar{c}^2 t}{8T^5})^{1/3} = (\frac{D\lambda_0^2 t}{8(2\nu)^5})^{1/3}$$

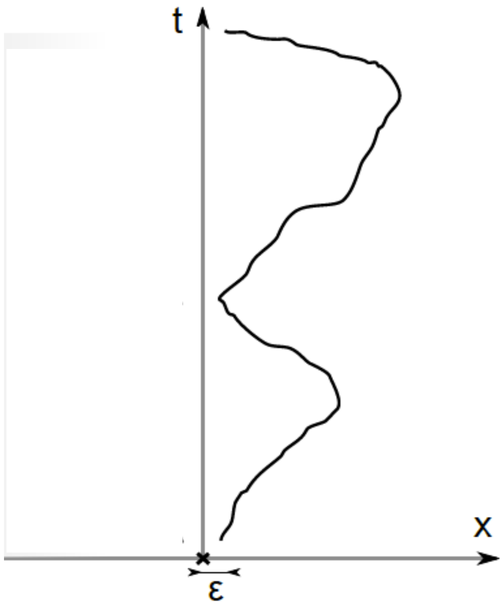


$$\ln Z = \frac{\lambda_0}{2\nu} \tilde{h}(0, t) = v_\infty t + 2^{2/3} \lambda \chi_4$$

χ_4 distributed as $F_4(s)$

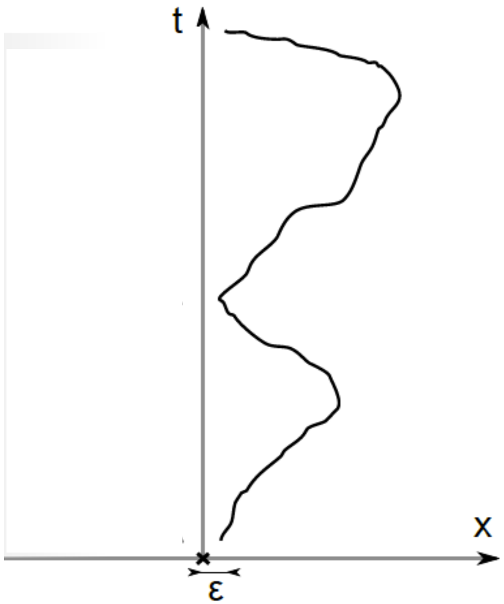
Gaussian Symplectic Ensemble

Probability that a polymer (starting near the wall) does not cross the wall



$$q_\eta(t) = \frac{Z_\eta^{\text{half space}}(t)/\epsilon^2}{Z_\eta^{\text{full space}}(t)}$$

Probability that a polymer (starting near the wall) does not cross the wall



$$q_\eta(t) = \frac{Z_\eta^{\text{half space}}(t)/\epsilon^2}{Z_\eta^{\text{full space}}(t)}$$

$$\overline{\ln q_\eta(t)} = -(\overline{\chi_2} - \overline{\chi_4})(\bar{c}^2 t)^{1/3} \approx -1.49134(\bar{c}^2 t)^{1/3}$$

$$\mu^{F_2} = -1.7710868 \qquad \mu^{F_4} = -3.2624279$$

gives q(t) in typical sample: decays sub-exponentially

Probability that N directed paths in a random potential do not cross

why interesting ?

- example of disorder + interaction

Kardar-Emig large N
vortex lines in SC
Cardy-Ostlund model

- BA with generalized statistics - beyond bosons

- connections to random matrix theory

Probability that N directed paths in a random potential do not cross

Partition sum of 1 path with endpoints y,x

$$Z_\eta(x;y|t)$$

Karlin McGregor formula

Partition sum of N non-crossing paths with endpoints

$y_1 < y_2 < ..y_N \qquad x_1 < x_2 < ..x_N$

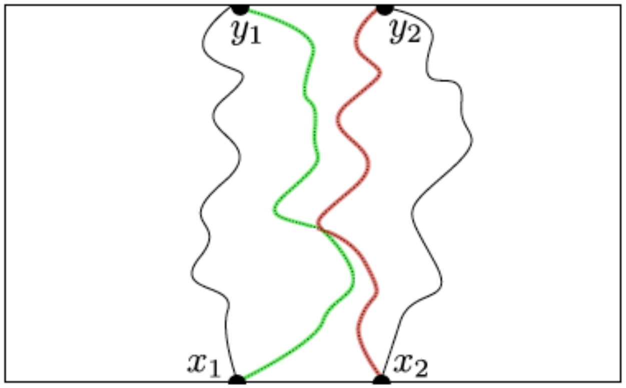
$$\det[Z_\eta(x_i;y_j|t)]_{N\times N}$$

(thermal) probability of no crossing

$$= \prod_{i=1}^{N} \frac{1}{Z_\eta(x_i;y_i|t)} \det[Z_\eta(x_i;y_j|t)]_{N\times N}$$

Probability that 2 directed polymers in same disorder do not cross

Andrea de Luca, PLD, arXiv 1505.04802



non-crossing probability

Karlin Mc Gregor

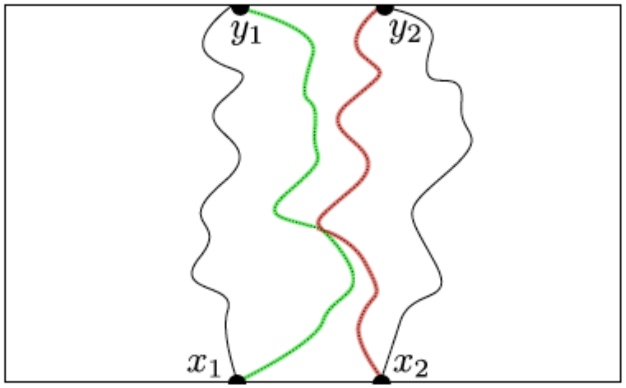
$$p_\eta(x_1, x_2; y_1, y_2|t) \equiv 1 - \frac{Z_\eta(x_2; y_1|t) Z_\eta(x_1; y_2|t)}{Z_\eta(x_1; y_1|t) Z_\eta(x_2; y_2|t)}$$

we will study limit at coinciding endpoints

$$p_\eta(t) \equiv \lim_{\epsilon \to 0} \frac{p_\eta(-\epsilon, \epsilon| -\epsilon, \epsilon; t)}{4\epsilon^2}$$

Probability that 2 directed polymers in same disorder do not cross

Andrea de Luca, PLD, arXiv 1505.04802



non-crossing probability

Karlin Mc Gregor

$$p_\eta(x_1, x_2; y_1, y_2|t) \equiv 1 - \frac{Z_\eta(x_2; y_1|t) Z_\eta(x_1; y_2|t)}{Z_\eta(x_1; y_1|t) Z_\eta(x_2; y_2|t)}$$

we will study limit at coinciding endpoints

$$p_\eta(t) \equiv \lim_{\epsilon \to 0} \frac{p_\eta(-\epsilon, \epsilon| - \epsilon, \epsilon; t)}{4\epsilon^2}$$

NC probability is related to single path free energy

$$p_\eta(x; y|t) = \partial_x \partial_y \ln Z_\eta(x; y|t)$$

$$p_\eta(t) = \partial_x \partial_y \ln Z_\eta(x; y|t)|_{\substack{x=0 \\ y=0}}$$

Moments of non-crossing probability via replica ..

$$\overline{p_\eta(t)^m} = \lim_{n\to 0} \Theta_{n,m}(t)$$

$$p_\eta(t) \equiv \lim_{\epsilon\to 0} \frac{p_\eta(-\epsilon,\epsilon|-\epsilon,\epsilon;t)}{4\epsilon^2}$$

$$\Theta_{n,m}(t) \equiv \lim_{\epsilon\to 0} \overline{[(2\epsilon)^{-2} Z_\eta^{(2)}(\epsilon)]^m [Z_\eta(0;0|t)]^{n-2m}}$$

$$Z_\eta^{(2)}(\epsilon) = Z_\eta(\epsilon;\epsilon|t) Z_\eta(-\epsilon;-\epsilon|t) - Z_\eta(-\epsilon;\epsilon|t) Z_\eta(\epsilon;-\epsilon|t)$$

Moments of non-crossing probability via replica ..

$$\overline{p_\eta(t)^m} = \lim_{n\to 0} \Theta_{n,m}(t)$$

$$p_\eta(t) \equiv \lim_{\epsilon\to 0} \frac{p_\eta(-\epsilon,\epsilon|-\epsilon,\epsilon;t)}{4\epsilon^2}$$

$$\Theta_{n,m}(t) \equiv \lim_{\epsilon\to 0} \overline{[(2\epsilon)^{-2} Z_\eta^{(2)}(\epsilon)]^m [Z_\eta(0;0|t)]^{n-2m}}$$

$$Z_\eta^{(2)}(\epsilon) = Z_\eta(\epsilon;\epsilon|t) Z_\eta(-\epsilon;-\epsilon|t) - Z_\eta(-\epsilon;\epsilon|t) Z_\eta(\epsilon;-\epsilon|t)$$

and quantum mechanics …

Lieb-Liniger model with general symmetry (beyond bosons)

$$H_n = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - 2\bar{c} \sum_{1\le i<j\le n} \delta(x_i - x_j)$$

$$\Theta_{n,m}(t) = \lim_{\epsilon\to 0} (2\epsilon)^{-2m} \langle \Psi_m(\epsilon)| e^{-tH_n} |\Psi_m(\epsilon)\rangle = \sum_\mu \frac{|\mathcal{D}_m \psi_\mu(\mathbf{x})|^2}{||\mu||^2} e^{-tE_\mu}$$

$$|\Psi_m(\epsilon)\rangle = 2^{-m/2} (\otimes_{j=1}^m |\epsilon,-\epsilon\rangle - |-\epsilon,\epsilon\rangle) \otimes |0\dots 0\rangle$$

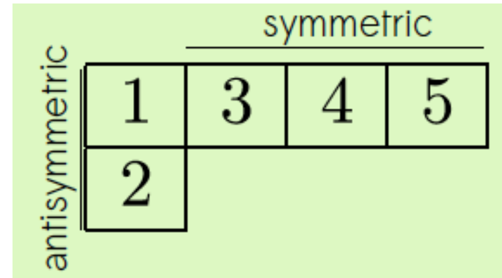
$$\mathcal{D}_1 = 2^{-1/2} (\partial_{x_1} - \partial_{x_2})\big|_{\mathbf{x}=0}$$

bosonic sector gives vanishing contribution

$$\overline{p_\eta(t)^m} \quad \text{m=1}$$

look for eigenfunctions antisymmetric in the first two variables

$$\Psi_\mu(x_1, x_2, ..x_n) \quad \longleftrightarrow$$



$\overline{p_\eta(t)^m}$ m=1

look for eigenfunctions antisymmetric in the first two variables

$$\Psi_\mu(x_1, x_2, ..x_n) \longleftrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 \\ \cline{1-1}\end{array}$$

symmetric

antisymmetric

more general Bethe ansatz

$$\psi_\mu(\mathbf{x}) = \sum_{P,Q\in\mathcal{S}_n} \vartheta_Q(\mathbf{x}) A_Q^P \exp[i \sum_{j=1}^{n} x_{Q_j}\mu_{P_j}]$$

$$\{\mu_1, \ldots, \mu_n\}$$

$$E_\mu = \sum_{j=1}^n \mu_j^2$$

$$x_{Q_1} \le x_{Q_2} \ldots \le x_{Q_n}$$

A_Q^P inside irreducible representation of S_n

2-row Young diagram $\xi = (n-m, m)$

example for m=3

$n = 8$ and $m = 3$

$$(5,3) \equiv \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 5 & 7 & 9 \\ \hline 2 & 4 & 6 \\ \cline{1-3}\end{array}$$

antisymmetry $\mathcal{D}_{m=3}$

$$x_1 \leftrightarrow x_2\,, x_3 \leftrightarrow x_4\,, x_5 \leftrightarrow x_6$$

Nested Bethe ansatz

C-N Yang PRL 19,1312 (1967)

Bethe equations

auxiliary spin chain

$$\mu_{\alpha\beta} = \mu_\alpha - \mu_\beta$$

auxiliary rapidities λ_a $a = 1, ..m$

one for every doubled column

they implement the symmetry of the wave-function

$$\prod_{\substack{b=1\\b\neq a}}^{m} \frac{\lambda_{ab} - ic}{\lambda_{ab} + ic} = \prod_{j=1}^{n} \frac{\lambda_a - \mu_j - ic/2}{\lambda_a - \mu_j + ic/2} ,$$

$$\prod_{\substack{k=1\\k\neq j}}^{n} \frac{\mu_{jk} + ic}{\mu_{jk} - ic} \times \prod_{a=1}^{m} \frac{\mu_j - \lambda_a - ic/2}{\mu_j - \lambda_a + ic/2} = e^{i\mu_j L}$$

Nested Bethe ansatz

C-N Yang PRL 19,1312 (1967)

Bethe equations

auxiliary spin chain

$$\mu_{\alpha\beta} = \mu_\alpha - \mu_\beta$$

auxiliary rapidities λ_a $a = 1, ..m$

one for every doubled column

they implement the symmetry of the wave-function

$$\prod_{\substack{b=1 \\ b\neq a}}^{m} \frac{\lambda_{ab} - ic}{\lambda_{ab} + ic} = \prod_{j=1}^{n} \frac{\lambda_a - \mu_j - ic/2}{\lambda_a - \mu_j + ic/2} \,,$$

$$\prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{\mu_{jk} + ic}{\mu_{jk} - ic} \times \prod_{a=1}^{m} \frac{\mu_j - \lambda_a - ic/2}{\mu_j - \lambda_a + ic/2} = e^{i\mu_j L}$$

solved at large L by strings again !

$$\mu_j^a = k_j + \frac{ic}{2}(m_j + 1 - 2a) + \delta_j^a$$

Nested Bethe ansatz

C-N Yang PRL 19,1312 (1967)

Bethe equations

auxiliary spin chain

$$\mu_{\alpha\beta} = \mu_\alpha - \mu_\beta$$

auxiliary rapidities $\lambda_a \quad a = 1, ..m$

one for every doubled column

they implement the symmetry of the wave-function

$$\prod_{\substack{b=1\\b\neq a}}^{m} \frac{\lambda_{ab} - ic}{\lambda_{ab} + ic} = \prod_{j=1}^{n} \frac{\lambda_a - \mu_j - ic/2}{\lambda_a - \mu_j + ic/2},$$

$$\prod_{\substack{k=1\\k\neq j}}^{n} \frac{\mu_{jk} + ic}{\mu_{jk} - ic} \times \prod_{a=1}^{m} \frac{\mu_j - \lambda_a - ic/2}{\mu_j - \lambda_a + ic/2} = e^{i\mu_j L}$$

solved at large L by strings again !

$$\mu_j^a = k_j + \frac{ic}{2}(m_j + 1 - 2a) + \delta_j^a$$

can one solve the polynomial equation for the auxiliary variable ?

example for m=1

$$\prod_{j=1}^{n} \frac{\lambda - \mu_j - ic/2}{\lambda - \mu_j + ic/2} = 1$$

→ not all string states contribute !

for instance the n-string does not

telescopic product, no solution

→ in general several roots => difficult

BUT: the sum over all solutions for λ can be written as a contour integral

$$\sum_{\lambda} \frac{|(\partial_{x_1} - \partial_{x_2})\Psi_\mu(x)|^2}{2||\mu||^2}$$

$\longrightarrow$ simplifies => expression very similar to bosonic case

- first moment, m =1

$$\Theta_{n,0}(t) = \mathcal{Z}_n(t) \equiv \mathcal{Z}_n(\mathbf{x} = \mathbf{0}; \mathbf{0}|t)$$

$$\Lambda_{n,1} \to \Lambda_{n,0} \equiv 1$$

$$\Theta_{n,1}(t) = \sum_{n_s=1}^{n} \frac{n!\bar{c}^n}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,..m_{n_s})_n} \prod_{j=1}^{n_s} \int_{-\infty}^{+\infty} \frac{dk_j e^{-A_2 t}}{m_j} \Phi(\mathbf{k},\mathbf{m}) \Lambda_{n,1}(\mathbf{k},\mathbf{m})$$

Lieb Lineger conserved charges

$$A_p = \sum_{j=1}^{n} \mu_j^p$$

$$\Lambda_{n,1} = \frac{1}{n(n-1)} \left(n A_2 - A_1^2 + \frac{n^2(n^2-1)}{12} \bar{c}^2 \right)$$

BUT: the sum over all solutions for λ can be written as a contour integral

$$\sum_{\lambda} \frac{|(\partial_{x_1} - \partial_{x_2})\Psi_\mu(x)|^2}{2||\mu||^2}$$

⟶ simplifies => expression very similar to bosonic case

- first moment, m =1

$$\Theta_{n,0}(t) = \mathcal{Z}_n(t) \equiv \mathcal{Z}_n(\mathbf{x} = \mathbf{0}; \mathbf{0}|t)$$

$$\Lambda_{n,1} \to \Lambda_{n,0} \equiv 1$$

$$\Theta_{n,1}(t) = \sum_{n_s=1}^{n} \frac{n!\bar{c}^n}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,..m_{n_s})_n} \prod_{j=1}^{n_s} \int_{-\infty}^{+\infty} \frac{dk_j e^{-A_2 t}}{m_j} \Phi(\mathbf{k},\mathbf{m}) \Lambda_{n,1}(\mathbf{k},\mathbf{m})$$

Lieb Lineger conserved charges

$$A_p = \sum_{j=1}^{n} \mu_j^p$$

$$\Lambda_{n,1} = \frac{1}{n(n-1)} \left(nA_2 - A_1^2 + \frac{n^2(n^2-1)}{12}\bar{c}^2 \right)$$

- does this extend to higher moments, m ? YES

$$\Theta_{n,m}(t) = \langle \Lambda_{n,m}(\boldsymbol{\mu}) \rangle_n \quad \Rightarrow \quad \Lambda_{n,m}(A_p)$$

1) how does one get the $\Lambda_{n,m}(A_p)$?

We calculated them from the Borodin-Corwin "conjecture"

BC arXiv11114408

$$\Theta_{n,m}(t) = \frac{1}{2^m} \int \frac{dz_1}{2\pi} \cdots \int \frac{dz_n}{2\pi} e^{-t\sum_{k=1}^{n} z_k^2} \times \left(\prod_{1 \le k < j \le n} f(z_{kj}) \right) \left(\prod_{q=1}^{m} h(z_{2q-1,2q}) \right)$$

$$h(u) = u(u - ic)$$
$$f(u) \equiv u/(u - ic)$$

$z_{kj} = z_k - z_j$ imaginary part C_j for z_j satisfying $C_{j+1} > C_j + \bar{c}$

1) how does one get the $\Lambda_{n,m}(A_p)$?

We calculated them from the Borodin-Corwin "conjecture"

BC arXiv11114408

$$\Theta_{n,m}(t) = \frac{1}{2^m} \int \frac{dz_1}{2\pi} \cdots \int \frac{dz_n}{2\pi} e^{-t\sum_{k=1}^n z_k^2} \times \left(\prod_{1 \le k < j \le n} f(z_{kj}) \right) \left(\prod_{q=1}^m h(z_{2q-1,2q}) \right)$$

$$h(u) = u(u - ic)$$
$$f(u) \equiv u/(u - ic)$$

$z_{kj} = z_k - z_j$ imaginary part C_j for z_j satisfying $C_{j+1} > C_j + \bar{c}$

2) what can one do with them?

Introduce GGE partition function $\mathcal{Z}_n^{\beta}(t)$ $e^{-A_2 t}$ => $e^{-A_2 t + \sum_{p \ge 1} \beta_p A_p}$

$$A_p \to \partial_p \equiv \partial_{\beta_p} \qquad \Theta_{n,m}(t) = \Lambda_{n,m}(\{\partial_p\})[\mathcal{Z}_n^{\beta}(t)]$$

$$\lim_{n \to 0} \partial_{i_1} \ldots \partial_{i_k} \frac{\mathcal{Z}_n^{\beta}(t) - 1}{n} \Big|_{\beta=0} = -\int_0^\infty \frac{du}{u} \partial_{i_1} \ldots \partial_{i_k} \operatorname{Det}(1 + \Pi_0 \mathcal{K}_u^{\beta} \Pi_0)$$

relate to Fredholm determinant !

Results: - first moment: simple from STS !

$$\overline{\ln Z_\eta(x;y|t)} = h(t) - (x-y)^2/(4t)$$

$$\overline{p_\eta(t)} = \lim_{n\to 0} \Theta_{n,1}(t) = \frac{1}{2t}$$

$$h(t) = \overline{\ln Z_\eta(0;0|t)}$$

$$\simeq -\frac{\bar{c}^2 t}{12} + \overline{\chi_2}(\bar{c}^2 t)^{1/3}$$

Results: - first moment: simple from STS !

$$\overline{\ln Z_\eta(x;y|t)} = h(t) - (x-y)^2/(4t)$$

$$\overline{p_\eta(t)} = \lim_{n\to 0} \Theta_{n,1}(t) = \frac{1}{2t}$$

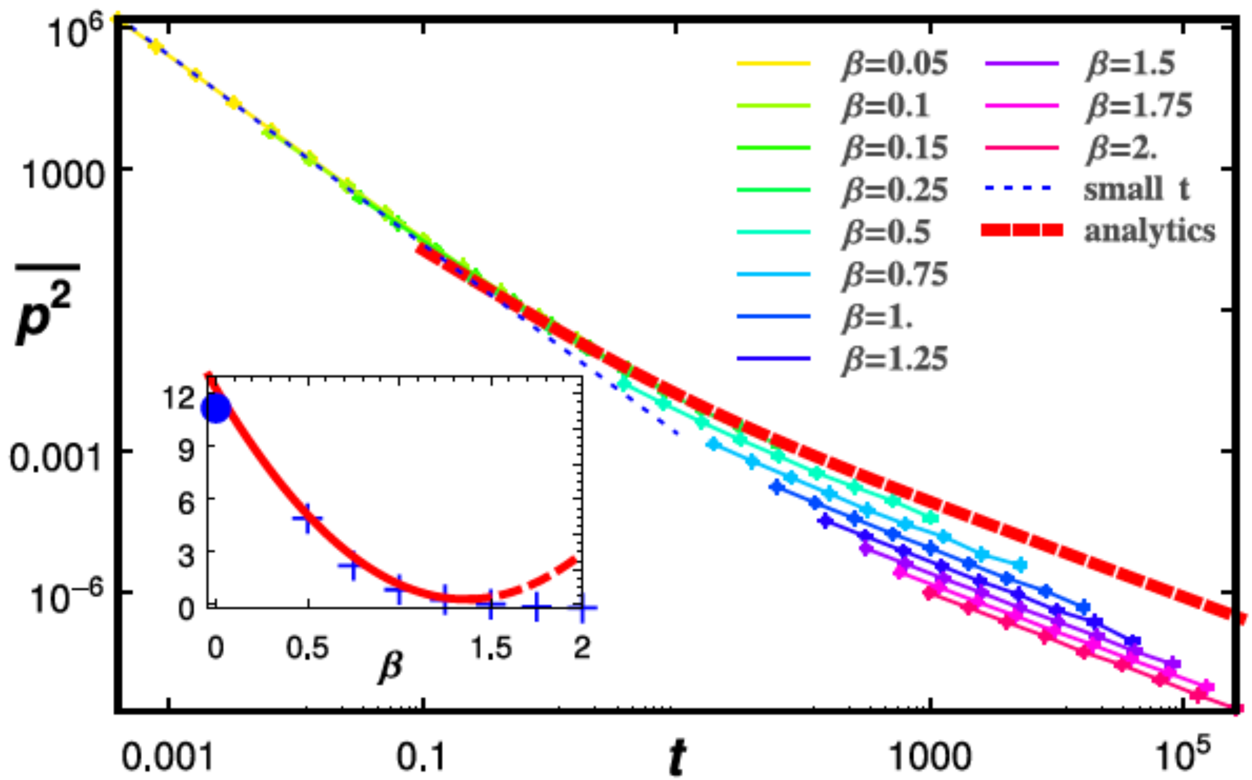
$$h(t) = \overline{\ln Z_\eta(0;0|t)}$$

$$\simeq -\frac{\bar{c}^2 t}{12} + \overline{\chi_2}(\bar{c}^2 t)^{1/3}$$

- second moment = we find exact relation to average free energy

$$\overline{p_\eta(t)^2} = -\left(\frac{1}{t}\partial_t + \frac{1}{2}\partial_t^2\right) h(t) \simeq \frac{\bar{c}^2}{12t} - \frac{2\overline{\chi_2}\bar{c}^{2/3}}{9t^{5/3}}$$

$$\overline{p_\eta(t)^2} = -\left(\frac{1}{t}\partial_t + \frac{1}{2}\partial_t^2\right) h(t) \simeq \frac{\bar{c}^2}{12t} - \frac{2\overline{\chi_2}\bar{c}^{2/3}}{9t^{5/3}}$$



Results: - first moment: simple from STS !

$$\overline{\ln Z_\eta(x;y|t)} = h(t) - (x-y)^2/(4t)$$

$$\overline{p_\eta(t)} = \lim_{n\to 0} \Theta_{n,1}(t) = \frac{1}{2t}$$

$$h(t) = \overline{\ln Z_\eta(0;0|t)}$$

- second moment = we find exact relation to average free energy

$$\overline{p_\eta(t)^2} = -\left(\frac{1}{t}\partial_t + \frac{1}{2}\partial_t^2\right) h(t) \simeq \frac{\bar{c}^2}{12t} - \frac{2\overline{\chi_2}\bar{c}^{2/3}}{9t^{5/3}}$$

- higher moments

$$\overline{p_\eta(t)^3} \simeq \frac{\bar{c}^4}{15t} - \frac{2\overline{\chi_2}\bar{c}^{8/3}}{9t^{5/3}}$$

+ complicated

Results:

- first moment: simple from STS !

$$\overline{\ln Z_\eta(x;y|t)} = h(t) - (x-y)^2/(4t)$$

$$\overline{p_\eta(t)} = \lim_{n\to 0} \Theta_{n,1}(t) = \frac{1}{2t}$$

$$h(t) = \overline{\ln Z_\eta(0;0|t)}$$

$$\simeq -\frac{\bar{c}^2 t}{12} + \overline{\chi_2}(\bar{c}^2 t)^{1/3}$$

- second moment = we find exact relation to average free energy

$$\overline{p_\eta(t)^2} = -\left(\frac{1}{t}\partial_t + \frac{1}{2}\partial_t^2\right) h(t) \simeq \frac{\bar{c}^2}{12t} - \frac{2\overline{\chi_2}\bar{c}^{2/3}}{9t^{5/3}}$$

- higher moments

$$\overline{p_\eta(t)^3} \simeq \frac{\bar{c}^4}{15t} - \frac{2\overline{\chi_2}\bar{c}^{8/3}}{9t^{5/3}}$$ + complicated

$$\gamma_1 = \frac{1}{2}\,, \quad \gamma_2 = \frac{1}{12}\,, \quad \gamma_3 = \frac{1}{15}$$

=> in arXiv1505.04802 we conjectured leading large time behavior

conjecture 1

$$\overline{p_\eta(t)^m} \simeq \gamma_m \bar{c}^{2(m-1)}/t$$

=> $p_\eta(t) \sim \bar{c}^2$ for a fraction $\sim 1/(\bar{c}^2 t)$ of environments

the PDF of p has a 1/t tail which controls the moments

What is p in a typical sample ?

What is p in a typical sample ?

remember proba q(t) of single DP not crossing a hard-wall at 0

T. Gueudre, PLD 2012

$$\overline{\ln q_\eta(t)} = -(\overline{\chi_2} - \overline{\chi_4})(\bar{c}^2 t)^{1/3} \approx -1.49134(\bar{c}^2 t)^{1/3}$$

=> typical p is sub-exponentially small

What is p in a typical sample ?

remember proba q(t) of single DP not crossing a hard-wall at 0

$$\overline{\ln q_\eta(t)} = -(\overline{\chi_2} - \overline{\chi_4})(\bar{c}^2 t)^{1/3} \approx -1.49134(\bar{c}^2 t)^{1/3}$$

T. Gueudre, PLD 2012

=> typical p is sub-exponentially small

=> conjecture 2

universality

$$\overline{\ln p_\eta(t)} \sim -a(\bar{c}^2 t)^{1/3} \qquad a = \overline{\chi_2} - \overline{\chi_2'} \approx 1.9043$$

$$\overline{\ln p_\eta(t)} < \overline{\ln q_\eta(t)}$$

known for semi-discrete DP

non-crossing paths and GUE eigenvalues

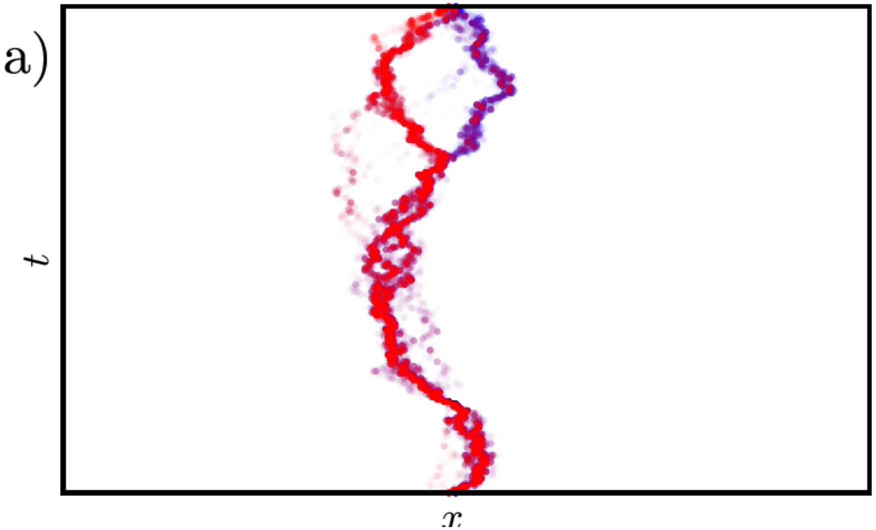
h_k max energy of k non crossing paths length N

$$h_1 =_d \lambda_1$$

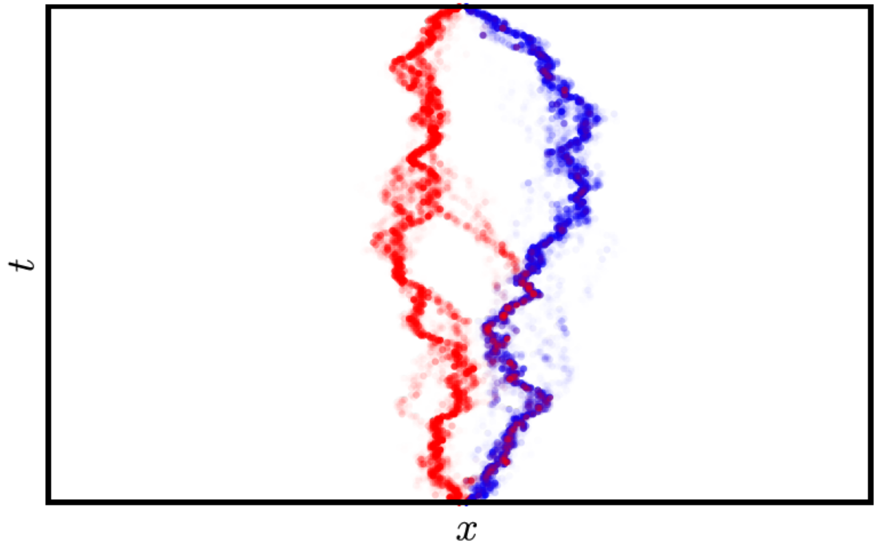
$$h_2 - h_1 =_d \lambda_2$$

O'Connell, Yor (2002) Doumerc (2003)

(a)



typical sample with small p



(b)

rare sample with p=O(1)

plot of $Z(\pm 1/2, \hat{x}|\hat{t}) \times Z(\hat{x}, \pm 1/2|\hat{\tau} - \hat{t})$

at inverse temperature

$\beta = 1.0$

Distribution of non-crossing probability

$$\mathcal{P}_t(p) \simeq_{t\to+\infty} \mathcal{P}^0\big(p/p_{typ}(t)\big) + \frac{\rho(p/\bar{c}^2)}{\bar{c}^4 t}$$

bulk of the PDF

tail
controls the moments

$$p_{typ}(t) \sim e^{-a(\bar{\bar{c}}^2 t)^{1/3}}$$

$$\int_0^\infty d\rho\, \rho(p) p^m = \gamma_m$$

$$\gamma_1 = \frac{1}{2}\,, \quad \gamma_2 = \frac{1}{12}\,, \quad \gamma_3 = \frac{1}{15}$$

Distribution of non-crossing probability

$$\mathcal{P}_t(p) \simeq_{t\to+\infty} \mathcal{P}^0\big(p/p_{typ}(t)\big) + \frac{\rho(p/\bar{c}^2)}{\bar{c}^4 t}$$

bulk of the PDF

tail
controls the moments

$$p_{typ}(t) \sim e^{-a(\bar{\bar{c}}^2 t)^{1/3}}$$

$$\int_0^\infty d\rho\, \rho(p) p^m = \gamma_m$$

$$\gamma_1 = \frac{1}{2}\,, \quad \gamma_2 = \frac{1}{12}\,, \quad \gamma_3 = \frac{1}{15}$$

FindSequenceFunction[..]

Distribution of non-crossing probability

$$\mathcal{P}_t(p) \simeq_{t\to+\infty} \mathcal{P}^0\big(p/p_{typ}(t)\big) + \frac{\rho(p/\bar{c}^2)}{\bar{c}^4 t}$$

bulk of the PDF

tail
controls the moments

$$p_{typ}(t) \sim e^{-a(\bar{c}^2 t)^{1/3}}$$

$$\int_0^\infty d\rho\, \rho(p) p^m = \gamma_m$$

$$\gamma_1 = \frac{1}{2}\,, \quad \gamma_2 = \frac{1}{12}\,, \quad \gamma_3 = \frac{1}{15}$$

FindSequenceFunction[..]

Recent result

A. De Luca, PLD in prep.

$$\overline{p_\eta(t)^m} \simeq_{t\to\infty} \gamma_m \frac{\bar{c}^{2(m-1)}}{t}$$

$$\gamma_m = \frac{\sqrt{\pi}\, 4^{-m}\Gamma(m)^3}{\Gamma\left(m+\frac{1}{2}\right)}$$

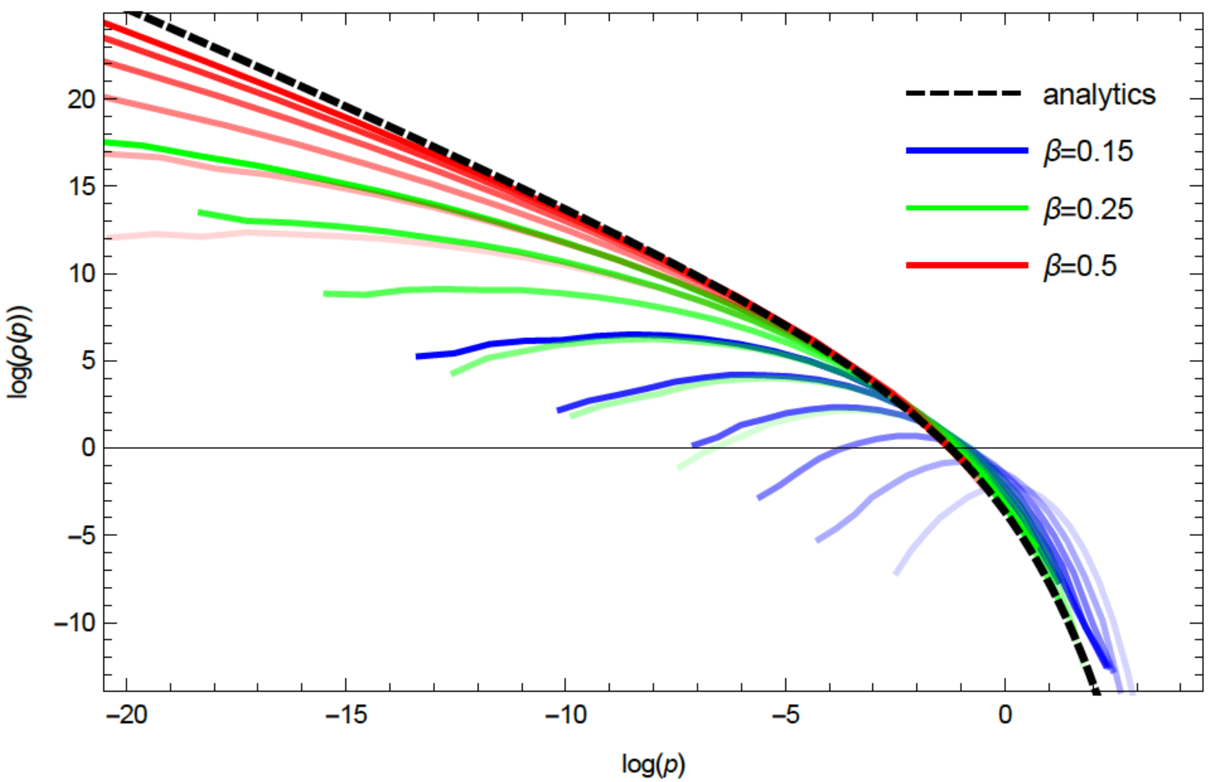
from the moments extract the density

$$\gamma_m = \frac{\sqrt{\pi}4^{-m}\Gamma(m)^3}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\rho(p) = \frac{2}{p}\int_0^{+\infty} \frac{du}{\sqrt{u(u+4)}} K_0(2\sqrt{p}\sqrt{u+4})$$

$$\rho(p) \simeq \frac{1}{2p}(\ln p)^2$$

numerical check



Idea of the method $\Theta_{n,m}(t) = \langle \Lambda_{n,m}(\boldsymbol{\mu}) \rangle_n$

1) after massaging of BC formula..

$$h(u) = u(u - i\bar{c})$$

$$\Lambda_{n,m}(\boldsymbol{\mu}) = \frac{1}{2^m} \operatorname{sym}_{\boldsymbol{\mu}} \left[\frac{\prod_{q=1}^{m} h(\mu_{2q-1,2q})}{\prod_{1 \le \alpha < \beta \le n} f(\mu_{\beta\alpha})} \right] \qquad f(u) = u/(u - i\bar{c})$$

2) algebra and guesses.. express in terms of elementary symmetric polynomials

$$\Lambda_{n,m}(\boldsymbol{\mu}) = \sum_{a=0}^{m} \bar{c}^{2a} \Omega^a_{n,m} \tilde{\Lambda}_{n,m-a}(\boldsymbol{\mu}) \qquad \Omega^a_{n,m} = \frac{m!(n-m+a)! B_{2a}^{(2m-2n-1)}(m-n)}{(2a)!(m-a)!(n-m)!}$$

$$\tilde{\Lambda}_{n,m}(\boldsymbol{\mu}) = \frac{1}{2^m} \operatorname{sym}_{\boldsymbol{\mu}} \left[\prod_{q=1}^{m} (\mu_{2q-1} - \mu_{2q})^2 \right] \qquad e_p(\boldsymbol{\mu}) = \sum_{1 \le \alpha_1 < \ldots < \alpha_p \le n} \mu_{\alpha_1} \ldots \mu_{\alpha_p}$$

$$\tilde{\Lambda}_{n,m}(\boldsymbol{\mu}) = \frac{m!}{n!(n-m)!} (-1)^m \sum_{p=0}^{2m} (-1)^p (n-p)!(n-2m+p)! e_p e_{2m-p}$$

3) take limit n -> 0 $\Lambda_{n,m}(\boldsymbol{\mu}) = \dfrac{\lambda_m(\boldsymbol{\mu})}{n} + O(n^0)$ $\qquad A_p = \sum_{j=1}^{n} \mu_j^p$

4) dominant term at large time $e_p^s(\mathbf{k}, \mathbf{m}) \longrightarrow (ic)^{p-1}(2^{2-p} - 1) B_{p-1} \mathcal{A}_1^s(\mathbf{k}, \mathbf{m})$

only one charge contributes $\langle (\mathcal{A}_1)^2 \rangle_n = n/(2t)$

remains to be done in progress ..

- obtain the bulk of the distribution of p (non-crossing proba)

- treat N > 2 non-crossing polymers

- extend generalized Bethe Ansatz calculations to m>1

Perspectives/other works

- replica BA method

Airy process

stationary KPZ Sasamoto Inamura $t \to \infty$ $A_2(y)$

2 space points $Prob(h(x_1,t),h(x_2,t))$ Prohlac-Spohn (2011), Dotsenko (2013)

2 times $Prob(h(0,t),h(0,t'))$ Dotsenko (2013)

endpoint distribution of DP Dotsenko (2012) Schehr, Quastel et al (2011)

- rigorous replica..

Borodin, Corwin, Quastel, O Neil, ..

q-TASEP $q \to 1$ avoids moment problem $\overline{Z^n} \sim e^{cn^3}$

WASEP Bose gas moments as nested contour integrals

- sine-Gordon FT

P. Calabrese, M. Kormos, PLD, EPL 10011 (2014)

- Lattice directed polymers

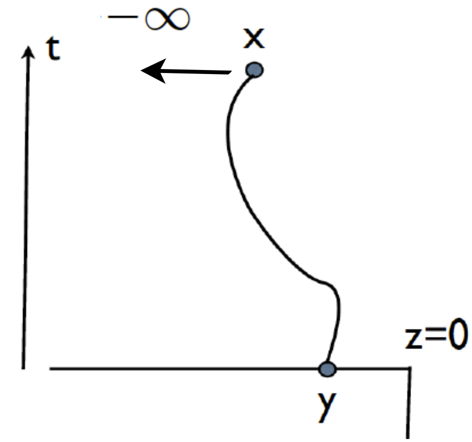
how to calculate $\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$

first method: flat as limit of half-flat (wedge)

$$\lim_{x\to-\infty, w\to 0} Z_{\rm hs}(x,t) \equiv Z_{\rm flat}(x,t)$$

$$Z_{\rm hs,w}(x,t) = \int_{-\infty}^{0} dy e^{wy} Z(x,t|y,0)$$

$$Z(x,t=0) = \theta(-x) e^{wx}$$



$$\Big(\prod_{\alpha=1}^{n} \int_{-\infty}^{0} dy_\alpha e^{wy_\alpha}\Big) \Psi_\mu(y_1\dots y_n) = \sum_P A_P G_{P\lambda}$$

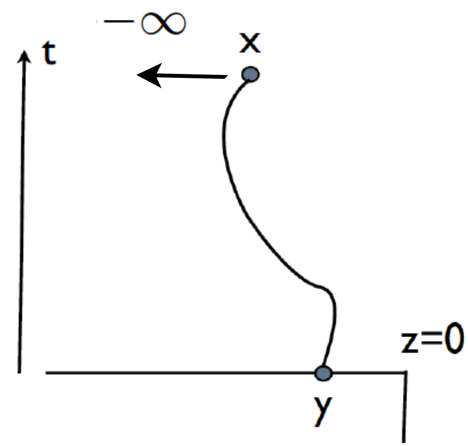
$$\Psi_\mu = \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P_\ell} x_\ell}$$

$$G_\lambda = \prod_{j=1}^{n} \frac{1}{jw + i\lambda_1 + .. + i\lambda_j}$$

how to calculate $\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$

first method: flat as limit of half-flat (wedge)

$$\lim_{x\to-\infty, w\to 0} Z_{\text{hs}}(x,t) \equiv Z_{\text{flat}}(x,t)$$



$$Z_{\text{hs,w}}(x,t) = \int_{-\infty}^{0} dy e^{wy} Z(x,t|y,0)$$

$$Z(x,t=0) = \theta(-x)e^{wx}$$

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P_\ell} x_\ell}$$

$$\Big(\prod_{\alpha=1}^n \int_{-\infty}^0 dy_\alpha e^{wy_\alpha}\Big)\Psi_\mu(y_1\dots y_n) = \sum_P A_P G_{P\lambda}$$

$$G_\lambda = \prod_{j=1}^n \frac{1}{jw + i\lambda_1 + .. + i\lambda_j}$$

miracle !

$$= \frac{n!}{\prod_{\alpha=1}^n (w+i\lambda_\alpha)} \prod_{1\le\alpha<\beta\le n} \frac{2w + i\lambda_\alpha + i\lambda_\beta - 1}{2w + i\lambda_\alpha + i\lambda_\beta}$$

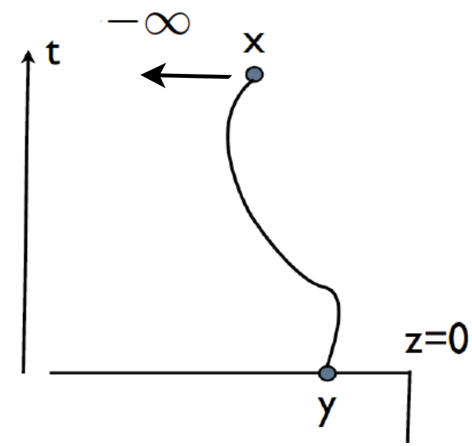
how to calculate $\int dy_1..dy_n \Psi_\mu(y_1,..y_n)$

first method: flat as limit of half-flat (wedge)

$$\lim_{x\to-\infty, w\to 0} Z_{\rm hs}(x,t) \equiv Z_{\rm flat}(x,t)$$

$$Z_{\rm hs,w}(x,t) = \int_{-\infty}^{0} dy e^{wy} Z(x,t|y,0)$$

$$Z(x,t=0) = \theta(-x) e^{wx}$$



$$\Psi_\mu = \sum_P A_P \prod_{j=1}^n e^{i\lambda_{P_\ell} x_\ell}$$

$$\Big(\prod_{\alpha=1}^n \int_{-\infty}^0 dy_\alpha e^{wy_\alpha}\Big)\Psi_\mu(y_1\dots y_n) = \sum_P A_P G_{P\lambda}$$

$$G_\lambda = \prod_{j=1}^n \frac{1}{jw + i\lambda_1 + .. + i\lambda_j}$$

miracle !

$$= \frac{n!}{\prod_{\alpha=1}^n (w+i\lambda_\alpha)} \prod_{1\le\alpha<\beta\le n} \frac{2w+i\lambda_\alpha+i\lambda_\beta-1}{2w+i\lambda_\alpha+i\lambda_\beta}$$

strings: $\lambda^{j,a} = k_j + \frac{i\bar{c}}{2}(j+1-2a)$

$$a = 1, ..., m_j$$

$$\int^w \Psi_\mu = n!(-2)^n \prod_{i=1}^{n_s} S^w_{m_i,k_i} \prod_{1\le i<j\le n_s} D^w_{m_i,k_i,m_j,k_j}$$

$$D^w_{m_1,k_1,m_2,k_2} = \quad (-1)^{m_2} \frac{\Gamma(1-z+\frac{m_1+m_2}{2})\Gamma(z+\frac{m_1-m_2}{2})}{\Gamma(1-z+\frac{m_1-m_2}{2})\Gamma(z+\frac{m_1+m_2}{2})} \qquad z = ik_1 + ik_2 + 2w$$

$$S^w_{m,k} = \frac{(-1)^m \Gamma(z)}{\Gamma(z+m)} \qquad z = 2ik + 2w.$$

in double limit $\displaystyle\lim_{x\to-\infty, w\to 0}$

$$S^w_{m_i,k_i} \to \frac{(-1)^{m_i}}{2\Gamma(m_i)} 2\pi\delta(k_i) + s^0_{m_i,k_i}$$

expand the product $\prod_i S_i \prod_{i<j} D_{ij}$

each momentum k_ℓ appears only in exactly one pole

$$D^w_{m_i,k_i,m_j,k_j} \to (-1)^{m_i} m_i \delta_{m_i,m_j}\; 2\pi\delta(k_i+k_j) + d^w_{m_i,k_i,m_j,k_j}$$

pairing of string momenta and pfaffian structure emerges second method:

calculate: $$\prod_{\alpha=1}^{n}\int_0^L dy_\alpha \Psi_\mu(y_1,..,y_n) = \langle \Phi_0|\mu\rangle$$

use Bethe equations: $e^{i\lambda_j L} = \prod_{\ell\neq j}\frac{\lambda_j-\lambda_\ell-i\bar{c}}{\lambda_j-\lambda_\ell+i\bar{c}}$

is the overlap with uniform state

$$\Phi_0(x_1,..x_n) = 1$$

=> integral vanishes for generic state

oberve: requires pairs opposite rapidities

Can be seen as interaction quench in Lieb-Liniger model with initial state BEC (c=0)

de Nardis et al., arXiv 1308.4310

overlap is non zero only for parity invariant states $\{\lambda_1,-\lambda_1,..,\lambda_{n/2},-\lambda_{n/2}\}$

$$\langle\Phi_0|\mu\rangle = n!c^{n/2}\prod_{\alpha=1}^{n/2}\frac{1}{\lambda_\alpha^2}\prod_{1\leq\alpha<\beta\leq n/2}\frac{(\lambda_\alpha-\lambda_\beta)^2+c^2}{(\lambda_\alpha-\lambda_\beta)^2}\frac{(\lambda_\alpha+\lambda_\beta)^2+c^2}{(\lambda_\alpha+\lambda_\beta)^2}\times\det G^Q.$$

$$G^Q_{\alpha\beta} = \delta_{\alpha\beta}(L+\sum_{\gamma=1}^{n/2}K^Q(\lambda_\alpha,\lambda_\gamma)) - K^Q(\lambda_\alpha,\lambda_\beta)$$

$$K^Q(x,y) = K(x-y)+K(x+y),$$

$$K(x) = \frac{2c}{x^2+c^2}.$$

Brockmann, arXiv1402.1471.

P. Calabrese, P. Le Doussal, arXiv 1402.1278

large L limit, overlap for strings

partially recovers the moments Z^n for flat