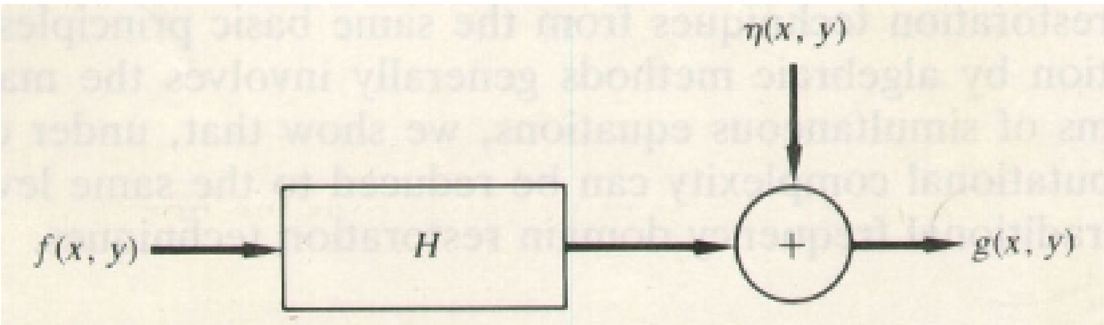
Chapter 5. Image Restoration

- A process to reconstruct or recover on image that has been degraded by using some a prior knowledge of the degradation phenomenon
- Modeling of degradation
  Inverse process
- Example: Removal of image blur by applying a deblurring function

5.1 Degradation Model



H : degradation operator
η : noise
f(x,y) : input image
g(x,y) : degraded output

- Restoration
  - the process obtaining an approximation to f(x,y), given g(x,y), H and η

5.1.1 Some Definitions

- $g(x,y) = H[f(x,y)] + \eta(x,y)$
  no noise case: $\eta(x,y)$ =0

- H : linear
  - $H[k_1 f_1 + k_2 f_2] = k_1 H[f_1] + k_2 H[f_2]$: additivity and homogeneity
  - position (or space) invariant
    $H[f(x-\alpha, y-\beta)] = g(x-\alpha, y-\beta)$

5.1.2 Degradation Model for Continuous Functions

- from the definition of impulse function

  $f(x,y)=\iint_{-\infty}^{\infty} f(\alpha,\beta)\delta(x-\alpha,y-\beta)d\alpha d\beta$; shifting property

  if $\eta(x,y)=0$

  $g(x,y)=H\left[\iint_{-\infty}^{\infty} f(\alpha,\beta)\delta(x-\alpha,y-\beta)d\alpha d\beta\right]$

  from additive property

  $$=\iint_{-\infty}^{\infty} H\left[f(\alpha,\beta)\delta(x-\alpha,y-\beta)\right]d\alpha d\beta$$

  from homogeneity property

  $$=\iint_{-\infty}^{\infty} f(\alpha,\beta)H\left[\delta(x-\alpha,y-\beta)\right]d\alpha d\beta$$

- Impulse response of H

  $H\left[\delta(x-\alpha,y-\beta)\right]=h(x,\alpha,y,\beta)$

  $= h(x-\alpha,y-\beta)$: position invariant

- General model with noise

  $g(x,y)=\iint_{-\infty}^{\infty} f(\alpha,\beta)h(x-\alpha,y-\beta)d\alpha d\beta+\eta(x,y)$

  noise : independent of position in image

5.1.3 Discrete Formulation

- **Discrete, space-invariant degradation model**

- 1-D sampled function

  f(x) : x=0, 1, 2, … , A-1

  g(x): x=0, 1, 2, … , B-1

- Convolution:
  - period of sampled function : M

    to avoid the overlap in convolution

    $M \ge A+B-1$; zero paddings
  - Extending the function with zero $\rightarrow f_e(x), h_e(x)$

$$g_e(x) = \sum_{m=0}^{M-1} f_e(m) h_e(x-m) \qquad \text{for } x=0, 1, 2, \ldots, M-1$$

- in matrix form ;

$$\mathbf{g} = \mathbf{H}\mathbf{f}$$

where
$$\mathbf{g} = \begin{bmatrix} g_e(0) \\ g_e(1) \\ \vdots \\ g_e(M-1) \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} f_e(0) \\ f_e(1) \\ \vdots \\ f_e(M-1) \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} h_e(0) & h_e(-1) & \cdots & h_e(-M+1) \\ h_e(1) & h_e(0) & \cdots & h_e(-M+2) \\ h_e(2) & h_e(1) & \cdots & h_e(-M+3) \\ \vdots & \vdots & \ddots & \vdots \\ h_e(M-1) & h_e(M-2) & \cdots & h_e(0) \end{bmatrix}$$; circulant matrix

- By periodicity assumption of $h_e(x)$

$h_e(x) = h_e(M+x)$; modulo M

$$\mathbf{H} = \begin{bmatrix} h_e(0) & h_e(M-1) & \cdots & h_e(1) \\ h_e(1) & h_e(0) & \cdots & h_e(2) \\ h_e(2) & h_e(1) & \cdots & h_e(3) \\ \vdots & \vdots & \ddots & \vdots \\ h_e(M-1) & h_e(M-2) & \cdots & h_e(0) \end{bmatrix}$$

- rows : circular shift to the right

ex.) if A=4 and B=3, M=6

In this case **f** and **g** are 6-D vectors and **H** is the 6 × 6 matrix

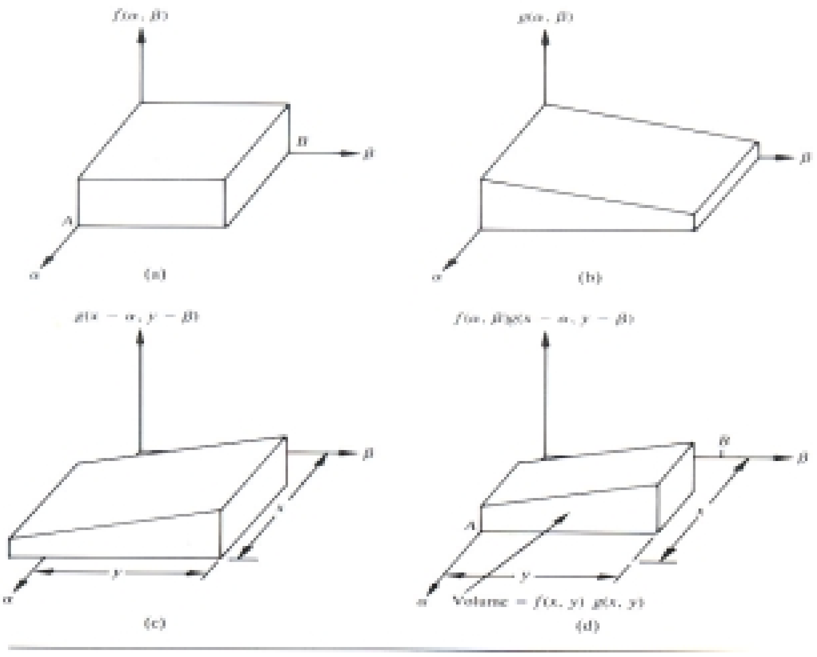
$$\mathbf{H} = \begin{bmatrix} h_e(0) & h_e(5) & h_e(4) & \cdots & h_e(1) \\ h_e(1) & h_e(0) & h_e(5) & \cdots & h_e(2) \\ h_e(2) & h_e(1) & h_e(0) & \cdots & h_e(3) \\ \vdots & & & & \\ h_e(5) & h_e(4) & h_e(3) & \cdots & h_e(0) \end{bmatrix}$$

However, as $h_e(x) = 0$ for $x = 3, 4, 5$, and $h_e(x) = h(x)$ for $x = 0, 1, 2$,

$$\mathbf{H} = \begin{bmatrix} h(0) & & & & h(2) & h(1) \\ h(1) & h(0) & & & & h(2) \\ h(2) & h(1) & h(0) & & & \\ & h(2) & h(1) & h(0) & & \\ & & h(2) & h(1) & h(0) & \\ & & & h(2) & h(1)) & h(0) \end{bmatrix}$$

where all elements not indicated in the matrix are zero.

- 2-D case : straightforward extension of 1-D case
  - f(x,y) : A×B
    h(x,y) : C×D



- Extended image of size M×N by zero padding

$$f_e(x,y)=\begin{cases} f(x,y) & 0\le x\le A-1 \quad and \quad 0\le y\le B-1 \\ 0 & A\le x\le M-1 \quad or \quad B\le y\le N-1 \end{cases}$$

$$h_e(x,y)=\begin{cases} h(x,y) & 0\le x\le C-1 \quad and \quad 0\le y\le D-1 \\ 0 & C\le x\le M-1 \quad or \quad D\le y\le N-1 \end{cases}$$

where $M\ge A+C-1$, $N\ge B+D-1$

- Convolution, in general ;

$$g_e(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f_e(m,n)h_e(x-m,y-n)+\eta_e(x,y)$$

for x=0, 1, … , M-1
y=0, 1, … , N-1

- in matrix form ;

$\mathbf{g}=\mathbf{Hf}+\mathbf{n}$; linear, space-invariant degradation process

where $\mathbf{f}, \mathbf{g}, \mathbf{n}$: MN-dimensional column vectors

✓ the first N elements of **f** : the element in the first row of $f_e(x,y)$

the second N elements of **f** : the element in the second row of $f_e(x,y)$

$$\mathbf{g} = \begin{bmatrix} g_e(0,0) \\ g_e(0,1) \\ \vdots \\ g_e(0,N-1) \\ g_e(1,0) \\ g_e(1,1) \\ \vdots \\ g_e(1,N-1) \\ \vdots \\ g_e(M-1,0) \\ g_e(M-1,1) \\ \vdots \\ g_e(M-1,N-1) \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} f_e(0,0) \\ f_e(0,1) \\ \vdots \\ f_e(0,N-1) \\ f_e(1,0) \\ f_e(1,1) \\ \vdots \\ f_e(1,N-1) \\ \vdots \\ f_e(M-1,0) \\ f_e(M-1,1) \\ \vdots \\ f_e(M-1,N-1) \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} H_0 & H_{M-1} & \cdots & H_1 \\ H_1 & H_0 & \cdots & H_2 \\ H_2 & H_1 & \cdots & H_3 \\ \vdots & \vdots & \ddots & \vdots \\ H_{M-1} & H_{M-2} & \cdots & H_0 \end{bmatrix}$$: MN×MN matrix, block circulant matrix

where $$\mathbf{H_j} = \begin{bmatrix} h_e(j,0) & h_e(j,N-1) & \cdots & h_e(j,1) \\ h_e(j,1) & h_e(j,0) & \cdots & h_e(j,2) \\ h_e(j,2) & h_e(j,1) & \cdots & h_e(j,3) \\ \vdots & \vdots & \ddots & \vdots \\ h_e(j,N-1) & h_e(j,N-2) & \cdots & h_e(j,0) \end{bmatrix}$$

- computational complexity
  : reduced considerably by taking advantage of the circulant properties of **H**

5.2 Diagonalization of circulant and block-circulant matrices

5.2.1 circulant matrices

For MxM circulant matrix :

$$\mathbf{H} = \begin{bmatrix} h_e(0) & h_e(M-1) & \Lambda & h_e(1) \\ h_e(1) & h_e(0) & \Lambda & h_e(2) \\ h_e(2) & h_e(1) & \Lambda & h_e(3) \\ \mathrm{M} & \mathrm{M} & \mathrm{O} & \mathrm{M} \\ h_e(M-1) & h_e(M-2) & \Lambda & h_e(0) \end{bmatrix}$$

- define a scalar

$$\lambda(k) = h_e(0) + h_e(M-1)\exp\left[j\frac{2\pi}{M}k \right] + h_e(M-2)\exp\left[j\frac{2\pi}{M}2k \right]\Lambda + h_e(1)\exp\left[j\frac{2\pi}{M}(M-1)k \right]$$

- define a vector ;

$$\mathbf{w}(k) = \begin{bmatrix} 1 \\ \exp\left[j\frac{2\pi}{M}k \right] \\ \exp\left[j\frac{2\pi}{M}2k \right] \\ \mathrm{M} \\ \exp\left[j\frac{2\pi}{M}(M-1)k \right] \end{bmatrix}$$

for k=0, 1, … , M-1

- $\mathbf{Hw}(k) = \lambda(k)\mathbf{w}(k)$;

where $\mathbf{w}(k)$: eigenvector of **H**

$\lambda(k)$: eigenvalue of **H**

$\lambda(k)$

- define MxM matrix **W ;**

$$\mathbf{W} = [\mathbf{w}(0)\ \ \mathbf{w}(1)\ \Lambda\ \ \mathbf{w}(M-1)]$$

ki-th element of **w**

$$w(k,i) = \exp\left[j\frac{2\pi}{M}ki \right]$$

ki-th element of inverse matrix $\mathbf{W}^{-1}$

$$w^{-1}(k,i) = \frac{1}{M}\exp\left[-j\frac{2\pi}{M}ki \right]$$

- $\mathbf{W}\mathbf{W}^{-1} = \mathbf{W}^{-1}\mathbf{W} = \mathbf{I}$

  because the column of **W** are linearly independent
- From elementary matrix theory

  $\mathbf{H} = \mathbf{W}\mathbf{D}\mathbf{W}^{-1}$

  $\Rightarrow \mathbf{D} = \mathbf{W}^{-1}\mathbf{H}\mathbf{W}$

  where $\mathbf{D}$: a diagonal matrix whose elements D(k,k) are the eigenvalue of $\mathbf{H}$

5.2.2 Block-Circulant Matrix

- $w_M \; i \; m \; = \quad \left[j\frac{2\pi}{M}im \right]$

  $$w_N(k,n) = \exp\left[j\frac{2\pi}{M}kn \right]$$

- $\mathbf{W}$: matrix of size MN×MN and containing M^2 partition of size N×N
- im-th partition of $\mathbf{W}$

  $\mathbf{W} \quad m) = \quad {}_M \quad m)\mathbf{W}_N$ for i=0, 1, … , M-1

  where $\mathbf{W}_N$ is NxN matrix with element of $W_N(k,n) = w_N(k,n)$ for k, n=0, 1, … , N-1

- for inverse matrix :matrix of size MN×MN and containing M^2 partition of size N×N

$$\mathbf{W}^{-1}(i,m)=\frac{1}{M}w^{-1}(i,m)\mathbf{W}_N^{-1}$$

where $\mathbf{W}_N^{-1}$ is NxN matrix with element of $W_N^{-1}(k,n)=\frac{1}{N}w_N^{-1}(k,n)$ for k, n=0, 1, … , N-1

and $w_N^{-1}\ i\ m\ =$ $\left[-j\frac{2\pi}{N}im\right]$ for i,m=0,1,…,N-1

- $\mathbf{W}\mathbf{W}^{-1}=\mathbf{W}^{-1}\mathbf{W}=\mathbf{I}$ (MNxMN matrix)

- $=\mathbf{W}\mathbf{D}\mathbf{W}^{-1}$

  $=\mathbf{W}^{-1}\mathbf{H}\mathbf{W}$

  where $\mathbf{D}$: a diagonal matrix whose elements D(k,k) are related to the discrete FT of $h_e(x,y)$

5.2.3 Effects of Diagonalization on the Degradation Model

● $=\mathbf{f}\quad=\qquad{}^{-1}\mathbf{f}$

$^{-1}\quad=\qquad{}^{-1}\mathbf{f}$

where $\mathbf{W}^{-1}\mathbf{f}$: M-dimensional column vector

k-th element is $F(k)=\frac{1}{M}\sum_{i=0}^{M-1}f_e(i)\exp\left[-j\frac{2\pi}{M}ki\right]$

for k=0, 1, … , M-1

$\mathbf{W}^{-1}\mathbf{f}$: Fourier Transform of $\mathbf{f}$ => F(k)

$^{-1}\mathbf{g}$: Fourier Transform of $\mathbf{g}$ => G(k)

- For eigenvalue :

$$\lambda\ k) = \ _e\ ) + \ _e\ \ -)\ \left[\ \frac{\pi}{}k\right] + \ _e\ \ -)\ \left[\ \frac{\pi}{}\ k\right]\Lambda + \ _e\ )\ \left[\ \frac{\pi}{}\ \ -)k\right]$$

- by $\exp\left[j\frac{2\pi}{M}(M-i)k\right] = \exp\left[-j\frac{2\pi}{M}ki\right]$

$$D(k,k) = \lambda(k) = \sum_{i=}^{M-1} h_e(i)\exp\left[-j\frac{2\pi}{M}ki\right] = MH(k)$$

for k= 0, 1, … , M-1

- in other words, 1-D discrete Formulation of degradation model is 1D Fourier transform

$G(k) = MH(k)F(k)$ for k=0, 1, … , M-1

5.3 Algebraic Approach to Restoration

5.3.1 Unconstrained Restoration

- in absence of any knowledge about $\mathbf{n}$

  $\mathbf{n} = \mathbf{g} - \mathbf{Hf}$; noise term in degradation model

- Euclidean Norm

  $\|\ \| = \|\ \ \mathbf{f}\ \ \| \longrightarrow \hat{\mathbf{f}}$

  by definition, squared norm.

  $\|\mathbf{n}\|^2 = \mathbf{n}^T\mathbf{n}, \|\mathbf{g} - \mathbf{Hf}\|^2 = (\mathbf{g} - \mathbf{H}\hat{\mathbf{f}})^T(\mathbf{g} - \mathbf{H}\hat{\mathbf{f}})$

- criterion function

  $\hat{\mathbf{f}}\ = \|\ \ \mathbf{Hf}\|^2$

  $\frac{\partial\ \ \hat{}\ )}{\partial\ \hat{}} = 0 = \mathbf{H}\ \ ^T\ \ \ \mathbf{H}\ \hat{}\ )$

  $\Rightarrow \hat{\mathbf{f}} = (\mathbf{H}^T\mathbf{H})\ \ \mathbf{H}^T\mathbf{g} = \mathbf{H}\ \ \ (\mathbf{H}^T)^{-1}\mathbf{H}^T\mathbf{g}$

- if **H** is square matrix

$$\Rightarrow \hat{\ } = \quad {}^{-1}\mathbf{g}$$

5.3.2 Constrained Restoration

- criterion function
  - least squares restoration as one of minimizing $\|\mathbf{Q}\hat{\mathbf{f}}\|^2$ subject to constraint

$$\|\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\|^2 = \|\mathbf{n}\|^2$$

  - Using Lagrange multiplier

$$\mathbf{J}(\hat{\mathbf{f}}) = \|\mathbf{Q}\hat{\mathbf{f}}\|^2 + \alpha \ \|\mathbf{g} - \mathbf{H}\mathbf{f}\|^2 - \|\mathbf{n}\|^2)$$

    where α : Lagrange multiplier

  - minimization

$$\frac{\partial J(\hat{\mathbf{f}})}{\partial \hat{\mathbf{f}}} = 0 = \mathbf{2Q}^T\mathbf{Q}\hat{\mathbf{f}} - \mathbf{2}\alpha\mathbf{H}^T(\mathbf{g} - \mathbf{H}\hat{\mathbf{f}})$$

$$\Rightarrow \hat{\mathbf{f}} = (\mathbf{H}^T\mathbf{H} + \gamma\mathbf{Q}^T\mathbf{Q})^{-1}\mathbf{Hg} \qquad \text{where } \gamma = 1/\alpha$$