Dispersive Navier-Stokes Systems for Gas Dynamics: Formal Derivations

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$\boxed{\textbf{Introduction}}$

Many physical systems of partial differential equations from physics that are dissipative have "corrections" that are dispersive. Examples include:

- so-called α-*models of turbulence* that have dispersive corrections proportional to a coefficient α,

- so-called dispersive Navier-Stokes systems that have dispersive corrections to classical Navier-Stokes systems of gas dynamics.

The later take two forms: those derived by ad-hoc continuum arguments and those derived from an underlying kinetic equation. Here we take the route of kinetic theory, which is a better foundation upon which to build.

Kinetic Setting

In a kinetic setting a gas of identical particles contained in a D-dimensional spatial domain Ω is described by a so-called kinetic density $F(v,x,t)$. For any subset A of the single-particle phase space $\mathbb{R}^D \times \Omega$, the mass of particles with velocity-position $(v,x) \in A$ at time t is given by

$$\iint_A F(v,x,t)\,\mathrm{d}v\,\mathrm{d}x\,.$$

The traditional gas dynamics variables of mass density $\rho(x,t)$, bulk velocity $u(x,t)$, and temperature $\theta(x,t)$ are recovered as

$$\rho(x,t) = \int_{\mathbb{R}^D} F(v,x,t)\,\mathrm{d}v\,, \quad \rho(x,t)u(x,t) = \int_{\mathbb{R}^D} v\,F(v,x,t)\,\mathrm{d}v\,,$$
$$\tfrac{1}{2}\rho(x,t)|u(x,t)|^2 + \tfrac{D}{2}\rho(x,t)\theta(x,t) = \int_{\mathbb{R}^D} \tfrac{1}{2}|v|^2 F(v,x,t)\,\mathrm{d}v\,.$$

These quantities are the densities of mass, momentum, and energy.

It is assumed the gas is sufficiently dilute that the potential energy between particles can be neglected.

Consider a Boltzmann-like collisional kinetic equation for $F(v, x, t)$ over a D-dimensional spatial domain:

$$\boxed{\partial_t F + v \cdot \nabla_x F = \frac{1}{\epsilon} \mathcal{C}(F) \,,}$$

where ϵ is the Knudsen number, which is small in fluid dynamical regimes. (Of course, $D = 3$ is usually the physical case!)

We assume that the collision operator $\mathcal{C}$ respects Galilean symmetry, locally conserves mass, momentum and energy, locally dissipates entropy, and has local Maxwellians as equilibria.

Local Conservation Laws

The collision operator $\mathcal{C}$ satisfies

$$\langle \mathbf{e}\, \mathcal{C}(F) \rangle = 0 \quad \text{for “every” } F \,,$$

where

$$\mathbf{e} = \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} \,, \quad \text{and} \quad \langle g \rangle = \int_{\mathbb{R}^D} g(v)\, \mathrm{d}v \,.$$

This yields the local conservation laws

$$\partial_t \langle \mathbf{e}\, F \rangle + \nabla_x \cdot \langle v\, \mathbf{e}\, F \rangle = 0 \,.$$

Local Entropy Dissipation Law

The collision operator $\mathcal{C}$ satisfies

$$\boxed{\langle \eta'(F)\,\mathcal{C}(F)\rangle \leq 0 \quad \text{for "every" } F\,,}$$

where $\eta(F) = F\log(F) - F$.

This yields the local entropy dissipation law

$$\boxed{\partial_t\langle \eta(F)\rangle + \nabla_x\cdot\,\langle v\,\eta(F)\rangle = \frac{1}{\epsilon}\,\langle \eta'(F)\,\mathcal{C}(F)\rangle \leq 0\,.}$$

Local Equilibra

For "every" F the following are equivalent:

(1) $\langle \eta'(F)\,\mathcal{C}(F)\rangle = 0\,;$

(2) $\mathcal{C}(F) = 0\,;$

(3) F has the Maxwellian form

$$F = \mathcal{E}(\boldsymbol{\rho}) = \frac{\rho}{(2\pi\theta)^{D/2}} \exp\left(-\frac{|v-u|^2}{2\theta} \right),$$

where $\boldsymbol{\rho} = \langle \mathbf{e}\, F \rangle = \begin{pmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho|u|^2 + \frac{D}{2}\rho\theta \end{pmatrix}.$

Fluid Dynamical Closures

We decompose F into its *local equilibrium* $\mathcal{E}$ and *deviation* $\widetilde{F}$ as

$$F = \mathcal{E} + \widetilde{F}\,, \quad \text{where } \mathcal{E} = \mathcal{E}(\boldsymbol{\rho}) \text{ with } \boldsymbol{\rho} = \langle \mathrm{e}\, F \rangle.$$

One sees that $\langle \mathrm{e}\, \mathcal{E} \rangle = \boldsymbol{\rho}$ and $\langle \mathrm{e}\, \widetilde{F} \rangle = 0$.

Expressed in terms of $\boldsymbol{\rho}$ and $\widetilde{F}$, the local conservation laws are

$$\partial_t \boldsymbol{\rho} + \nabla_x \cdot \langle v\, \mathbf{e}\, \mathcal{E} \rangle + \nabla_x \cdot \langle v\, \mathbf{e}\, \widetilde{F} \rangle = 0\,.$$

A fluid dynamical closure is specified by expressing $\widetilde{F}$ above in terms of $\boldsymbol{\rho}$ and its derivatives.

Fluid Dynamical System

The resulting fluid dynamical system takes the form

$$\begin{aligned}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0\,, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + pI + \widetilde{P}) &= 0\,, \\
\partial_t (\rho e) + \nabla_x \cdot (\rho e u + pu + \widetilde{P} u + \tilde{q}) &= 0\,,
\end{aligned}$$

where $p = \rho\theta$, $e = \frac{1}{2}|u|^2 + \frac{D}{2}\theta$, and

$$\widetilde{P} = \langle A\, \widetilde{F} \rangle\,, \qquad \tilde{q} = \langle B\, \widetilde{F} \rangle\,,$$

with A and B defined by

$$\begin{aligned}
A &= (v-u) \otimes (v-u) - \tfrac{1}{D}|v-u|^2 I\,, \\
B &= \tfrac{1}{2}|v-u|^2 (v-u) - \tfrac{D+2}{2}\theta (v-u)\,.
\end{aligned}$$

Euler Approximation

Maxwell first argued that in fluid regimes F should be near its local equilibrium $\mathcal{E}$. He then observed that $\widetilde{F} \approx 0$ yields the compressible Euler system. Its solutions formally dissipate the so-called Euler entropy $\langle \eta(\mathcal{E}) \rangle$ as

$$\boxed{\partial_t \langle \eta(\mathcal{E}) \rangle + \nabla_x \cdot \langle v \, \eta(\mathcal{E}) \rangle \leq 0 \, .}$$

If one wants to improve upon the Euler approximation, one needs a better approximation for $\widetilde{F}$.

Deviation Equation

The deviation $\widetilde{F}$ satisfies

$$\partial_t \widetilde{F} + \widetilde{\mathcal{P}} v \cdot \nabla_x \widetilde{F} + \widetilde{\mathcal{P}} v \cdot \nabla_x \mathcal{E} = \frac{1}{\epsilon} \mathcal{C}(\mathcal{E} + \widetilde{F}) \,,$$

where $\widetilde{\mathcal{P}} = \mathcal{I} - \mathcal{P}$ with $\mathcal{P}$ defined by

$$\mathcal{P}(\boldsymbol{\rho}) f = \mathcal{E}_{\boldsymbol{\rho}}(\boldsymbol{\rho}) \langle \mathrm{e}\, f \rangle \,.$$

One can show $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are orthogonal projections over $L^2(\mathcal{E}^{-1} \mathrm{d}v)$.

Navier-Stokes Approximation

Maxwell (1867) first argued that $\widetilde{F} \approx \widetilde{F}_{NS}$ where

$$\widetilde{\mathcal{P}} v \cdot \nabla_x \mathcal{E} = \frac{1}{\epsilon} D\mathcal{C}(\mathcal{E}) \widetilde{F}_{NS} \,.$$

This yields the compressible Navier-Stokes system. Its solutions formally dissipate the Euler entropy as

$$\partial_t \langle \eta(\mathcal{E}) \rangle + \nabla_x \cdot \langle v\, \eta(\mathcal{E}) \rangle + \nabla_x \cdot \langle v\, \eta'(\mathcal{E}) \widetilde{F}_{NS} \rangle = \frac{1}{\epsilon} \langle \eta''(\mathcal{E}) \widetilde{F}_{NS}\, D\mathcal{C}(\mathcal{E}) \widetilde{F}_{NS} \rangle \,.$$

Beyond Navier-Stokes Approximations

Hilbert (1912) introduced a derivation of the Navier-Stokes system based on a systematic expansion in the (small) Knudsen number ϵ. This so-called Hilbert expansion yields the Euler system at leading order, and corrections that satisfy linearized Euler systems driven by lower order terms. These have to be summed through order ϵ to obtain the Navier-Stokes system.

Short afterward, Enskog (1917) introduced a slightly different expansion in ϵ, subsequently dubbed the Chapman-Enskog expansion, that led directly to the Navier-Stokes system at order ϵ.

Both these approaches fail to systematically yield corrections to the Navier-Stokes system that are formally well-posed.

An Alternative Approach

We will not use either the Hilbert expansion or the Chapman-Enskog expansion. Rather, we consider three approximations:

(1) Small Deviation (a kind of linearization)

(2) Material-Frame Stationary Balance (a temporal approximation)

(3) Small Gradient Expansion (a spatial approximation)

All these approximations yield solutions that formally dissipate an entropy, and are therefore formally well-posed.

(1) Small Deviation Approximation

We argue that $\widetilde{F} \approx \widetilde{F}_{SD}$ where

$$\widetilde{\mathcal{A}}\widetilde{F}_{SD} + \widetilde{\mathcal{P}}v \cdot \nabla_x \mathcal{E} = \frac{1}{\epsilon} D\mathcal{C}(\mathcal{E})\widetilde{F}_{SD} \, ,$$

where $\widetilde{\mathcal{A}} = \widetilde{\mathcal{P}}\mathcal{A}\widetilde{\mathcal{P}}$ with $\mathcal{A}$ defined by

$$\mathcal{A}f = \frac{1}{2}\left[(\partial_t + v \cdot \nabla_x)f + \mathcal{E}\,(\partial_t + v \cdot \nabla_x)\frac{f}{\mathcal{E}}\right] \, .$$

The small deviation system dissipates an entropy as

$$\boxed{\begin{gathered}\partial_t\langle \eta(\mathcal{E})\rangle + \nabla_x \cdot \langle v\,\eta(\mathcal{E})\rangle + \nabla_x \cdot \langle v\,\eta'(\mathcal{E})\widetilde{F}_{SD}\rangle \\ +\partial_t\langle \tfrac{1}{2}\eta''(\mathcal{E})\widetilde{F}_{SD}^2\rangle + \nabla_x \cdot \langle v\,\tfrac{1}{2}\eta''(\mathcal{E})\widetilde{F}_{SD}^2\rangle = \frac{1}{\epsilon}\langle \eta''(\mathcal{E})\widetilde{F}_{SD}\, D\mathcal{C}(\mathcal{E})\widetilde{F}_{SD}\rangle \, .\end{gathered}}$$

Remark: Second Order Entropy Law

If one places $F = \mathcal{E} + \widetilde{F}$ into the terms of the exact entropy dissipation law and expand through second order in $\widetilde{F}$ one sees that

$$
\begin{gathered}
\langle \eta(\mathcal{E} + \widetilde{F}) \rangle = \langle \eta(\mathcal{E}) \rangle + \langle \tfrac{1}{2} \eta''(\mathcal{E}) \widetilde{F}^2 \rangle + \cdots \,, \\
\langle v\, \eta(\mathcal{E} + \widetilde{F}) \rangle = \langle v\, \eta(\mathcal{E}) \rangle + \langle v\, \eta'(\mathcal{E}) \widetilde{F} \rangle + \langle v\, \tfrac{1}{2} \eta''(\mathcal{E}) \widetilde{F}^2 \rangle + \cdots \,, \\
\langle \eta'(\mathcal{E} + \widetilde{F})\, \mathcal{C}(\mathcal{E} + \widetilde{F}) \rangle = \langle \eta''(\mathcal{E}) \widetilde{F}\, D\mathcal{C}(\mathcal{E}) \widetilde{F} \rangle + \cdots \,.
\end{gathered}
$$

Truncated at second order, the exact entropy dissipation law then becomes

$$
\boxed{
\begin{gathered}
\partial_t \langle \eta(\mathcal{E}) \rangle + \nabla_x \cdot \langle v\, \eta(\mathcal{E}) \rangle + \nabla_x \cdot \langle v\, \eta'(\mathcal{E}) \widetilde{F} \rangle \\
+ \partial_t \langle \tfrac{1}{2} \eta''(\mathcal{E}) \widetilde{F}^2 \rangle + \nabla_x \cdot \langle v\, \tfrac{1}{2} \eta''(\mathcal{E}) \widetilde{F}^2 \rangle = \frac{1}{\epsilon} \langle \eta''(\mathcal{E}) \widetilde{F}\, D\mathcal{C}(\mathcal{E}) \widetilde{F} \rangle \,.
\end{gathered}
}
$$

Remark: Relation to Linearization

When $\mathcal{E}$ is an exact solution of the kinetic equation then the small deviation approximation is simply the linearization of that equation about $\mathcal{E}$. Indeed, one sees that

$$\begin{aligned}\mathcal{E}\text{ is a solution of the kinetic equation } &\iff \big(\partial_t + v\cdot\nabla_x\big)\mathcal{E} = 0\\ &\iff \mathcal{A}f = \partial_t f + v\cdot\nabla_x f\,.\end{aligned}$$

The small deviation approximation is thereby $\widetilde{F} \approx \widetilde{F}_{SD}$ where

$$\partial_t \widetilde{F}_{SD} + v\cdot\nabla_x \widetilde{F}_{SD} = \frac{1}{\epsilon} D\mathcal{C}(\mathcal{E})\widetilde{F}_{SD}\,.$$

In this case $\widetilde{F}_{SD}$ contains only initial and boundary layers.

(2) Material-Frame Stationary Balance Approximation

We argue that $\widetilde{F} \approx \widetilde{F}_{MS}$ where

$$\widetilde{\mathcal{A}}_{MS}\widetilde{F}_{MS} + \widetilde{\mathcal{P}}v \cdot \nabla_x \mathcal{E} = \frac{1}{\epsilon} D\mathcal{C}(\mathcal{E})\widetilde{F}_{MS} \,,$$

where $\widetilde{\mathcal{A}}_{MS} = \widetilde{\mathcal{P}}\mathcal{A}_{MS}\widetilde{\mathcal{P}}$ with $\mathcal{A}_{MS}$ defined by

$$\mathcal{A}_{MS}f = \frac{1}{2}\left[(v-u)\cdot\nabla_x f + \mathcal{E}\,\nabla_x\cdot\frac{(v-u)f}{\mathcal{E}}\right] \,.$$

The resulting system dissipates the Euler entropy as

$$\boxed{\begin{aligned} &\partial_t\langle\eta(\mathcal{E})\rangle + \nabla_x\cdot\langle v\,\eta(\mathcal{E})\rangle + \nabla_x\cdot\langle v\,\eta'(\mathcal{E})\widetilde{F}_{MS}\rangle \\ &\qquad + \nabla_x\cdot\langle(v-u)\,\tfrac{1}{2}\eta''(\mathcal{E})\widetilde{F}_{MS}^2\rangle = \frac{1}{\epsilon}\langle\eta''(\mathcal{E})\widetilde{F}_{MS}\,D\mathcal{C}(\mathcal{E})\widetilde{F}_{MS}\rangle \,. \end{aligned}}$$

(3) Small Gradient Expansion

The equation for $\widetilde{F}_{MS}$ can be written as

$$DC(\mathcal{E})\widetilde{F}_{MS} - \epsilon\,\widetilde{\mathcal{A}}_{MS}\widetilde{F}_{MS} = \epsilon\,\widetilde{\mathcal{P}}v \cdot \nabla_x \mathcal{E}\ ,$$

which has a formal solution by Neumann series

$$\boxed{\begin{aligned} \widetilde{F}_{MS} &= \epsilon\, DC(\mathcal{E})^{-1} v \cdot \nabla_x \mathcal{E} \\ &\quad + \epsilon^2 DC(\mathcal{E})^{-1} \mathcal{A}_{MS} DC(\mathcal{E})^{-1} v \cdot \nabla_x \mathcal{E} \\ &\quad + \cdots\ . \end{aligned}}$$

Truncations of this series at orders 0, 1, 4, 5, ... in $\mathcal{A}_{MS}$ lead to fluid dynamical closures that dissipate the Euler entropy.

These closures have spatial derivatives of orders 2, 3, 6, 7, ...

Fluid Dynamical Systems Recalled

The recall that a fluid dynamical system takes the form

$$
\begin{aligned}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0\,, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + pI + \widetilde{P}) &= 0\,, \\
\partial_t (\rho e) + \nabla_x \cdot (\rho e u + p u + \widetilde{P} u + \tilde{q}) &= 0\,,
\end{aligned}
$$

where $p = \rho\theta$, $e = \frac{1}{2}|u|^2 + \frac{D}{2}\theta$, and

$$
\widetilde{P} = \langle A\, \widetilde{F} \rangle\,, \qquad \tilde{q} = \langle B\, \widetilde{F} \rangle\,,
$$

with A and B defined by

$$
\begin{aligned}
A &= (v-u) \otimes (v-u) - \tfrac{1}{D}|v-u|^2 \delta\,, \\
B &= \tfrac{1}{2}|v-u|^2 (v-u) - \tfrac{D+2}{2}\theta (v-u)\,.
\end{aligned}
$$

Navier-Stokes Closure - 1

The Navier-Stokes closure takes the form

$$\widetilde{F}_{NS} = -\frac{1}{\theta}\,\mathcal{E}\widehat{A} : \nabla_x u - \frac{1}{\theta^2}\,\mathcal{E}\widehat{B} \cdot \nabla_x \theta\,,$$

where $\widehat{A}$ and $\widehat{B}$ solve

$$D\mathcal{C}(\mathcal{E})\mathcal{E}\widehat{A} = \mathcal{E}A\,, \qquad D\mathcal{C}(\mathcal{E})\mathcal{E}\widehat{B} = \mathcal{E}B\,.$$

By using Galilean symmetry it can be shown that

$$\widehat{A} = \tau_A A\,, \qquad \widehat{B} = \tau_B B\,,$$

where $\tau_A = \tau_A(|v-u|/\sqrt{\theta}, \rho, \theta)$ and $\tau_B = \tau_B(|v-u|/\sqrt{\theta}, \rho, \theta)$ are positive and have units of time.

Navier-Stokes Closure - 2

One finds that

$$\widetilde{P} = \epsilon\, \widetilde{P}_{NS}\,, \qquad \tilde{q} = \epsilon\, \tilde{q}_{NS}\,,$$

where

$$\widetilde{P}_{NS} = -\mu[\nabla_x u + (\nabla_x u)^T - \tfrac{2}{D}\nabla_x \cdot u\delta]\,, \qquad \tilde{q}_{NS} = -\kappa\nabla_x\theta\,,$$

with

$$\mu(\rho,\theta) = \rho\theta\,\frac{\langle\tau_A\,|A|^2\mathcal{E}\rangle}{\langle|A|^2\mathcal{E}\rangle}\,,$$

$$\kappa(\rho,\theta) = \tfrac{D+2}{2}\rho\theta\,\frac{\langle\tau_B\,|B|^2\mathcal{E}\rangle}{\langle|B|^2\mathcal{E}\rangle}\,.$$

First Correction to Navier-Stokes - 1

The first correction to the Navier-Stokes system is obtained as

$$\widetilde{P} = \epsilon\, \widetilde{P}_{NS} + \epsilon^2 \widetilde{P}_{FC} + \cdots\,, \qquad \tilde{q} = \epsilon\, \tilde{q}_{NS} + \epsilon^2 \tilde{q}_{FC} + \cdots\,,$$

where $\widetilde{P}_{NS}$ and $\tilde{q}_{NS}$ are as before, while

$$\widetilde{P}_{FC} = \widetilde{P}_{FC}^{AA} + \widetilde{P}_{FC}^{AB}\,, \qquad \tilde{q}_{FC} = \tilde{q}_{FC}^{BA} + \tilde{q}_{FC}^{BB}\,,$$

with

$$\begin{aligned} \widetilde{P}_{FC}^{AA} &= \tfrac{1}{2}\Big\langle \tau_A^2\, A\, \mathcal{E}(v-u)\nabla_x A \Big\rangle : \frac{\nabla_x u}{\theta} \\ &\quad - \tfrac{1}{2}\left\langle \tau_A^2\, A : \frac{\nabla_x u}{\theta}\, \mathcal{E}(v-u)\cdot\nabla_x A \right\rangle, \end{aligned}$$

First Correction to Navier-Stokes - 2

$$
\begin{aligned}
\widetilde{P}_{FC}^{AB} = \tfrac{1}{2}\nabla_x \cdot \left[\left\langle \tau_A \tau_B \,(v-u)\,\mathcal{E}\, A\, B \right\rangle \cdot \frac{\nabla_x \theta}{\theta^2} \right] \\
+ \tfrac{1}{2} \left\langle \tau_A \tau_B \, A\, B\, \mathcal{E} (v-u) \right\rangle : \nabla_x \left(\frac{\nabla_x \theta}{\theta^2} \right) \\
+ \tfrac{1}{2} \left\langle \tau_A \tau_B \, A\, \mathcal{E}\, (v-u) \cdot \nabla_x B \right\rangle \cdot \frac{\nabla_x \theta}{\theta^2} \,,
\end{aligned}
$$

First Correction to Navier-Stokes - 3

$$
\begin{aligned}
\tilde{q}_{FC}^{BA} = \tfrac{1}{2}\nabla_x \cdot \left[\Big\langle \tau_B \tau_A \, (v-u)\, \mathcal{E}\, B\, A \Big\rangle : \frac{\nabla_x u}{\theta} \right] \\
+ \tfrac{1}{2} \Big\langle \tau_B \tau_A \, B\, A\, \mathcal{E} (v-u) \Big\rangle \cdot \nabla_x \left(: \frac{\nabla_x u}{\theta} \right) \\
+ \tfrac{1}{2} \Big\langle \tau_B \tau_A \, A \cdot \nabla_x \theta \, \mathcal{E}\, A \Big\rangle : \frac{\nabla_x u}{\theta} \,,
\end{aligned}
$$

$$
\tilde{q}_{FC}^{BB} = \tfrac{1}{2} \Big\langle \tau_B^2 B\, \mathcal{E}\, B \cdot \nabla_x u \Big\rangle \cdot \frac{\nabla_x \theta}{\theta^2} - \tfrac{1}{2} \Big\langle \tau_B^2 B \cdot \nabla_x u \, \mathcal{E}\, B \Big\rangle \cdot \frac{\nabla_x \theta}{\theta^2} \,.
$$

First Correction to Navier-Stokes - 4

There are four new scalar transport coefficients — one that averages τ_A^2, one that averages τ_B^2, and two that average $\tau_A \tau_B$.

It is easily checked that

$$\widetilde{P}_{FC} : \frac{\nabla_x u}{\theta} + \tilde{q}_{FC} \cdot \frac{\nabla_x \theta}{\theta^2} = \text{divergence}\,,$$

so the new terms formally conserve the entropy.

Linearized First Correction to Navier-Stokes

If the above equations are linearized about the homogeneous state with unit density, zero velocity, and unit temperature, the corresponding fluctuations satisfy

$$
\begin{aligned}
\partial_t \rho + \nabla_x \cdot u &= 0\,, \\
\partial_t u + \nabla_x(\rho + \theta) &= \mu\Big[\Delta_x u + \tfrac{D-2}{D}\nabla_x(\nabla_x \cdot u)\Big] - \eta \nabla_x \Delta_x \theta\,, \\
\tfrac{D}{2}\partial_t \theta + \nabla_x \cdot u &= \kappa \Delta_x \theta - \eta \Delta_x (\nabla_x \cdot u)\,,
\end{aligned}
$$

where

$$
\mu = \frac{\langle \tau_A |v|^4 \mathcal{E}\rangle}{\langle |v|^4 \mathcal{E}\rangle}\,, \qquad \kappa = \frac{D+2}{2}\,\frac{\langle \tau_B |B|^2 \mathcal{E}\rangle}{\langle |B|^2 \mathcal{E}\rangle}\,,
$$

$$
\eta = 2\frac{\langle \tau_A \tau_B (\frac{1}{2}|v|^2 - \frac{D+2}{2})|v|^4 \mathcal{E}\rangle}{\langle (\frac{1}{2}|v|^2 - \frac{D+2}{2})|v|^4 \mathcal{E}\rangle}\,.
$$

Conclusion: Some Open Questions

- What are the correct boundary conditions for these dispersive systems?

- Are these dispersive systems an improvement? This can be investigated numerically on periodic domains.

- Does one gain more regularity from the additional dispersive terms than one would expect from the dissipative terms alone? This can be asked even for the linear system on the previous slide.

- What are good local well-posedness results for classical solutions?