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Topología Algebraica de Espacios Topológicos Finitos y Aplicaciones

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Resumen

El objetivo principal de esta Tesis es estudiar y profundizar el desarrollo de la teoría de espacios topológicos finitos e investigar sus aplicaciones a la teoría de homotopía y homotopía simple de poliedros y espacios topológicos más generales. Utilizamos en particular varios de los resultados obtenidos para analizar dos conjeturas abiertas muy importantes de topología algebraica y geométrica: La conjetura de Quillen sobre el poset de p-subgrupos de un grupo finito y la conjetura geométrica de Andrews-Curtis.

Los tipos homotópicos de espacios finitos pueden ser descriptos a través de movimientos elementales que consisten en agregar o quitar un tipo especial de puntos a los espacios, llamados *beat points*. Por otro lado, es más importante comprender los tipos homotópicos débiles de espacios finitos, ya que estos se corresponden con los tipos homotópicos de los poliedros asociados. Un acercamiento a la resolución de este problema viene dado por los puntos que denominamos *weak points*. Estos puntos dan lugar a una noción de colapso entre espacios finitos que se corresponde exactamente con el concepto de colapso simplicial de los complejos asociados. De este modo obtenemos una correspondencia entre los tipos homotópicos simples de espacios finitos y los de complejos simpliciales finitos. Este resultado fundamental nos permite estudiar problemas geométricos conocidos desde una nueva óptica, utilizando toda la maquinaria combinatoria y topológica propia de los espacios finitos.

La conjetura de Quillen sobre el poset de p-subgrupos investiga la relación entre las propiedades algebraicas de un grupo finito y las propiedades topológicas de un poliedro asociado al grupo. Por medio de nuestros resultados, veremos que esta conjetura puede ser reformulada y analizada en términos puramente topológicos, utilizando homotopía simple equivariante.

La conjetura de Andrews-Curtis es una de las conjeturas más importantes de topología geométrica y está muy relacionada con la conjetura de Zeeman, y, por lo tanto, con la conjetura de Poincaré. Como consecuencia de la demostración de Perelman de la conjetura de Poincaré, se deduce que esta conjetura es verdadera para ciertos complejos, llamados *standard spines*, pero el problema todavía permanece abierto para los poliedros de dimensión 2 en general. Utilizando los resultados desarrollados en esta Tesis extenderemos sustancialmente la clase de complejos para los cuales la conjetura se sabe cierta.

Palabras clave: Espacios toplógicos finitos, complejos simpliciales, tipos homotópicos, equivalencias débiles, homotopía simple, colapsos.

Algebraic Topology of Finite Topological Spaces and Applications

Abstract

The main goal of this Thesis is to study and to delve deeper into the development of the theory of finite spaces and to investigate their applications to the homotopy theory and simple homotopy theory of polyhedra and general topological spaces. We use, in particular, some of the results that we obtain, to analize two important open conjectures of algebraic and geometric topology: Quillen's conjecture on the poset of p-subgroups of a group and the geometric Andrews-Curtis conjecture.

Homotopy types of finite spaces can be described through elemental moves which consist in adding or removing a particular kind of points from the spaces, called *beat points*. On the other hand, it is more important to understand the weak homotopy types of finite spaces, since they correspond to the homotopy types of the associated polyhedra. One step in this direction is given by the points that we called *weak points*. These points lead to a notion of collapse of finite spaces which corresponds exactly to the concept of simplicial collapse of the associated simplicial complexes. In this way we obtain a correspondence between simple homotopy types of finite spaces and of simplicial complexes. This fundamental result allows us to study well-known geometrical problems from a new point of view, using all the combinatorial and topological machinery proper of finite spaces.

Quillen's conjecture on the poset of p-subgroups of a group investigates the relationship between algebraic properties of a finite group and topological properties of a polyhedron associated to the group. As an application of our results, we will see that this conjecture can be restated and analized in purely topological terms, using equivariant simple homotopy theory.

The Andrews-Curtis conjecture is one of the most important conjectures in geometric topology and it is closely related to Zeeman's conjecture, and, therefore, to Poincaré conjecture. As a consequence of Perelman's proof of Poincaré conjecture, one deduces that this conjecture is true for some complexes called *standard spines*, but the problem is still open for general polyhedra of dimension 2. With the results developed in this Thesis we substantially extend the class of complexes for which the conjecture is known to be true.

Key words: Finite topological spaces, simplicial complexes, homotopy types, weak homotopy equivalences, simple homotopy, collapses.

Introduction

Topology allows to handle structures more flexible than metric spaces, however, most of the spaces studied in Algebraic Topology, such as CW-complexes or manifolds, are Hausdorff spaces. In contrast, finite topological spaces are rarely Hausdorff: a topological space with finitely many points, each of which is closed, must be discrete. Mathematically speaking, finite spaces are in many senses more natural than CW-complexes. Their combinatorics and their apparent simplicity, make them attractive and tractable, as much as finite partially ordered sets are, but it is the conjuction between their combinatorial and topological structures what makes them so fascinating and useful. At first glance, one could think that such spaces with a finite number of points and non Hausdorff are uninteresting, but we will see that the theory of finite spaces can be used to investigate deep known problems in Topology, Algebra and Geometry.

In 1937, P.S. Alexandroff [1] described the combinatorics of finite spaces, comparing it with the one of finite partially ordered sets (posets). He proved that finite spaces and finite posets are essentially the same objects considered from different points of view. However, it was not until 1966 that strong and deep results on the homotopy theory of finite spaces appeared, shaped in the two foundational and independent papers [37] and [26]. R. E. Stong [37] used the combinatorics of finite spaces to explain their homotopy types. This astounding article would have probably gone unnoticed if in the same year, M.C. McCord had not discovered the relationship between finite spaces and compact polyhedra. Given a finite topological space X, there exists an associated simplicial complex $\mathcal{K}(X)$ (the order complex) which has the same weak homotopy type as X, and, for each finite simplicial complex K, there is a finite space $\mathcal{X}(K)$ (the face poset) weak homotopy equivalent to K. Therefore, in contrast to what one could have expected at first sight, weak homotopy types of finite spaces coincide with homotopy types of finite CW-complexes. In this way, Stong and McCord put finite spaces in the game, showing implicitely that the composite between their combinatorics and topology can be used to study homotopy invariants of well-known Hausdorff spaces.

Despite the importance of those papers, finite spaces remained in the shadows for many years more. During that time, the relationship between finite posets and finite simplicial complexes was exploited, but in most cases ignoring or unknowing the intrinsic topology of the posets. A clear example of this is the case of D. Quillen [33], who, in 1978 investigates the connection between algebraic properties of a finite group G and homotopy properties of the simplicial complex associated to the poset of p-subgroups of G. In that article, Quillen develops powerful tools and proves very nice results about this subject, and he leaves a very interesting conjecture which remains open until these days. However, it seems that he was unaware of Stong's and McCord's results on finite spaces. We will see that the finite space point of view adds a completely new dimension to his conjecture and allows to attack the problem with new topological and combinatorial tools. We will show that Whitehead's Theorem does not hold for finite spaces: there are weak homotopy equivalent finite spaces with different homotopy types. This distinction between weak homotopy types and homotopy types is lost when we look into the associated polyhedra (because of Whitehead's Theorem) and, in fact, the essence of Quillen's conjecture lies precisely in the distinction between weak homotopy types and homotopy types of finite spaces.

In the last decades, a few interesting papers on finite spaces appeared [20, 31, 38], but the subject certainly did not receive the attention it required. In 2003, Peter May writes a series of unpublished notes [24, 23, 22] in which he synthesizes the most important ideas on finite spaces until that time. In these articles, May also formulates some natural and interesting questions and conjectures which arise from his own research. May was one of the first to note that Stong's combinatorial point of view and the bridge constructed by McCord could be used together to attack algebraic topology problems using finite spaces. Those notes came to the hands of my advisor Gabriel Minian, who proposed me to work on this subject. May's notes and problems, jointly with Stong's and McCord's papers were the starting point of our research on the Algebraic Topology of Finite Topological Spaces and Applications. In this Dissertation I will try to set the basis of the theory of finite spaces, recalling the development previous to ours and then I will exhibit the most important results of our work along these years.

Almost all the results presented in this Thesis are new and original. Some of them appear in our publications [6, 8, 7, 5]. The previous results on finite spaces appear in Chapter 1 and in the introduction of some Sections. The Chapter 5 (on strong homotopy types of polyhedra), Chapter 8 (on equivariant simple homotopy types and Quillen's conjecture) and Chapter 9 (on the Andrews-Curtis conjecture), which contain some of the strongest results of this Dissertation, are still unpublished and subjects of future papers.

Given a finite space X, there exists a homotopy equivalent finite space X_0 which is T_0. That means that for any two points of X_0 there exists an open set which contains only one of them. Therefore, when studying homotopy types of finite spaces, we can restrict our attention to T_0-spaces.

In [37], Stong defines the notion of *linear* and *colinear points*, which we call *up beat* and *down beat points* following May's terminology. Stong proves that removing a beat point from a finite space does not affect its homotopy type. Moreover, two finite spaces are homotopy equivalent if and only if it is possible to obtain one from the other just by adding or removing beat points. On the other hand, McCord results suggest that weak homotopy types of finite spaces are more important to be understood than homotopy types. In this direction, we generalized Stong's definition of beat points introducing the notion of *weak point* (see Definition 4.2.2). If one removes a weak point x from a finite space X, the resulting space need not be homotopy equivalent to X, however we prove that in this case the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence. As an application of this result, we exhibit an example (4.2.1) of a finite space which is homotopically trivial, i.e., weak homotopy equivalent to a point, but which is not contractible. This shows that

Whitehead's Theorem does not hold for finite spaces, not even for homotopically trivial spaces.

T. Osaki proves in [31] that if x is a beat point of a finite space X, there is a simplicial collapse from the associated complex $\mathcal{K}(X)$ to $\mathcal{K}(X \smallsetminus \{x\})$. In particular, if two finite spaces are homotopy equivalent, their associated complexes have the same simple homotopy type. However, we noticed that the converse is not true. There are easy examples of non-homotopy equivalent finite spaces with simple homotopy equivalent associated complexes. Removing beat points constitute a fundamental move of finite spaces, which gives rise to homotopy types. Whitehead's notion of simplicial collapse is the fundamental move of complexes which leads to simple homotopy types. We asked whether there existed another kind of fundamental move of finite spaces, which corresponded exactly to the simple homotopy types of complexes. We found out that weak points were the key to answer this question. We say that there is a *collapse* from a finite space X to a subspace Y if we can obtain Y from X by removing weak points, and we say that two finite spaces have the same *simple homotopy type* if we can obtain one from the other by adding and removing weak points. In the first case we denote $X \searrow Y$ and in the second, $X \diagup\!\!\searrow Y$. The following result, which appears in Chapter 4, says that simple homotopy types of finite spaces correspond precisely to simple homotopy types of the associated complexes.

Theorem 4.2.12.

(a) Let X and Y be finite T_0-spaces. Then, X and Y are simple homotopy equivalent if and only if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type. Moreover, if $X \searrow Y$ then $\mathcal{K}(X) \searrow \mathcal{K}(Y)$.

(b) Let K and L be finite simplicial complexes. Then, K and L are simple homotopy equivalent if and only if $\mathcal{X}(K)$ and $\mathcal{X}(L)$ are simple homotopy equivalent. Moreover, if $K \searrow L$ then $\mathcal{X}(K) \searrow \mathcal{X}(L)$.

This result allows one to use finite spaces to study problems of classical simple homotopy theory. Indeed, we will use some stronger version of Theorem 4.2.12 to investigate Quillen's conjecture on the poset of p-subgroups of a finite group, mentioned above.

It is relatively easy to know whether two finite spaces are homotopy equivalent using Stong's ideas, however it is very difficult to distinguish whether two finite spaces have the same weak homotopy type. Note that this is as hard as recognize if the associated polyhedra have the same homotopy type. Our results on simple homotopy types give a first approach in this direction. If two finite spaces have trivial Whitehead group, then they are weak homotopy equivalent if and only if they are simple homotopy equivalent. In particular, a finite space X is homotopically trivial if and only if it is possible to add and remove weak points from X to obtain the singleton $*$. The importance of recognizing homotopically trivial spaces will be evident when we study the conjecture of Quillen. Note that the fundamental move of finite spaces induced by weak points is easier to handle and describe than the simplicial one because it consists in removing just one single point of the space.

In the Third Section of Chapter 4 we study an analogue of Theorem 4.2.12 for simple homotopy equivalences. We give a description of the maps between finite spaces which correspond to simple homotopy equivalences at the level of complexes. The main result of this Section is Theorem 4.3.12. In contrast to the classical situation where simple homotopy equivalences are particular cases of homotopy equivalences, homotopy equivalences between finite spaces are a special kind of simple homotopy equivalences.

As an interesting application of our methods on simple homotopy types, we will prove the following simple homotopy version of Quillen's famous Theorem A.

Theorem 4.3.14. *Let $\varphi : K \to L$ be a simplicial map between finite simplicial complexes. If $\varphi^{-1}(\sigma)$ is collapsible for every simplex σ of L, then $|\varphi|$ is a simple homotopy equivalence.*

The fundamental moves described by beat or weak points are what we call *methods of reduction*. A reduction method is a technique that allows to change a finite space to obtain a smaller one, preserving some homotopy properties, such as homotopy type, simple homotopy type, weak homotopy type or the homology groups. In [31], Osaki introduce two methods of this kind which preserve the weak homotopy type, and he asks whether these moves are effective in the following sense: given a finite space X, is it always possible to obtain a space of minimum cardinality weak homotopy equivalent to X by applying repeatedly these methods? In Chapter 6 we give an example to show that the answer to this question is negative. In fact, it is a very difficult problem to find *minimal finite models* of spaces (i.e. a space weak homotopy equivalent with minimum cardinality) since this question is directly related to the problem of distinguish weak homotopy equivalent spaces.

In Chapter 6 we study Osaki's methods of reduction and we prove that in fact they preserve the simple homotopy type. In this Chapter we also study *one-point reduction methods* which consist in removing just one point of the space. For instance, beat points and weak points lead to one-point methods of reduction. In the second Section of that Chapter, we define the notion of γ-point which generalizes the concept of weak point and provides a more appliable method which preserves the weak homotopy type. The impotance of this new method is that it is almost the most general possible one-point reduction method. More specifically, we prove the following result

Theorem 6.2.5. *Let X be a finite T_0-space, and $x \in X$ a point which is neither maximal nor minimal and such that $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence. Then x is a γ-point.*

In some sense, one-point methods are not sufficient to describe weak homotopy types of finite spaces. Concretely, if $x \in X$ is such that the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence, then $X \smallsetminus \{x\} \diagup\!\!\searrow X$ (see Theorem 6.2.7). Therefore, these methods cannot be used to obtain weak homotopy equivalent spaces which are not simple homotopy equivalent.

McCord finds in [26] a *finite model* of the n-sphere S^n (i.e. a finite space weak homotopy equivalent to S^n) with only $2n+2$ points. May conjectures in his notes that this space is, in our language, a minimal finite model of S^n. In Chapter 3 we prove that May's conjecture is true. Moreover, the minimal finite model of S^n is unique up to homeomorphism (see Theorem 3.1.2). In this Chapter we also study minimal finite models of finite graphs (CW-complexes of dimension 1) and give a full description of them in Theorem 3.2.7. In this case the uniqueness of the minimal finite models depends on the graph. The reason to study finite models of spaces instead of finite spaces with the same homotopy type is that homotopy types of finite complexes rarely occur in the setting of finite spaces (see Corollary 2.3.4).

In Chapter 5 we study the relationship between homotopy equivalent finite spaces and the associated complexes. The concept of contiguity classes of simplicial maps leads to the notion of *strong homotopy equivalence* (Definition 5.0.4) and *strong homotopy types* of simplicial complexes. This equivalence relation is generated by *strong collapses* which are more restrictive than the usual simplicial collapses. We proved the following result.

Theorem 5.0.15.

(a) If two finite T_0-spaces are homotopy equivalent, their associated complexes have the same strong homotopy type.

(b) If two finite complexes have the same strong homotopy type, the associated finite spaces are homotopy equivalent.

Another of the problems originally stated by May in [23] consists on extending McCord's ideas in order to model, with finite spaces, not only simplicial complexes, but general CW-complexes. We give an approach to this question in Chapter 7, where the notion of h-regular CW-complex is defined. It was already known that regular CW-complexes could be modeled by their face posets. The class of h-regular complexes extends considerably the class of regular complexes and we explicitly construct for each h-regular complex K, a weak homotopy equivalence $K \to \mathcal{X}(K)$. Our results on h-regular complexes allow the construction of a lot of new and interesting examples of finite models. We also apply these results to investigate quotients of finite spaces and derive a long exact sequence of reduced homology for finite spaces.

Given a finite group G and a prime integer p, we denote by $S_p(G)$ the poset of nontrivial p-subgroups of G. In [33], Quillen proves that if G has a nontrivial normal p-subgroup, then $\mathcal{K}(S_p(G))$ is contractible and he conjectures the converse: if the complex $\mathcal{K}(S_p(G))$ is contractible, G has a nontrivial p-subgroup. Quillen himself proves his conjecture for the case of solvable groups, but the general problem still remains open. Some important advances were achived in [3]. As we said above, Quillen never regards $S_p(G)$ as a topological space. In 1984, Stong [38] publishes a second article on finite spaces. He proves some results on the equivariant homotopy theory of finite spaces, which he uses to attack Quillen's conjecture. He shows that G has a nontrivial normal p-subgroup if and only if $S_p(G)$ is a contractible finite space. Therefore, the conjecture can be restated in terms of finite spaces as follows: $S_p(G)$ is contractible if and only if it is homotopically trivial. In Chapter 8 we study an equivariant version of simple homotopy types of simplicial complexes and finite spaces and we prove an analogue of Theorem 4.2.12 in this case. Using this result we obtain some new formulations of the conjecture, but which are exclusively written in terms of simplicial complexes. Finite spaces are used in this case as a tool to obtain the result, but they do not appear in the final formulation which is the following:

$\mathcal{K}(S_p(G))$ is contractible if and only if it has trivial equivariant simple homotopy type. We also obtain formulations of the conjecture in terms of the polyhedron associated to the much smaller poset $A_p(G)$ of the elementary abelian p-subgroups.

In the last Chapter of the Thesis we prove some advances concerning the Andrews-Curtis conjecture. The geometric Andrews-Curtis conjecture states that if K is a contractible complex of dimension 2, then it 3-deforms to a point, i.e. it can be deformed into a point by a sequence of collapses and expansions which involve complexes of dimension not greater than 3. This very known problem stated in the sixties, is closely related to Zeeman's conjecture and hence, to the famous Poincaré conjecture. With the proof of the Poincaré conjecture by G. Perelman, and by [17], we know now that the geometric Andrews-Curtis conjecture is true for *standard spines* ([34]), but it still remains open for general 2-complexes. Inspired by our results on simple homotopy theory of finite spaces and simplicial complexes, we define the notion of *quasi constructible* 2*-complexes* which generalizes the concept of constructible complexes. Using techniques of finite spaces we prove that contractible constructible complexes 3-deform to a point. In this way we substantially enlarge the class of complexes which are known to satisfy the conjecture.

Other results of this Dissertation include a description of the fundamental group of a finite space, an alternative proof of the homotopy invariance of Euler Characteristic, a result on the realizability of a group as automorphism group of a poset and some results on fixed point theory of finite spaces and the Lefschetz number.

I hope that after this work it will be clear that the combinatorics of finite spaces along with their topology make of these objects a very powerful and suitable tool which can be used to study well-known problems of algebra, algebraic topology, combinatorics and discrete geometry, giving more information than simplicial complexes.

Contents

Chapter 1

Preliminaries

In this First Chapter, we will recall the most important results on finite spaces which are previous to our work. They originally appeared in [1, 37, 26]. We will describe the correspondence between finite spaces and finite posets and its relationship with basic topological properties: continuous maps, connectedness, homotopies. Then, we will study the homotopy types of finite spaces from Stong's angle and compare weak homotopy types of finite spaces with homotopy types of compact polyhedra using McCord's results.

Homotopy types of finite spaces are conclusively characterized by Stong and homotopy equivalences are well understood as well. However, it is much more difficult to characterize weak homotopy equivalences between finite spaces. One of the most important tools to identify weak homotopy equivalences is the Theorem of McCord 1.4.2. However, we will see in following Chapters that in some sense this result is not sufficient to describe all weak equivalences. The problem of distinguishing weak homotopy equivalences between finite spaces is directly related to the problem of recognizing homotopy equivalences between polyhedra.

1.1 Finite spaces and posets

A finite topological space is a topological space with finitely many points and a finite preordered set is a finite set with a transitive and refexive relation. We will see that finite spaces and finite preordered sets are basically the same objects seen from different perspectives. Given a finite topological space X, we define for every point $x \in X$ the *minimal open set* U_x as the intersection of all the open sets which contain x. These sets are again open. In fact arbitrary intersections of open sets in finite spaces are open. It is easy to see that minimal open sets constitute a basis for the topology of X which is called the *minimal basis* of X. Define a preorder on X by $x \leq y$ if $x \in U_y$.

Conversely, if X is a finite preordered set, there is a topology on X given by the basis $\{y \leq x\}_{x \in X}$. These two applications relating topologies and preorders of a finite set are mutually inverse. This simple remark made in first place by Alexandroff [1] allows us to use algebraic topology to study finite spaces as well as combinatorics araising from their intrinsic preorder structures.

The antisimetry of a finite preorder corresponds exactly to the T_0 separation axiom.

Recall that a topological space X is said to be T_0 if for any two points of X there exists an open set containing one and only one of them. Therefore finite T_0-spaces are in correspondence with finite partially ordered sets (posets).

Example 1.1.1. Let $X = \{a, b, c, d\}$ be a finite space whose open sets are $\emptyset$, $\{a, b, c, d\}$ $\{b, d\}$, $\{c\}$, $\{d\}$, $\{b, c, d\}$ and $\{c, d\}$. This space is T_0, and therefore it is a poset. The first figure (Figure 1.1) is a scheme of X with its open sets represented by the interiors of the closed curves. A more useful way to represent finite T_0-spaces is with their Hasse diagrams. The Hasse diagram of a poset X is a digraph whose vertices are the points of X and whose edges are the ordered pairs (x, y) such that $x < y$ and there exists no $z \in X$ such that $x < z < y$. In the graphical representation of a Hasse diagram we will not write an arrow from x to y, but a segment with y over x (see Figure 1.2).

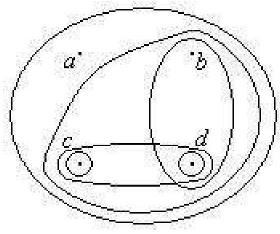


Figure 1.1: Open sets of X.

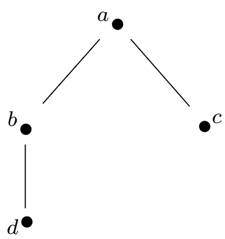


Figure 1.2: Hasse diagram of X.

If (x, y) is an edge of the Hasse diagram of a finite poset X, we say that y *covers* x and write $x \prec y$.

Open sets of finite spaces correspond to *down-sets* and closed sets to *up-sets*. A subset U of a preordered set X is a down-set if for every $x \in U$ and $y \leq x$, it holds that $y \in U$. The notion of up-set is defined dually. If X is T_0, the open sets of X are in bijection with its antichains.

If x is a point of a finite space X, $F_x = \{y \in X \mid y \geq x\}$ denotes the closure of the set $\{x\}$ in X. If a point x belongs to finite spaces X and Y, we write U_x^X, U_x^Y, F_x^X and F_x^Y so as to distinguish whether the minimal open sets and closures are considered in X or in Y.

Note that the set of closed subspaces of a finite space X is also a topology on the underlying set of X. The finite space with this topology is the *opposite* of X (or *dual*) and it is denoted by X^{op}. The order of X^{op} is the inverse order of X. If $x \in X$, then $U_x^{X^{op}} = F_x^X$.

The following remark is easy to check.

Remark 1.1.2.

(a) Let A be a subspace of a finite space X and let $a, a' \in A$. Then $a \leq_A a'$ if and only if $a \leq_X a'$. Here $\leq_A$ denotes the preorder corresponding to the subspace topology of A and $\leq_X$ the corresponding to the topology of X.

(b) Let X and Y be two finite spaces and let $(x, y), (x', y') \in X \times Y$ with the product topology. Then $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$.

1.2 Maps, homotopies and connectedness

Proposition 1.2.1. *A function $f : X \to Y$ between finite spaces is continuous if and only if it is order preserving.*

Proof. Suppose f is continuous and $x \leq x'$ in X. Then $f^{-1}(U_{f(x')}) \subseteq X$ is open and since $x' \in f^{-1}(U_{f(x')})$, $x \in U_{x'} \subseteq f^{-1}(U_{f(x')})$. Therefore $f(x) \leq f(x')$.

Now assume that f is order preserving. To prove that f is continuous it suffices to show that $f^{-1}(U_y)$ is open for every set U_y of the minimal basis of Y. Let $x \in f^{-1}(U_y)$ and let $x' \leq x$. Then $f(x') \leq f(x) \leq y$ and $x' \in f^{-1}(U_y)$. This proves that $f^{-1}(U_y)$ is a down-set. □

If $f : X \to Y$ is a function between finite spaces, the map $f^{op} : X^{op} \to Y^{op}$ is the map which coincides with f in the underlying sets. It easy to see that f is continuous if and only if f^{op} is continuous.

Remark 1.2.2. If X is a finite space, a one-to-one continuous map $f : X \to X$ is a homemorphism. In fact, since f is a permutation of the set X, there exists $n \in \mathbb{N}$ such that $f^n = 1_X$.

Lemma 1.2.3. *Let x, y be two comparable points of a finite space X. Then, there exists a path from x to y in X, i.e. a map α from the unit interval I to X such that $\alpha(0) = x$ and $\alpha(1) = y$.*

Proof. Assume $x \leq y$ and define $\alpha : I \to X$, $\alpha(t) = x$ if $0 \leq t < 1$, $\alpha(1) = y$. If $U \subseteq X$ is open and contains y, then it contains x also. Therefore $\alpha^{-1}(U)$ is one of the following sets, $\emptyset$, I or $[0, 1)$, which are all open in I. Thus, α is a continuous path from x to y. □

Let X be a finite preordered set. A *fence* in X is a sequence $x_0, x_1, \ldots, x_n$ of points such that any two consecutive are comparable. X is *order-connected* if for any two points $x, y \in X$ there exists a fence starting in x and ending in y.

Proposition 1.2.4. *Let X be a finite space. Then, the following are equivalent:*

1. *X is a connected topological space.*

2. X is an order-connected preorder.

3. X is a path-connected topological space.

Proof. If X is order-connected, it is path-connected by Lemma 1.2.3. We only have to prove that connectedness implies order-connectedness. Suppose X is connected and let $x \in X$. Let $A = \{y \in X \mid \text{there is a fence from } x \text{ to } y\}$. If $y \in A$ and $z \leq y$, then $z \in A$. Therefore A is a down-set. Analogously, it is an up-set and then, $A = X$. □

If X and Y are finite spaces we can consider the finite set Y^X of continuous maps from X to Y with the pointwise order: $f \leq g$ if $f(x) \leq g(x)$ for every $x \in X$.

Proposition 1.2.5. *The pointwise order on Y^X corresponds to the compact-open topology.*

Proof. Let $S(K,W) = \{f \in Y^X \mid f(K) \subseteq W\}$ be a set of the subbase of the compact-open topology, where K is a (compact) subset of X and W an open set of Y. If $g \leq f$ and $f \in S(K,W)$, then $g(x) \leq f(x) \in W$ for every $x \in K$ and therefore, $g \in S(K,W)$. Thus, $S(K,W)$ is a down-set in Y^X. Conversely if $f \in Y^X$, $\{g \in Y^X \mid g \leq f\} = \bigcap\limits_{x \in X} S(\{x\}, U_{f(x)})$. Therefore both topologies coincide. □

If X and Y are topological spaces, a sufficient condition for the compact-open topology of Y^X being exponential is that every point of X has a basis of compact neighborhoods. If X is a finite space, every subspace of X is compact and this condition is trivialy satisfied. In particular, if X is a finite space and Y is a topological space not necessarily finite, there is a natural correspondence between the set of homotopies $\{H : X \times I \to Y\}$ and the set of paths $\{\alpha : I \to Y^X\}$. From now on we consider the map spaces Y^X with the compact-open topology, unless we say otherwise.

Corollary 1.2.6. *Let $f, g : X \to Y$ be two maps between finite spaces. Then $f \simeq g$ if and only if there is a fence $f = f_0 \leq f_1 \geq f_2 \leq \ldots f_n = g$. Moreover, if $A \subseteq X$, then $f \simeq g$ rel A if and only if there exists a fence $f = f_0 \leq f_1 \geq f_2 \leq \ldots f_n = g$ such that $f_i|_A = f|_A$ for every $0 \leq i \leq n$.*

Proof. There exists a homotopy $H : f \simeq g$ *rel* A if and only if there is a path $\alpha : I \to Y^X$ from f to g such that $\alpha(t)|_A = f|_A$ for every $0 \leq t \leq 1$. This is equivalent to say that there is a path $\alpha : I \to M$ from f to g where M is the subspace of Y^X of maps which coincide with f in A. By 1.2.4 this means that there is a fence from f to g in M. The order of M is the one induced by Y^X, which is the pointwise order by 1.2.5. □

Remark 1.2.7. Any finite space X with maximum or minimum is contractible since, in that case, the identity map 1_X is comparable with a constant map c and therefore $1_X \simeq c$.

For example, the space of Figure 1.2 has a maximum and therefore it is contractible.

Note that if X and Y are finite spaces and Y is T_0, then Y^X is T_0 since $f \leq g$, $g \leq f$ implies $f(x) = g(x)$ for every $x \in X$.

1.3 Homotopy types

In this Section we will recall the beautiful ideas of R. Stong [37] about homotopy types of finite spaces. Stong introduced the notion of *linear* and *colinear points* that later were called *up beat* and *down beat points* by P. May [24]. Removing such kind of points from a finite space does not affect its homotopy type. Therefore any finite space is homotopy equivalent to a space without beat points, which is called a *minimal finite space.* Moreover two minimal finite spaces are homotopy equivalent only if they are homeomorphic.

The next result essentially shows that, when studying homotopy types of finite spaces, we can restrict ourselves to T_0-spaces.

Proposition 1.3.1. *Let X be a finite space. Let X_0 be the quotient $X/\sim$ where $x \sim y$ if $x \leq y$ and $y \leq x$. Then X_0 is T_0 and the quotient map $q : X \to X_0$ is a homotopy equivalence.*

Proof. Take any section $i : X_0 \to X$, i.e. $qi = 1_{X_0}$. The composition iq is order preserving and therefore i is continuous. Moreover, since $iq \leq 1_X$, i is a homotopy inverse of q.

Let $x, y \in X_0$ such that $q(x) \leq q(y)$, then $x \leq iq(x) \leq iq(y) \leq y$. If in addition $q(y) \leq q(x)$, $y \leq x$ and then $q(x) = q(y)$. Therefore the preorder of X_0 is antisymmetric. □

Remark 1.3.2. Note that the map $i : X_0 \to X$ of the previous proof is a subspace map since $qi = 1_{X_0}$. Moreover, since $iq \leq 1_X$ and the maps iq and 1_X coincide on X_0, then by 1.2.6, $iq \simeq 1_X$ *rel* X_0. Therefore X_0 is a strong deformation retract of X.

Definition 1.3.3. A point x of a finite T_0-space X is a *down beat point* if x covers one and only one element of X. This is equivalent to say that the set $\hat{U}_x = U_x \smallsetminus \{x\}$ has a maximum. Dually, $x \in X$ is an *up beat point* if x is covered by a unique element or equivalently if $\hat{F}_x = F_x \smallsetminus \{x\}$ has a minimum. In any of this cases we say that x is a *beat point* of X.

Its easy to recognize beat points looking into the Hasse diagram of the space. A point $x \in X$ is a down beat point if and only if there is one and just one edge with x at its top. It is an up beat point if and only if there is one and only one edge with x at the bottom. In example of Figure 1.2, a is not a beat point: it is not a down beat point because there are two segments with a at the top and it is not an up beat point either because there is no segment with a at the bottom. The point b is both a down and an up beat point, and c is an up beat point but not a down beat point.

If X is a finite T_0-space, and $x \in X$, then x is a down beat point of X if and only if it is an up beat point of X^{op}. In particular x is a beat point of X if and only if it is a beat point of X^{op}.

Proposition 1.3.4. *Let X be a finite T_0-space and let $x \in X$ be a beat point. Then $X \smallsetminus \{x\}$ is a strong deformation retract of X.*

Proof. Assume that x is a down beat point and let y be the maximum of $\hat{U}_x$. Define the retraction $r : X \to X \smallsetminus \{x\}$ by $r(x) = y$. Clearly, r is order-preserving. Moreover if $i : X \smallsetminus \{x\} \hookrightarrow X$ denotes the canonical inclusion, $ir \leq 1_X$. By 1.2.6, $ir \simeq 1_X$ *rel* $X \smallsetminus \{x\}$. If x is an up beat point the proof is similar. □

Definition 1.3.5. A finite T_0-space is a *minimal finite space* if it has no beat points. A *core* of a finite space X is a strong deformation retract which is a minimal finite space.

By Remark 1.3.2 and Proposition 1.3.4 we deduce that every finite space has a core. Given a finite space X, one can find a T_0-strong deformation retract $X_0 \subseteq X$ and then remove beat points one by one to obtain a minimal finite space. The amazing thing about this construction is that in fact the core of a finite space is unique up to homeomorphism, moreover: two finite spaces are homotopy equivalent if and only if their cores are homeomorphic.

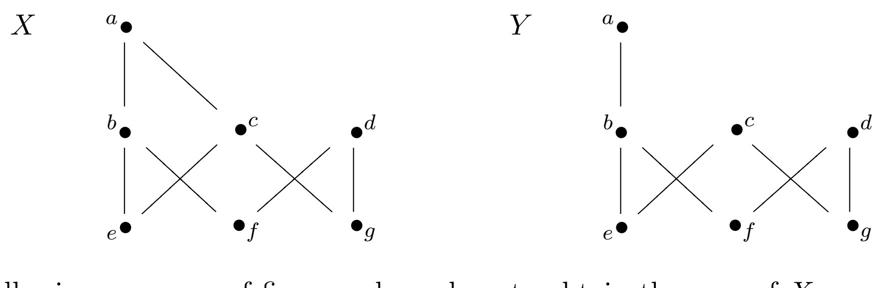
Theorem 1.3.6. *Let X be a minimal finite space. A map $f : X \to X$ is homotopic to the identity if and only if $f = 1_X$.*

Proof. By 1.2.6 we may suppose that $f \leq 1_X$ or $f \geq 1_X$. Assume $f \leq 1_X$. Let $x \in X$ and suppose by induction that $f|\hat{U}_x = 1_{\hat{U}_x}$. If $f(x) \neq x$, then $f(x) \in \hat{U}_x$ and for every $y < x$, $y = f(y) \leq f(x)$. Therefore, $f(x)$ is the maximum of $\hat{U}_x$ which is a contradiction since X has no down beat points. Therefore $f(x) = x$. The case $f \geq 1_X$ is similar. □

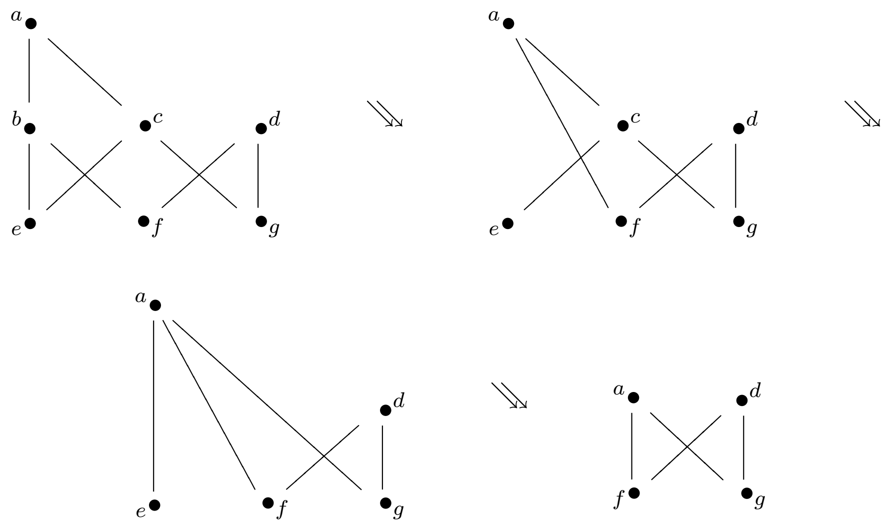
Corollary 1.3.7. *A homotopy equivalence between minimal finite spaces is a homeomorphism. In particular the core of a finite space is unique up to homeomorphism and two finite spaces are homotopy equivalent if and only if they have homeomorphic cores.*

Proof. Let $f : X \to Y$ be a homotopy equivalence between finite spaces and let $g : Y \to X$ be a homotopy inverse. Then $gf = 1_X$ and $fg = 1_Y$ by Theorem 1.3.6. Thus, f is a homeomorphism. If X_0 and X_1 are two cores of a finite space X, then they are homotopy equivalent minimal finite spaces, and therefore, homeomorphic. Two finite spaces X and Y have the same homotopy type if and only if their cores are homotopy equivalent, but this is the case only if they are homeomorphic. □

Example 1.3.8. Let X and Y be the following finite T_0-spaces:



The following sequence of figures, shows how to obtain the core of X removing beat points. Note that b is an up beat point of X, c is an up beat point of $X \smallsetminus \{b\}$ and e an up beat point of $X \smallsetminus \{b, c\}$. The subspace $X \smallsetminus \{b, c, e\}$ obtained in this way is a minimal finite space and then it is the core of X.



On the other hand, a is a beat point of Y and $Y \smallsetminus \{a\}$ is minimal. Therefore the cores of X and Y are not homeomorphic, so X and Y are not homotopy equivalent.

To finish this Section, we exhibit the following characterization of minimal finite spaces.

Proposition 1.3.9. *Let X be a finite T_0-space. Then X is a minimal finite space if and only if there are no $x, y \in X$ with $x \neq y$ such that if $z \in X$ is comparable with x, then so is it with y.*

Proof. If X is not minimal, there exists a beat point x. Without loss of generality assume that x is a down beat point. Let y be the maximum of $\hat{U}_x$. Then if $z \geq x$, $z \geq y$ and if $z < x$, $z \leq y$.

Conversely, suppose that there exists x and y as in the statement. In particular x is comparable with y. We may assume that $x \geq y$. Let $A = \{z \in X \mid z \geq y$ and for every $w \in X$ comparable with z, w is comparable with $y\}$. This set is non-empty since $x \in A$. Let x' be a minimal element of A. We show that x' is a down beat point with $y = max(\hat{U}_{x'})$. Let $z < x'$, then z is comparable with y since $x' \in A$. Suppose $z > y$. Let $w \in X$. If $w \leq z$, then $w \leq x'$ and so, w is comparable with y. If $w \geq z$, $w \geq y$. Therefore $z \in A$, contradicting the minimality of x'. Then $z \leq y$. Therefore y is the maximum of $\hat{U}_{x'}$. □

1.4 Weak homotopy types: The theory of McCord

In the previous Section we have studied homotopy types of finite spaces. On the other hand we will see in the next Chapter, that Hausdorff spaces do not have in general the homotopy type of any finite space. However finite CW-complexes do have the weak homotopy type of finite spaces. In 1966 M. C. McCord proved that every compact polyhedron has an associated finite space with the same weak homotopy type and every finite space has a weak equivalent associated polyhedron.

Recall that a continuous map $f : X \to Y$ between topological spaces is said to be a weak homotopy equivalence if it induces isomorphisms in all homotopy groups, i.e. if the maps

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

are isomorphisms for every $n \geq 0$ and every base point $x_0 \in X$. Note that, homotopy equivalences are weak homotopy equivalences, but the converse is not true. Whitehead's Theorem says that the converse holds when both spaces are CW-complexes. We will see many examples of weak homotopy equivalences which are not homotopy equivalences using finite spaces. If $f : X \to Y$ is a weak homotopy equivalence, it induces isomorphisms in all homology groups, that is to say $f_* : H_n(X) \to H_n(Y)$ are isomorphisms for every $n \geq 0$.

Next, we will state the Theorem of McCord 1.4.2 which plays an essential role in the homotopy theory of finite spaces. This result basically says that if a continuous map is locally a weak homotopy equivalence, then it is a weak homotopy equivalence itself. The original proof by McCord is in [26], Theorem 6, and it is based on an analogous result for quasifibrations by A. Dold and R. Thom. A proof for finite covers can be also obtained from Corollary 4K.2 of [21].

Definition 1.4.1. Let X be a topological space. An open cover $\mathcal{U}$ of X is called a *basis-like open cover* if $\mathcal{U}$ is a basis for a topology which is coarser than the topology of X (or, equivalently, if for any $U_1, U_2 \in \mathcal{U}$ and $x \in U_1 \cap U_2$, there exists $U_3 \in \mathcal{U}$ such that $x \in U_3 \subseteq U_1 \cap U_2$).

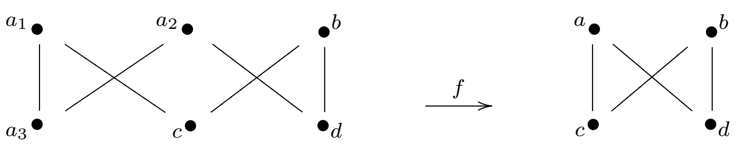
For instance, if X is a finite space, the minimal basis $\{U_x\}_{x \in X}$ is a basis like open cover of X.

Theorem 1.4.2 (McCord)**.** *Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. Suppose that there exists a basis-like open cover $\mathcal{U}$ of Y such that each restriction*

$$f|_{f^{-1}(U)} : f^{-1}(U) \to U$$

is a weak homotopy equivalence for every $U \in \mathcal{U}$. Then $f : X \to Y$ is a weak homotopy equivalence.

Example 1.4.3. Consider the following map between finite spaces



defined by $f(a_1) = f(a_2) = f(a_3) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$. It is order preserving and therefore continuous. Moreover, the preimage of each minimal open set U_y, is contractible, and then the restrictions $f|_{f^{-1}(U_y)} : f^{-1}(U_y) \to U_y$ are (weak) homotopy equivalences. Since the minimal basis is a basis like open cover, by Theorem 1.4.2 f is a weak homotopy equivalence. However, f is not a homotopy equivalence since its source and target are non homeomorphic minimal spaces.

Definition 1.4.4. Let X be a finite T_0-space. The *simplicial complex* $\mathcal{K}(X)$ *associated* to X (also called the *order complex*) is the simplicial complex whose simplices are the non-empty chains of X. Moreover, if $f : X \to Y$ is a continuous map between finite T_0-spaces, the *associated simplicial map* $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ is defined by $\mathcal{K}(f)(x) = f(x)$.

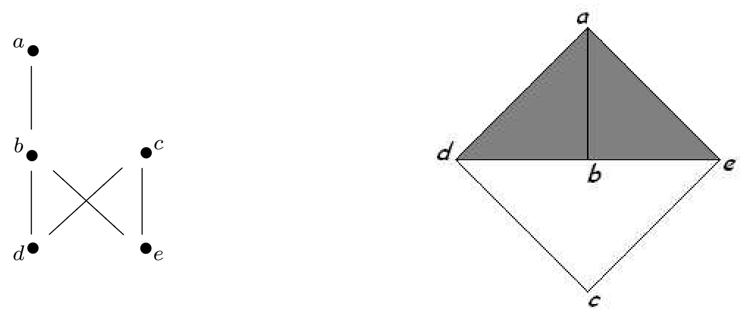


Figure 1.3: A finite space and its associated simplicial complex.

Note that if $f : X \to Y$ is a continuous map between finite T_0-spaces, the vertex map $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ is simplicial since f is order preserving and maps chains to chains.

If X is a finite T_0-space, $\mathcal{K}(X)=\mathcal{K}(X^{op})$. Moreover, if $f : X \to Y$ is a continuous map between finite T_0-spaces, $\mathcal{K}(f) = \mathcal{K}(f^{op})$.

Let X be a finite T_0-space. A point α in the geometric realization $|\mathcal{K}(X)|$ of $\mathcal{K}(X)$ is a convex combination $\alpha = t_1x_1 + t_2x_2 + \ldots + t_rx_r$ where $\sum\limits_{i=1}^{r} t_i = 1$, $t_i > 0$ for every $1 \le i \le r$ and $x_1 < x_2 < \ldots < x_r$ is a chain of X. The support or carrier of α is the set $support(\alpha) = \{x_1, x_2, \ldots, x_r\}$. We will see that the map $\alpha \mapsto x_1$ plays a fundamental role in this theory.

Definition 1.4.5. Let X be a finite T_0-space. Define the $\mathcal{K}$*-McCord map* $\mu_X : |\mathcal{K}(X)| \to X$ by $\mu_X(\alpha) = min(support(\alpha))$.

Theorem 1.4.6. *The $\mathcal{K}$-McCord map μ_X is a weak homotopy equivalence for every finite T_0-space X.*

Proof. Notice that the minimal open sets U_x are contractible because they have maximum. We will prove that for each $x \in X$, $\mu_X^{-1}(U_x)$ is open and contractible. This will show that μ_X is continuous and that the restrictions $\mu_X|_{\mu_X^{-1}(U_x)} : \mu_X^{-1}(U_x) \to U_x$ are weak homotopy equivalences. Therefore, by Theorem 1.4.2, μ_X is a weak homotopy equivalence.

Let $x \in X$ and let $L = \mathcal{K}(X \smallsetminus U_x) \subseteq \mathcal{K}(X)$. In other words, L is the full subcomplex of K (possibly empty) spanned by the vertices which are not in U_x. We claim that

$$\mu_X^{-1}(U_x) = |\mathcal{K}(X)| \smallsetminus |L|.$$

If $\alpha \in \mu_X^{-1}(U_x)$, then $min(support(\alpha)) \in U_x$. In particular, the support of α contains a vertex of U_x and then $\alpha \notin |L|$. Conversely, if $\alpha \notin |L|$, there exists $y \in support(\alpha)$ such that

$y \in U_x$. Then $min(support(\alpha)) \leq y \leq x$ and therefore $\mu_X(\alpha) \in U_x$. Since $|L| \subseteq |\mathcal{K}(X)|$ is closed, $\mu_X^{-1}(U_x)$ is open.

Now we show that $|\mathcal{K}(U_x)|$ is a strong deformation retract of $|\mathcal{K}(X)| \smallsetminus |L|$. This is a more general fact. Let $i : |\mathcal{K}(U_x)| \hookrightarrow |\mathcal{K}(X)| \smallsetminus |L|$ be the inclusion. If $\alpha \in |\mathcal{K}(X)| \smallsetminus |L|$, $\alpha = t\beta + (1-t)\gamma$ for some $\beta \in |\mathcal{K}(U_x)|$, $\gamma \in |L|$ and $0 < t \leq 1$. Define $r : |\mathcal{K}(X)| \smallsetminus |L| \to |\mathcal{K}(U_x)|$ by $r(\alpha) = \beta$. Note that r is continuous since $r|_{(|\mathcal{K}(X)| \smallsetminus |L|) \cap \overline{\sigma}} : (|\mathcal{K}(X)| \smallsetminus |L|) \cap \overline{\sigma} \to \overline{\sigma}$ is continuous for every $\sigma \in \mathcal{K}(X)$. Here, $\overline{\sigma} \subseteq |\mathcal{K}(X)|$ denotes the closed simplex. Now, let $H : (|\mathcal{K}(X)| \smallsetminus |L|) \times I \to |\mathcal{K}(X)| \smallsetminus |L|$ be the linear homotopy between $1_{|\mathcal{K}(X)| \smallsetminus |L|}$ and ri, i.e.

$$H(\alpha, s) = (1-s)\alpha + s\beta.$$

Then H is well defined and is continuous since each restriction

$$H|_{((|\mathcal{K}(X)| \smallsetminus |L|) \cap \overline{\sigma}) \times I} : ((|\mathcal{K}(X)| \smallsetminus |L|) \cap \overline{\sigma}) \times I \to \overline{\sigma}$$

is continuous for every simplex σ of $\mathcal{K}(X)$. To prove the continuity of r and of H we use that $|\mathcal{K}(X)| \smallsetminus |L|$ has the final topology with respect to the subspaces $(|\mathcal{K}(X)| \smallsetminus |L|) \cap \overline{\sigma}$ for $\sigma \in \mathcal{K}(X)$.

Since every element of U_x is comparable with x, $\mathcal{K}(U_x) = x\mathcal{K}(U_x \smallsetminus \{x\})$ is a simplicial cone (see Section 2.7). In particular $|\mathcal{K}(U_x)|$ is contractible and then, so is $\mu_X^{-1}(U_x) = |\mathcal{K}(X)| \smallsetminus |L|$. □

Remark 1.4.7. If $f : X \to Y$ is a continuous map between finite T_0-spaces, the following diagram commutes

$$\begin{array}{ccc} |\mathcal{K}(X)| & \xrightarrow{|\mathcal{K}(f)|} & |\mathcal{K}(Y)| \\ \downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

since, for $\alpha \in |\mathcal{K}(X)|$,

$$f\mu_X(\alpha) = f(min(support(\alpha))) = min(f(support(\alpha))) =$$

$$= min(support(|\mathcal{K}(f)|(\alpha))) = \mu_Y|\mathcal{K}(f)|(\alpha).$$

Corollary 1.4.8. *Let $f : X \to Y$ be a map between finite T_0-spaces. Then f is a weak homotopy equivalence if and only if $|\mathcal{K}(f)| : |\mathcal{K}(X)| \to |\mathcal{K}(Y)|$ is a homotopy equivalence.*

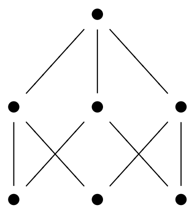
Proof. From Remark 1.4.7, we deduce that $f_*\mu_{X*} = \mu_{Y*}|\mathcal{K}(f)|_* : \pi_n(|\mathcal{K}(X)|) \to \pi_n(Y)$ for every $n \geq 0$. Since μ_{X*} and μ_{Y*} are isomorphisms, f_* is an isomorphism if and only if $|\mathcal{K}(f)|_*$ is an isomorphism. □

Corollary 1.4.9. *Let $f : X \to Y$ be a map between finite T_0-spaces. Then f is a weak homotopy equivalence if and only if f^{op} is a weak homotopy equivalence.*

Proof. Follows immediately from the previous result since $\mathcal{K}(f) = \mathcal{K}(f^{op})$. □

Definition 1.4.10. Let K be a finite simplicial complex. The finite T_0-space $\mathcal{X}(K)$ *associated* to K (also called the face poset of K) is the poset of simplices of K ordered by inclusion. If $\varphi : K \to L$ is a simplicial map between finite simplicial complexes, there is a continuous map $\mathcal{X}(\varphi) : \mathcal{X}(K) \to \mathcal{X}(L)$ defined by $\mathcal{X}(\varphi)(\sigma) = \varphi(\sigma)$ for every simplex σ of K.

Example 1.4.11. If K is the 2-simplex, the associated finite space is the following

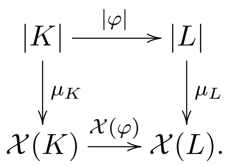


If K is a finite complex, $\mathcal{K}(\mathcal{X}(K))$ is the first barycentric subdivision K' of K. Let $s_K : |K'| \to |K|$ be the linear homeomorphism defined by $s_K(\sigma) = b(\sigma)$ for every simplex σ of K. Here, $b(\sigma) \in |K|$ denotes the barycenter of σ. Define the *$\mathcal{X}$-McCord map* $\mu_K = \mu_{\mathcal{X}(K)} s_K^{-1} : |K| \to \mathcal{X}(K)$

From 1.4.6 we deduce immediately the following result.

Theorem 1.4.12. *The $\mathcal{X}$-McCord map μ_K is a weak homotopy equivalence for every finite simplicial complex K.*

Proposition 1.4.13. *Let $\varphi : K \to L$ be a simplicial map between finite simplicial complexes. Then, the following diagram commutes up to homotopy*



Proof. Let $S = \{\sigma_1, \sigma_2, \dots, \sigma_r\}$ be a simplex of K', where $\sigma_1 \subsetneq \sigma_2 \subsetneq \dots \subsetneq \sigma_r$ is a chain of simplices of K. Let α be a point in the closed simplex $\overline{S}$. Then $s_K(\alpha) \in \overline{\sigma_r} \subseteq |K|$ and $|\varphi| s_K(\alpha) \in \overline{\varphi(\sigma_r)} \subseteq |L|$. On the other hand, $|\varphi'|(\alpha) \in \overline{\{\varphi(\sigma_1), \varphi(\sigma_2), \dots, \varphi(\sigma_r)\}}$ and then $s_L|\varphi'|(\alpha) \in \overline{\varphi(\sigma_r)}$. Therefore, the linear homotopy $H : |K'| \times I \to |L|$, $(\alpha, t) \mapsto (1-t)|\varphi| s_K(\alpha) + t s_L |\varphi'|(\alpha)$ is well defined and continuous. Then $|\varphi| s_K \simeq s_L |\varphi'|$ and, by 1.4.7,

$$\mu_L |\varphi| = \mu_{\mathcal{X}(L)} s_L^{-1} |\varphi| \simeq \mu_{\mathcal{X}(L)} |\varphi'| s_K^{-1} =$$

$$= \mathcal{X}(\varphi) \mu_{\mathcal{X}(K)} s_K^{-1} = \mathcal{X}(\varphi) \mu_K.$$

□

Remark 1.4.14. An explicit homotopy between $\mu_L|\varphi|$ and $\mathcal{X}(\varphi)\mu_K$ is $\widetilde{H} = \mu_L H (s_K^{-1} \times 1_I)$. If $K_1 \subseteq K$ and $L_1 \subseteq L$ are subcomplexes and $\varphi(K_1) \subseteq L_1$ then $\widetilde{H}(|K_1| \times I) \subseteq \mathcal{X}(L_1) \subseteq \mathcal{X}(L)$.

Similarly as in Corollary 1.4.8 one can prove the following

Corollary 1.4.15. *Let $\varphi : K \to L$ be a simplicial map between finite simplicial complexes. Then $|\varphi|$ is a homotopy equivalence if and only if $\mathcal{X}(\varphi) : \mathcal{X}(K) \to \mathcal{X}(L)$ is a weak homotopy equivalence.*

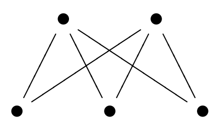
From now on we will call McCord maps to both $\mathcal{K}$-McCord maps and $\mathcal{X}$-McCord maps, and it will be clear from the context which we are referring to.

Two topological spaces X and Y, not necessarily finite, are weak homotopy equivalent (or they are said to have the same weak homotopy type) if there exists a sequence of spaces $X = X_0, X_1, \ldots, X_n = Y$ such that there are weak homotopy equivalences $X_i \to X_{i+1}$ or $X_{i+1} \to X_i$ for every $0 \leq i \leq n-1$. Clearly this defines an equivalence relation. If two topological spaces X and Y are weak homotopy equivalent, we write $X \overset{we}{\approx} Y$. If X and Y are homotopy equivalent we write $X \overset{he}{\simeq} Y$.

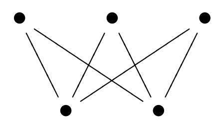
If two topological spaces X and Y are weak homotopy equivalent, there exists a CW-complex Z and weak homotopy equivalences $Z \to X$ and $Z \to Y$. CW-complexes are weak homotopy equivalent if and only if they are homotopy equivalent. As we have seen, for finite spaces, weak homotopy equivalences are not in general homotopy equivalences. Moreover, there exist weak homotopy equivalent finite spaces such that there is no weak homotopy equivalence between them.

Example 1.4.16. The non-Hausdorff suspension $\mathbb{S}(D_3)$ (see the paragraph below Definition 2.7.1) of the discrete space with three elements and its opposite $\mathbb{S}(D_3)^{op}$ have the same weak homotopy type, because there exist weak homotopy equivalences

$$\mathbb{S}(D_3) \leftarrow |\mathcal{K}(\mathbb{S}(D_3))| = |\mathcal{K}(\mathbb{S}(D_3)^{op})| \to \mathbb{S}(D_3)^{op}.$$



$\mathbb{S}(D_3)$



$\mathbb{S}(D_3)^{op}$

However there is no weak homotopy equivalence between $\mathbb{S}(D_3)$ and $\mathbb{S}(D_3)^{op}$. In fact one can check that every map $\mathbb{S}(D_3) \to \mathbb{S}(D_3)^{op}$ factors through its image, which is a subspace of $\mathbb{S}(D_3)^{op}$ with trivial fundamental group or isomorphic to $\mathbb{Z}$. We exhibit a more elegant proof in 8.4.22.

From Theorems 1.4.6 and 1.4.12 we immediately deduce the following result.

Corollary 1.4.17.

(a) Let X and Y be finite T_0-spaces. Then, $X \overset{we}{\approx} Y$ if and only if $|\mathcal{K}(X)| \overset{he}{\simeq} |\mathcal{K}(Y)|$.

(b) Let K and L be finite simplicial complexes. Then, $|K| \overset{he}{\simeq} |L|$ if and only if $\mathcal{X}(K) \overset{we}{\approx} \mathcal{X}(L)$.

McCord's Theorem 1.4.2 is one of the most useful tools to distinguish weak homotopy equivalences. Most of the times, we will apply this result to maps $f : X \to Y$ with Y finite, using the open cover given by the minimal basis of Y. The particular case of Theorem 1.4.2 for X, Y finite and T_0 and the cover $\{U_y\}_{y \in Y}$ is also a particular case of the celebrated Quillen's Theorem A applied to categories which are finite posets (see [32, 33]).

The simplicial version of Quillen's Theorem A follows from this particular case for posets and it states that if $\varphi : K \to L$ is a simplicial map and $|\varphi|^{-1}(\overline{\sigma})$ is contractible for every closed simplex $\overline{\sigma} \in |L|$, then $|\varphi|$ is a homotopy equivalence (see [32], page 93).

Using this result, we prove a similar result to Theorem 1.4.2. A topological space is said to be homotopically trivial if all its homotopy groups are zero. In virtue of Whitehead's Theorem, homotopically trivial CW-complexes are contractible.

Proposition 1.4.18. *Let $f : X \to Y$ be a map between finite T_0-spaces such that $f^{-1}(c) \subseteq X$ is homotopically trivial for every chain c of Y. Then f is a weak homotopy equivalence.*

Proof. If c is a chain of Y or, equivalently, a simplex of $\mathcal{K}(Y)$, then $|\mathcal{K}(f)|^{-1}(\overline{c}) = |\mathcal{K}(f^{-1}(c))|$, which is contractible since $f^{-1}(c)$ is homotopically trivial. By Theorem A, $|\mathcal{K}(f)|$ is a homotopy equivalence and then f is a weak homotopy equivalence. □

In fact, if the hypothesis of Proposition 1.4.18 hold, then $f^{-1}(U_y)$ is homotopically trivial for every $y \in Y$ and, by McCord Theorem, f is a weak homotopy equivalence. Therefore the proof of Proposition 1.4.18 is apparently superfluous. However, the proof of the first fact is a bit twisted, because it uses the very Proposition 1.4.18. If $f : X \to Y$ is such that $f^{-1}(c)$ is homotopically trivial for every chain c of Y, then each restriction $f|_{f^{-1}(U_y)} : f^{-1}(U_y) \to U_y$ satisfies the same hypothesis. Therefore, by Proposition 1.4.18, $f|_{f^{-1}(U_y)}$ is a weak homotopy equivalence and then $f^{-1}(U_y)$ is homotopically trivial.

In Section 4.3 we will prove, as an application of the simple homotopy theory of finite spaces, a simple homotopy version of Quillen's Theorem A for simplicial complexes.

Chapter 2

Basic topological properties of finite spaces

2.1 Homotopy and contiguity

Recall that two simplicial maps $\varphi, \psi : K \to L$ are said to be contiguous if for every simplex $\sigma \in K$, $\varphi(\sigma) \cup \psi(\sigma)$ is a simplex of L. Two simplicial maps $\varphi, \psi : K \to L$ lie in the same contiguity class if there exists a sequence $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_n = \psi$ such that φ_i and φ_{i+1} are contiguous for every $0 \le i < n$.

If $\varphi, \psi : K \to L$ lie in the same contiguity class, the induced maps in the geometric realizations $|\varphi|, |\psi| : |K| \to |L|$ are homotopic. For more datails we refer the reader to [36].

In this Section we study the relationship between contiguity classes of simplicial maps and homotopy classes of the associated maps between finite spaces.

Lemma 2.1.1. *Let $f, g : X \to Y$ be two homotopic maps between finite T_0-spaces. Then there exists a sequence $f = f_0, f_1, \ldots, f_n = g$ such that for every $0 \le i < n$ there is a point $x_i \in X$ with the following properties:*

1. *f_i and f_{i+1} coincide in $X \smallsetminus \{x_i\}$, and*
2. *$f_i(x_i) \prec f_{i+1}(x_i)$ or $f_{i+1}(x_i) \prec f_i(x_i)$.*

Proof. Without loss of generality, we may assume that $f = f_0 \le g$ by 1.2.6. Let $A = \{x \in X \mid f(x) \neq g(x)\}$. If $A = \emptyset$, $f = g$ and there is nothing to prove. Suppose $A \neq \emptyset$ and let $x = x_0$ be a maximal point of A. Let $y \in Y$ such that $f(x) \prec y \le g(x)$ and define $f_1 : X \to Y$ by $f_1|_{X \smallsetminus \{x\}} = f|_{X \smallsetminus \{x\}}$ and $f_1(x) = y$. Then f_1 is continuous for if $x' > x$, $x' \notin A$ and therefore

$$f_1(x') = f(x') = g(x') \ge g(x) \ge y = f_1(x).$$

Repeating this construction for f_i and g, we define f_{i+1}. By finiteness of X and Y this process ends. □

Proposition 2.1.2. *Let $f, g : X \to Y$ be two homotopic maps between finite T_0-spaces. Then the simplicial maps $\mathcal{K}(f), \mathcal{K}(g) : \mathcal{K}(X) \to \mathcal{K}(Y)$ lie in the same contiguity class.*

Proof. By the previous lemma, we can assume that there exists $x \in X$ such that $f(y) = g(y)$ for every $y \neq x$ and $f(x) \prec g(x)$. Therefore, if C is a chain in X, $f(C) \cup g(C)$ is a chain on Y. In other words, if $\sigma \in \mathcal{K}(X)$ is a simplex, $\mathcal{K}(f)(\sigma) \cup \mathcal{K}(g)(\sigma)$ is a simplex in $\mathcal{K}(Y)$. $\square$

Proposition 2.1.3. *Let $\varphi, \psi : K \to L$ be simplicial maps which lie in the same contiguity class. Then $\mathcal{X}(\varphi) \simeq \mathcal{X}(\psi)$.*

Proof. Assume that φ and ψ are contiguous. Then the map $f : \mathcal{X}(K) \to \mathcal{X}(L)$, defined by $f(\sigma) = \varphi(\sigma) \cup \psi(\sigma)$ is well-defined and continuous. Moreover $\mathcal{X}(\varphi) \leq f \geq \mathcal{X}(\psi)$, and then $\mathcal{X}(\varphi) \simeq \mathcal{X}(\psi)$. $\square$

2.2 Minimal pairs

In this Section we generalize Stong ideas on homotopy types to the case of pairs of finite spaces. As a consequence, we will deduce that every core of a finite T_0-space can be obtained by removing beat points from X. Here we introduce the notion of *strong collapse* which plays a central role in Chapter 5.

Definition 2.2.1. A pair (X, A) of finite T_0-spaces is a *minimal pair* if all the beat points of X are in A.

The next result generalizes the result of Stong (the case $A = \emptyset$) studied in Section 1.3 and its proof is very similar to the original one.

Proposition 2.2.2. *Let (X, A) be a minimal pair and let $f : X \to X$ be a map such that $f \simeq 1_X$ rel A. Then $f = 1_X$.*

Proof. Suppose that $f \leq 1_X$ and $f|_A = 1_A$. Let $x \in X$. If $x \in X$ is minimal, $f(x) = x$. In general, suppose we have proved that $f|_{\hat{U}_x} = 1|_{\hat{U}_x}$. If $x \in A$, $f(x) = x$. If $x \notin A$, x is not a down beat point of X. However $y < x$ implies $y = f(y) \leq f(x) \leq x$. Therefore $f(x) = x$. The case $f \geq 1_X$ is similar, and the general case follows from 1.2.6. $\square$

Corollary 2.2.3. *Let (X, A) and (Y, B) be minimal pairs, $f : X \to Y$, $g : Y \to X$ such that $gf \simeq 1_X$ rel A, $gf \simeq 1_Y$ rel B. Then f and g are homeomorphisms.*

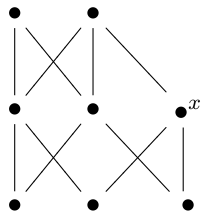
Definition 2.2.4. If x is a beat point of a finite T_0-space X, we say that there is an *elementary strong collapse* from X to $X \smallsetminus x$ and write $X \searrow\!\!\searrow^{e} X \smallsetminus x$. There is a *strong collapse* $X \searrow\!\!\searrow Y$ (or a *strong expansion* $Y \nearrow\!\!\nearrow X$) if there is a sequence of elementary strong collapses starting in X and ending in Y.

Stong's results show that two finite T_0-spaces are homotopy equivalent if and only if there exists a sequence of strong collapses and strong expansions from X to Y (since the later is true for homeomorphic spaces).

Corollary 2.2.5. *Let X be a finite T_0-space and let $A \subseteq X$. Then, $X \searrow\!\!\searrow A$ if and only if A is a strong deformation retract of X.*

Proof. If $X \searrow\!\!\searrow A$, $A \subseteq X$ is a strong deformation retract. This was already proved by Stong (see Section 1.3). Conversely, suppose $A \subseteq X$ is a strong deformation retract. Perform arbitrary elementary strong collapses removing beat points which are not in A. Suppose $X \searrow\!\!\searrow Y \supseteq A$ and that all the beat points of Y lie in A. Then (Y, A) is a minimal pair. Since A and Y are strong deformation retracts of X, the minimal pairs (A, A) and (Y, A) are in the hypothesis of Corollary 2.2.3. Therefore A and Y are homeomorphic and so, $X \searrow\!\!\searrow Y = A$. □

Example 2.2.6. The space X



is contractible, but the point x is not a strong deformation retract of X, because $(X, \{x\})$ is a minimal pair.

Corollary 2.2.7. *Let (X, A) be a minimal pair such that A is a minimal finite space and $f \simeq 1_{(X,A)} : (X, A) \to (X, A)$. Then $f = 1_X$.*

If X and Y are homotopy equivalent finite T_0-spaces, the associated polyhedra $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ also have the same homotopy type. However the converse is obviously false, since the associated polyhedra are homotopy equivalent if and only if the finite spaces are weak homotopy equivalent.

In Chapter 5 we will study the notion of *strong homotopy types* of simplicial complexes which have a very simple description and corresponds exactly to the concept of homotopy types of the associated finite spaces.

2.3 T_1-spaces

We will prove that Hausdorff spaces do not have in general the homotopy type of any finite space. Recall that a topological space X satisfies the T_1-separation axiom if for any two distinct points $x, y \in X$ there exist open sets U and V such that $x \in U$, $y \in V$, $y \notin U$, $x \notin V$. This is equivalent to say that the points are closed in X. All Hausdorff spaces are T_1, but the converse is false.

If a finite space is T_1, then every subset is closed and so, X is discrete.

Since the core X_c of a finite space X is the disjoint union of the cores of its connected components, we can deduce the following

Lemma 2.3.1. *Let X be a finite space such that X_c is discrete. Then X is a disjoint union of contractible spaces.*

Theorem 2.3.2. *Let X be a finite space and let Y be a T_1-space homotopy equivalent to X. Then X is a disjoint union of contractible spaces.*

Proof. Since $X \simeq Y$, $X_c \simeq Y$. Let $f : X_c \to Y$ be a homotopy equivalence with homotopy inverse g. Then $gf = 1_{X_c}$ by 1.3.6. Since f is a one to one map from X_c to a T_1-space, it follows that X_c is also T_1 and therefore discrete. Now the result follows from the previous lemma. □

Remark 2.3.3. The proof of the previous Theorem can be done without using 1.3.6, showing that any map $f : X \to Y$ from a finite space to a T_1-space must be locally constant.

Corollary 2.3.4. *Let Y be a connected and non contractible T_1-space. Then Y does not have the same homotopy type as any finite space.*

Proof. Follows immediately from the previous Theorem. □

For example, for any $n \geq 1$, the n-dimensional sphere S^n does not have the homotopy type of any finite space. Although, S^n does have, as any finite polyhedron, the same weak homotopy type as some finite space.

2.4 Loops in the Hasse diagram and the fundamental group

In this Section we give a full description of the fundamental group of a finite T_0-space in terms of its Hasse diagram. This characterization is induced from the well known description of the fundamental group of a simplicial complex. The Hasse diagram of a finite T_0-space X will be denoted $\mathcal{H}(X)$, and $\mathtt{E}(\mathcal{H}(X))$ will denote the set of edges of the digraph $\mathcal{H}(X)$.

Recall that an edge path in a simplicial complex K, is a sequence $(v_0, v_1), (v_1, v_2), \ldots,$ (v_{r-1}, v_r) in which $\{v_i, v_{i+1}\}$ is a simplex for every i. If an edge path contains a subsequence $(v_i, v_{i+1}), (v_{i+1}, v_{i+2})$ where $\{v_i, v_{i+1}, v_{i+2}\}$ is a simplex, we can replace it by the subsequence (v_i, v_{i+2}) to obtain an equivalent edge path. The equivalence classes of edge paths are the morphisms of a grupoid called the edge-path grupoid of K, which is denoted by $E(K)$. The full subcategory of edge paths with origin and end v_0 is the edge path-group $E(K, v_0)$ which is isomorphic to the fundamental group $\pi_1(|K|, v_0)$ (see [36] for more details).

Definition 2.4.1. Let (X, x_0) be a finite pointed T_0-space. An ordered pair of points $e = (x, y)$ is called an *$\mathcal{H}$-edge of X* if $(x, y) \in \mathtt{E}(\mathcal{H}(X))$ or $(y, x) \in \mathtt{E}(\mathcal{H}(X))$. The point x is called the *origin* of e and denoted $x = \mathfrak{o}(e)$, the point y is called the *end* of e and denoted $y = \mathfrak{e}(e)$. The *inverse* of an $\mathcal{H}$-edge $e = (x, y)$ is the $\mathcal{H}$-edge $e^{-1} = (y, x)$.

An *$\mathcal{H}$-path* in (X, x_0) is a finite sequence (possibly empty) of $\mathcal{H}$-edges $\xi = e_1 e_2 \ldots e_n$ such that $\mathfrak{e}(e_i) = \mathfrak{o}(e_{i+1})$ for all $1 \leq i \leq n-1$. The *origin* of a non empty $\mathcal{H}$-path ξ is $\mathfrak{o}(\xi) = \mathfrak{o}(e_1)$ and its *end* is $\mathfrak{e}(\xi) = \mathfrak{e}(e_n)$. The origin and the end of the empty $\mathcal{H}$-path is $\mathfrak{o}(\emptyset) = \mathfrak{e}(\emptyset) = x_0$. If $\xi = e_1 e_2 \ldots e_n$, we define $\overline{\xi} = e_n^{-1} e_{n-1}^{-1} \ldots e_1^{-1}$. If ξ, ξ' are $\mathcal{H}$-paths such that $\mathfrak{e}(\xi) = \mathfrak{o}(\xi')$, we define the product $\mathcal{H}$-path $\xi\xi'$ as the concatenation of the sequence ξ followed by the sequence ξ'.

An $\mathcal{H}$-path $\xi = e_1 e_2 \ldots e_n$ is said to be *monotonic* if $e_i \in \mathtt{E}(\mathcal{H}(X))$ for all $1 \leq i \leq n$ or $e_i^{-1} \in \mathtt{E}(\mathcal{H}(X))$ for all $1 \leq i \leq n$.

A *loop at* x_0 is an $\mathcal{H}$-path that starts and ends in x_0. Given two loops ξ, ξ' at x_0, we say that they are *close* if there exist $\mathcal{H}$-paths $\xi_1, \xi_2, \xi_3, \xi_4$ such that ξ_2 and ξ_3 are monotonic and the set $\{\xi, \xi'\}$ coincides with $\{\xi_1\xi_2\xi_3\xi_4, \xi_1\xi_4\}$.

We say that two loops ξ, ξ' at x_0 are $\mathcal{H}$-*equivalent* if there exists a finite sequence of loops $\xi = \xi_1, \xi_2, \ldots, \xi_n = \xi'$ such that any two consecutive are close. We denote by $\langle \xi \rangle$ the $\mathcal{H}$-equivalence class of a loop ξ and $\mathscr{H}(X, x_0)$ the set of these classes.

Theorem 2.4.2. *Let (X, x_0) be a pointed finite T_0-space. Then the product $\langle \xi \rangle \langle \xi' \rangle = \langle \xi\xi' \rangle$ is well defined and induces a group structure on $\mathscr{H}(X, x_0)$.*

Proof. It is easy to check that the product is well defined, associative and that $\langle \emptyset \rangle$ is the identity. In order to prove that the inverse of $\langle e_1 e_2 \ldots e_n \rangle$ is $\langle e_n^{-1} e_{n-1}^{-1} \ldots e_1^{-1} \rangle$ we need to show that for any composable $\mathcal{H}$-paths ξ, ξ' such that $\mathfrak{o}(\xi) = \mathfrak{e}(\xi') = x_0$ and for any $\mathcal{H}$-edge e, composable with ξ, one has that $\langle \xi e e^{-1} \xi' \rangle = \langle \xi\xi' \rangle$. But this follows immediately from the definition of close loops since e and e^{-1} are monotonic. □

Theorem 2.4.3. *Let (X, x_0) be a pointed finite T_0-space. Then the edge-path group $E(\mathcal{K}(X), x_0)$ of $\mathcal{K}(X)$ with base vertex x_0 is isomorphic to $\mathscr{H}(X, x_0)$.*

Proof. Let us define

$$\begin{aligned} \varphi : \mathscr{H}(X, x_0) &\longrightarrow E(\mathcal{K}(X), x_0), \\ \langle e_1 e_2 \ldots e_n \rangle &\longmapsto [e_1 e_2 \ldots e_n], \\ \langle \emptyset \rangle &\longmapsto [(x_0, x_0)], \end{aligned}$$

where $[\xi]$ denotes the class of ξ in $E(\mathcal{K}(X), x_0)$.

To prove that φ is well defined, let us suppose that the loops $\xi_1\xi_2\xi_3\xi_4$ and $\xi_1\xi_4$ are close, where $\xi_2 = e_1 e_2 \ldots e_n$, $\xi_3 = e'_1 e'_2 \ldots e'_m$ are monotonic $\mathcal{H}$-paths. By induction, it can be proved that $[\xi_1\xi_2\xi_3\xi_4] = [\xi_1 e_1 e_2 \ldots e_{n-j}(\mathfrak{o}(e_{n-j+1}), \mathfrak{e}(e_n))\xi_3\xi_4]$ for $1 \leq j \leq n$. In particular $[\xi_1\xi_2\xi_3\xi_4] = [\xi_1(\mathfrak{e}(\xi_1), \mathfrak{e}(e_n))\xi_3\xi_4]$.

Analogously,

$$[\xi_1(\mathfrak{e}(\xi_1), \mathfrak{e}(e_n))\xi_3\xi_4] = [\xi_1(\mathfrak{e}(\xi_1), \mathfrak{e}(e_n))(\mathfrak{o}(e'_1), \mathfrak{o}(\xi_4))\xi_4]$$

and then

$$\begin{aligned} [\xi_1\xi_2\xi_3\xi_4] &= [\xi_1(\mathfrak{e}(\xi_1), \mathfrak{e}(e_n))(\mathfrak{o}(e'_1), \mathfrak{o}(\xi_4))\xi_4] = [\xi_1(\mathfrak{e}(\xi_1), \mathfrak{e}(e_n))(\mathfrak{e}(e_n), \mathfrak{e}(\xi_1))\xi_4] = \\ &= [\xi_1(\mathfrak{e}(\xi_1), \mathfrak{e}(\xi_1))\xi_4] = [\xi_1\xi_4]. \end{aligned}$$

If $\xi = (x_0, x_1)(x_1, x_2) \ldots (x_{n-1}, x_n)$ is an edge path in $\mathcal{K}(X)$ with $x_n = x_0$, then x_{i-1} and x_i are comparable for all $1 \leq i \leq n$. In this case, we can find monotonic $\mathcal{H}$-paths $\xi_1, \xi_2, \ldots, \xi_n$ such that $\mathfrak{o}(\xi_i) = x_{i-1}$, $\mathfrak{e}(\xi_i) = x_i$ for all $1 \leq i \leq n$. Let us define

$$\begin{aligned} \psi : E(\mathcal{K}(X), x_0) &\longrightarrow \mathscr{H}(X, x_0), \\ [\xi] &\longmapsto \langle \xi_1\xi_2 \ldots \xi_n \rangle. \end{aligned}$$

This definition does not depend on the choice of the $\mathcal{H}$-paths ξ_i since if two choices differ only for $i = k$ then $\xi_1 \dots \xi_k \dots \xi_n$ and $\xi_1 \dots \xi'_k \dots \xi_n$ are $\mathcal{H}$-equivalent because both of them are close to $\xi_1 \dots \xi_k \xi_k^{-1} \xi'_k \dots \xi_n$.

The definition of ψ does not depend on the representative. Suppose that $\xi'(x,y)(y,z)\xi''$ and $\xi'(x,z)\xi''$ are simply equivalent edge paths in $\mathcal{K}(X)$ that start and end in x_0, where ξ and ξ' are edge paths and x, y, z are comparable.

In the case that y lies between x and z, we can choose the monotonic $\mathcal{H}$-path corresponding to (x,z) to be the juxtaposition of the corresponding to (x,y) and (y,z), and so ψ is equally defined in both edge paths.

In the case that $z \leq x \leq y$ we can choose monotonic $\mathcal{H}$-paths α, β from x to y and from z to x, and then α will be the corresponding $\mathcal{H}$-path to (x,y), $\overline{\alpha}\overline{\beta}$ that corresponding to (y,z) and $\overline{\beta}$ to (x,z). It only remains to prove that $\langle \gamma' \alpha \overline{\alpha} \overline{\beta} \gamma'' \rangle = \langle \gamma' \overline{\beta} \gamma'' \rangle$ for $\mathcal{H}$-paths γ' and γ'', which is trivial.

The other cases are analogous to the last one.

It remains to verify that φ and ψ are mutually inverses, but this is clear. □

Since $E(\mathcal{K}(X), x_0)$ is isomorphic to $\pi_1(|\mathcal{K}(X)|, x_0)$ (cf. [36]), we obtain the following result.

Corollary 2.4.4. *Let (X, x_0) be a pointed finite T_0-space, then $\mathscr{H}(X, x_0) = \pi_1(X, x_0)$.*

Remark 2.4.5. Since every finite space is homotopy equivalent to a finite T_0-space, this computation of the fundamental group can be applied to any finite space.

2.5 Euler characteristic

If the homology of a topological space X is finitely generated as a graded group, the Euler characteristic of X is defined by $\chi(X) = \sum\limits_{n \geq 0} (-1)^n rank(H_n(X))$. If Z is a compact CW-complex, its homology is finitely generated and $\chi(Z) = \sum\limits_{n \geq 0} (-1)^n \alpha_n$ where α_n is the number of n-cells of Z. A weak homotopy equivalence induces isomorphisms in homology groups and therefore weak homotopy equivalent spaces have the same Euler characteristic.

Since any finite T_0-space X is weak homotopy equivalent to the geometric realization of $\mathcal{K}(X)$, whose simplices are the non empty chains of X, the Euler characteristic of X is

$$\chi(X) = \sum_{C \in \mathcal{C}(X)} (-1)^{\#C+1} \tag{2.1}$$

where $\mathcal{C}(X)$ is the set of non empty chains of X and $\#C$ is the cardinality of C.

We will give a basic combinatorial proof of the fact that the Euler characteristic is a homotopy invariant in the setting of finite spaces, using only the formula 2.1 as definition.

Theorem 2.5.1. *Let X and Y be finite T_0-spaces with the same homotopy type. Then $\chi(X) = \chi(Y)$.*

Proof. Let X_c and Y_c be cores of X and Y. Then, there exist two sequences of finite T_0-spaces $X = X_0 \supseteq \ldots \supseteq X_n = X_c$ and $Y = Y_0 \supseteq \ldots \supseteq Y_m = Y_c$, where X_{i+1} is constructed from X_i by removing a beat point and Y_{i+1} is constructed from Y_i, similarly. Since X and Y are homotopy equivalent, X_c and Y_c are homeomorphic. Thus, $\chi(X_c) = \chi(Y_c)$.

It suffices to show that the Euler characteristic does not change when a beat point is removed. Let P be a finite poset and let $p \in P$ be a beat point. Then there exists $q \in P$ such that if r is comparable with p then r is comparable with q.

Hence we have a bijection

$$\varphi : \{C \in \mathcal{C}(P) \mid p \in C,\ q \notin C\} \longrightarrow \{C \in \mathcal{C}(P) \mid p \in C,\ q \in C\},$$
$$C \longmapsto C \cup \{q\}.$$

Therefore

$$\chi(P) - \chi(P \smallsetminus \{p\}) = \sum_{p \in C \in \mathcal{C}P} (-1)^{\#C+1} = \sum_{q \notin C \ni p} (-1)^{\#C+1} + \sum_{q \in C \ni p} (-1)^{\#C+1} =$$

$$= \sum_{q \notin C \ni p} (-1)^{\#C+1} + \sum_{q \notin C \ni p} (-1)^{\#\varphi(C)+1} = \sum_{q \notin C \ni p} (-1)^{\#C+1} + \sum_{q \notin C \ni p} (-1)^{\#C} = 0.$$

□

The Euler characteristic of finite T_0-spaces is intimately related to the Möbius function of posets, which is a generalization of the classical Möbius function of number theory. We will say just a few words about this. For proofs and applications we refer the reader to [16].

Given a finite poset P, we define the *incidence algebra* $\mathfrak{A}(P)$ of P as the set of functions $P \times P \to \mathbb{R}$ such that $f(x, y) = 0$ if $x \nleq y$ with the usual structure of $\mathbb{R}$-vector space and the product given by

$$fg(x, y) = \sum_{z \in P} f(x, z) g(z, y).$$

The element $\zeta_P \in \mathfrak{A}(P)$ defined by $\zeta_P(x, y) = 1$ if $x \leq y$ and 0 in other case, is invertible in $\mathfrak{A}(P)$. The *Möbius fuction* $\mu_P \in \mathfrak{A}(P)$ is the inverse of ζ_P.

The Theorem of Hall states that if P is a finite poset and $x, y \in P$, then $\mu_P(x, y) = \sum\limits_{n \geq 0} (-1)^{n+1} c_n$, where c_n is the number of chains of n-elements which start in x and end in y.

Given a finite poset P, $\hat{P} = P \cup \{0, 1\}$ denotes the poset obtained when adjoining a minimum 0 and a maximum 1 to P. In particular, Equation 2.1 and the Theorem of Hall, give the following

Corollary 2.5.2. *Let P be a finite poset. Then*

$$\widetilde{\chi}(P) = \mu_{\hat{P}}(0, 1),$$

where $\widetilde{\chi}(P) = \chi(P) - 1$ denotes the reduced Euler characteristic of the finite space P.

One of the motivations of the Möbius function is the following inversion formula.

Theorem 2.5.3 (Möbius inversion formula)**.** *Let P be a finite poset and let $f, g : P \to \mathbb{R}$. Then*

$$g(x) = \sum_{y \leq x} f(y) \text{ if and only if } f(x) = \sum_{y \leq x} \mu_P(y, x) g(y).$$

Analogously,

$$g(x) = \sum_{y \geq x} f(y) \text{ if and only if } f(x) = \sum_{y \geq x} \mu_P(y, x) g(y).$$

Beautiful applications of these formulae are: (1) the Möbius inversion of number theory which is obtained when applying Theorem 2.5.3 to the order given by divisibility of the integer numbers; (2) the inclusion-exclusion formula obtained from the power set of a set ordered by inclusion.

2.6 Automorphism groups of finite posets

It is well known that any finite group G can be realized as the automorphism group of a finite poset. In 1946 Birkhoff [9] proved that if the order of G is n, G can be realized as the automorphisms of a poset with $n(n+1)$ points. In 1972 Thornton [39] improved slightly Birkhoff's result: He obtained a poset of $n(2r+1)$ points, when the group is generated by r elements.

We present here a result which appears in [5]. Following Birkhoff's and Thornton's ideas, we exhibit a simple proof of the following fact which improves their results

Theorem 2.6.1. *Given a group G of finite order n with r generators, there exists a poset X with $n(r+2)$ points such that $Aut(X) \simeq G$.*

Proof. Let $\{h_1, h_2, \ldots, h_r\}$ be a set of r generators of G. We define the poset $X = G \times \{-1, 0, \ldots, r\}$ with the following order

- $(g, i) \leq (g, j)$ if $-1 \leq i \leq j \leq r$
- $(gh_i, -1) \leq (g, j)$ if $1 \leq i \leq j \leq r$

Define $\phi : G \to Aut(X)$ by $\phi(g)(h, i) = (gh, i)$. It is easy to see that $\phi(g) : X \to X$ is order preserving and that it is an automorphism with inverse $\phi(g^{-1})$. Therefore ϕ is a well defined homomorphism. Clearly ϕ is a monomorphism since $\phi(g) = 1$ implies $(g, -1) = \phi(g)(e, -1) = (e, -1)$.

It remains to show that ϕ is an epimorphism. Let $f : X \to X$ be an automorphism. Since $(e, -1)$ is minimal in X, so is $f(e, -1)$ and therefore $f(e, -1) = (g, -1)$ for some $g \in G$. We will prove that $f = \phi(g)$.

Let $Y = \{x \in X \mid f(x) = \phi(g)(x)\}$. Y is non-empty since $(e, -1) \in Y$. We prove first that Y is an open subspace of X. Suppose $x = (h, i) \in Y$. Then the restrictions

$$f|_{U_x}, \phi(g)|_{U_x} : U_x \to U_{f(x)}$$

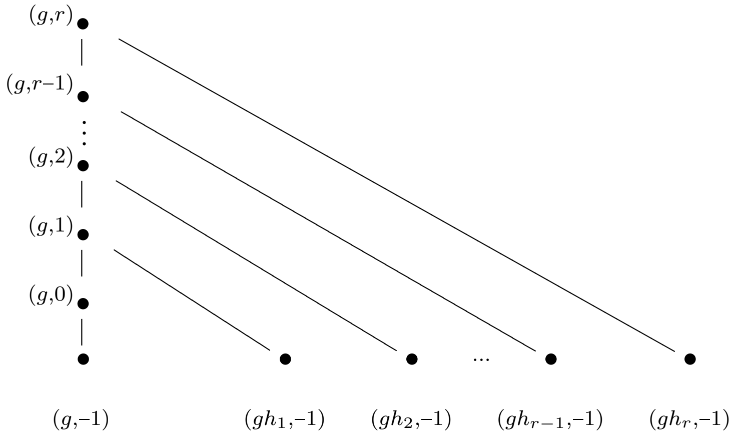


Figure 2.1: $U_{(g,r)}$.

are isomorphisms. On the other hand, there exists a unique automorphism $U_x \to U_x$ since the unique chain of $i+2$ elements must be fixed by any such automorphism. Thus, $f|_{U_x}^{-1}\phi(g)|_{U_x} = 1_{U_x}$, and then $f|_{U_x} = \phi(g)|_{U_x}$, which proves that $U_x \subseteq Y$. Similarly we see that $Y \subseteq X$ is closed. Assume $x = (h,i) \notin Y$. Since $f \in Aut(X)$, it preserves the height $ht(y)$ of any point y. In particular $ht(f(x)) = ht(x) = i+1$ and therefore $f(x) = (k,i) = \phi(kh^{-1})(x)$ for some $k \in G$. Moreover $k \neq gh$ since $x \notin Y$. As above, $f|_{U_x} = \phi(kh^{-1})|_{U_x}$, and since $kh^{-1} \neq g$ we conclude that $U_x \cap Y = \emptyset$.

We prove now that X is connected. It suffices to prove that any two minimal elements of X are in the same connected component. Given $h,k \in G$, we have $h = kh_{i_1}h_{i_2}\dots h_{i_m}$ for some $1 \leq i_1,\ i_2 \dots i_m \leq r$. On the other hand, $(kh_{i_1}h_{i_2}\dots h_{i_s},-1)$ and $(kh_{i_1}h_{i_2}\dots h_{i_{s+1}},-1)$ are connected via $(kh_{i_1}h_{i_2}\dots h_{i_s},-1) < (kh_{i_1}h_{i_2}\dots h_{i_s},r) > (kh_{i_1}h_{i_2}\dots h_{i_{s+1}},-1)$. This implies that $(k,-1)$ and $(h,-1)$ are in the same connected component.

Finally, since X is connected and Y is closed, open and nonempty, $Y = X$, i.e. $f = \phi(g)$. Therefore ϕ is an epimorphim, and then $G \simeq Aut(X)$. □ □

If the generators $h_1, h_2, \dots, h_r$ are non-trivial, the open sets $U_{(g,r)}$ look as in Figure 2.1. In that case it is not hard to prove that the finite space X constructed above is weak homotopy equivalent to a wedge of $n(r-1)+1$ circles, or in other words, that the order complex of X is homotopy equivalent to a wedge of $n(r-1)+1$ circles. The space X deformation retracts to the subspace $Y = G \times \{-1, r\}$ of its minimal and maximal points. A retraction is given by the map $f : X \to Y$, defined as $f(g,i) = (g,r)$ if $i \geq 0$ and $f(g,-1) = (g,-1)$. Now the order complex $\mathcal{K}(Y)$ of Y is a connected simplicial complex of dimension 1, so its homotopy type is completely determined by its Euler Characteristic. This complex has $2n$ vertices and $n(r+1)$ edges, which means that it has the homotopy type of a wedge of $1-\chi(\mathcal{K}(Y)) = n(r-1)+1$ circles.

On the other hand, note that in general the automorphism group of a finite space, does not say much about its homotopy type as we state in the following

Remark 2.6.2. Given a finite group G and a finite space X, there exists a finite space Y which is homotopy equivalent to X and such that $Aut(Y) \simeq G$.

We make this construction in two steps. First, we find a finite T_0-space $\tilde{X}$ homotopy equivalent to X and such that $Aut(\tilde{X}) = 0$. To do this, assume that X is T_0 and consider a linear extension $x_1, x_2, \ldots, x_n$ of the poset X. Now, for each $1 \leq k \leq n$ attach a chain of length kn to X with minimum x_{n-k+1}. The resulting space $\tilde{X}$ deformation retracts to X and every automorphism $f : \tilde{X} \to \tilde{X}$ must fix the unique chain C_1 of length n^2 (with minimum x_1). Therefore f restricts to a homeomorphism $\tilde{X} \smallsetminus C_1 \to \tilde{X} \smallsetminus C_1$ which must fix the unique chain C_2 of length $n(n-1)$ of $\tilde{X} \smallsetminus C_1$ (with minimum x_2). Applying this reasoning repeatedly, we conclude that f fixes every point of $\tilde{X}$. On the other hand, we know that there exists a finite T_0-space Z such that $Aut(Z) = G$.

Now the space Y is constructed as follows. Take one copy of $\tilde{X}$ and of Z, and put every element of Z under $x_1 \in \tilde{X}$. Clearly Y deformation retracts to $\tilde{X}$. Moreover, if $f : Y \to Y$ is an automorphism, $f(x_1) \notin Z$ since $f(x_1)$ cannot be comparable with x_1 and distinct from it. Since there is only one chain of n^2 elements in $\tilde{X}$, it must be fixed by f. In particular $f(x_1) = x_1$, and then $f|_Z : Z \to Z$. Thus f restricts to automorphisms of $\tilde{X}$ and of Z and therefore $Aut(Y) \simeq Aut(Z) \simeq G$.

2.7 Joins, products, quotients and wedges

In this Section we will study some basic constructions of finite spaces, simplicial complexes and general topological spaces. We will investigate the relationship between the simplicial and the finite space constructions and we will see how they are related to the homotopy and weak homotopy type of the spaces involved.

Recall that the *simplicial join* $K * L$ of two simplicial complexes K and L is the complex

$$K * L = K \cup L \cup \{\sigma \cup \tau |\ \sigma \in K, \tau \in L\}.$$

The cone aK of a simplicial complex K is the join of K with a vertex $a \notin K$. It is well known that for finite simplicial complexes K and L, the geometric realization $|K * L|$ is homeomorphic to the topological join $|K| * |L|$. If K is the 0-complex with two vertices, $|K * L| = |K| * |L| = S^0 * |L| = \Sigma |L|$ is the suspension of $|L|$. Here, S^0 denotes the discrete space on two points (0-sphere).

There is an analogous construction for finite spaces.

Definition 2.7.1. The *(non-Hausdorff) join* $X \oplus Y$ of two finite T_0-spaces X and Y is the disjoint union $X \sqcup Y$ keeping the giving ordering within X and Y and setting $x \leq y$ for every $x \in X$ and $y \in Y$.

Special cases of joins are the *non-Hausdorff cone* $\mathbb{C}(X) = X \oplus D^0$ and the *non-Hausdorff suspension* $\mathbb{S}(X) = X \oplus S^0$ of any finite T_0-space X. Here D^0 denotes the singleton (0-cell).

Remark 2.7.2. $\mathcal{K}(X \oplus Y) = \mathcal{K}(X) * \mathcal{K}(Y)$.

Given a point x in a finite T_0-space X, the *star* C_x of x consists of the points which are comparable with x, i.e. $C_x = U_x \cup F_x$. Note that C_x is always contractible since $1_{C_x} \leq f \geq g$ where $f : C_x \to C_x$ is the map which is the identity on F_x and the constant map x on U_x, and g is the constant map x. The *link* of x is the subspace $\hat{C}_x = C_x \smallsetminus \{x\}$. In case we need to specify the ambient space X, we will write $\hat{C}_x^X$. Note that $\hat{C}_x = \hat{U}_x \oplus \hat{F}_x$.

Proposition 2.7.3. *Let X and Y be finite T_0-spaces. Then $X \oplus Y$ is contractible if and only if X or Y is contractible.*

Proof. Assume X is contractible. Then there exists a sequence of spaces

$$X = X_n \supsetneq X_{n-1} \supsetneq \ldots \supsetneq X_1 = \{x_1\}$$

with $X_i = \{x_1, x_2, \ldots, x_i\}$ and such that x_i is a beat point of X_i for every $2 \leq i \leq n$. Then x_i is a beat point of $X_i \oplus Y$ for each $2 \leq i \leq n$ and therefore, $X \oplus Y$ deformation retracts to $\{x_1\} \oplus Y$ which is contractible. Analogously, if Y is contractible, so is $X \oplus Y$.

Now suppose $X \oplus Y$ is contractible. Then there exists a sequence

$$X \oplus Y = X_n \oplus Y_n \supsetneq X_{n-1} \oplus Y_{n-1} \supsetneq \ldots \supsetneq X_1 \oplus Y_1 = \{z_1\}$$

with $X_i \subseteq X$, $Y_i \subseteq Y$, $X_i \oplus Y_i = \{z_1, z_2 \ldots, z_i\}$ such that z_i is a beat point of $X_i \oplus Y_i$ for $i \geq 2$.

Let $i \geq 2$. If $z_i \in X_i$, z_i is a beat point of X_i unless it is a maximal point of X_i and Y_i has a minimum. In the same way, if $z_i \in Y_i$, z_i is a beat point of Y_i or X_i has a maximum. Therefore, for each $2 \leq i \leq n$, either $X_{i-1} \subseteq X_i$ and $Y_{i-1} \subseteq Y_i$ are deformation retracts (in fact, one inclusion is an identity and the other inclusion is strict), or one of them, X_i or Y_i, is contractible. This proves that X or Y is contractible. □

In 4.2.19 we will prove a result which is the analogous of 2.7.3 for collapsible finite spaces.

If X and Y are finite spaces, the preorder corresponding to the topological product $X \times Y$ is the product of the preorders of X and Y (Remark 1.1.2), i.e. $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. If X and Y are two topological spaces, not necessarily finite, and A is strong deformation retract of a X, then $A \times Y$ is a strong deformation retract of $X \times Y$.

Proposition 2.7.4. *Let X_c and Y_c be cores of finite spaces X and Y. Then $X_c \times Y_c$ is a core of $X \times Y$.*

Proof. Since $X_c \subseteq X$ is a strong deformation retract, so is $X_c \times Y \subseteq X \times Y$. Analogously $X_c \times Y_c$ is a strong deformation retract of $X_c \times Y$ and then, so is $X_c \times Y_c \subseteq X \times Y$. We have to prove that the product of minimal finite spaces is also minimal. Let $(x, y) \in X_c \times Y_c$. If there exist $x' \in X_c$ with $x' \prec x$ and $y' \in Y_c$ with $y' \prec y$, (x, y) covers at least two elements (x', y) and (x, y'). If x is minimal in X_c, $\hat{U}_{(x,y)}$ is homeomorphic to $\hat{U}_y$. Analogously if y is minimal. Therefore, (x, y) is not a down beat point. Similarly, $X_c \times Y_c$ does not have up beat points. Thus, it is a minimal finite space. □

In particular $X \times Y$ is contractible if and only if each space X and Y is contractible. In fact this result holds in general, when X and Y are not necessarily finite.

Recall that the product of two non-empty spaces is T_0 if and only if each space is.

Proposition 2.7.5. *Let X and Y be finite T_0-spaces. Then $|\mathcal{K}(X \times Y)|$ is homeomorphic to $|\mathcal{K}(X)| \times |\mathcal{K}(Y)|$.*

Proof. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the canonical projections. Define $f : |\mathcal{K}(X \times Y)| \to |\mathcal{K}(X)| \times |\mathcal{K}(Y)|$ by $f = |\mathcal{K}(p_X)| \times |\mathcal{K}(p_Y)|$. In other words, if $\alpha = \sum\limits_{i=0}^{k} t_i(x_i, y_i) \in |\mathcal{K}(X \times Y)|$ where $(x_0, y_0) < (x_1, y_1) < \ldots < (x_k, y_k)$ is a chain in $X \times Y$, define $f(\alpha) = (\sum\limits_{i=0}^{k} t_i x_i, \sum\limits_{i=0}^{k} t_i y_i)$.

Since $|\mathcal{K}(p_X)|$ and $|\mathcal{K}(p_Y)|$ are continuous, so is f. $|\mathcal{K}(X \times Y)|$ is compact and $|\mathcal{K}(X)| \times |\mathcal{K}(Y)|$ is Hausdorff, so we only need to show that f is a bijection. Details will be left to the reader. An explicit formula for $g = f^{-1}$ is given by

$$g(\sum_{i=0}^{k} u_i x_i, \sum_{i=0}^{l} v_i y_i) = \sum_{i,j} t_{ij}(x_i, y_j),$$

where $t_{ij} = max\{0, min\{u_0 + u_1 + \ldots + u_i, v_0 + v_1 + \ldots v_j\} - max\{u_0 + u_1 + \ldots + u_{i-1}, v_0 + v_1 + \ldots v_{j-1}\}\}$. The idea is very simple. Consider the segments $U_0, U_1, \ldots, U_k \subseteq I = [0,1]$, each U_i of length u_i, $U_i = [u_0 + u_1 + \ldots + u_{i-1}, u_0 + u_1 + \ldots + u_i]$. Analogously, define $V_j = [v_0 + v_1 + \ldots + v_{j-1}, v_0 + v_1 + \ldots + v_j] \subseteq I$ for $0 \le j \le l$. Then t_{ij} is the length of the segment $U_i \cap V_j$. It is not hard to see that $g : |\mathcal{K}(X)| \times |\mathcal{K}(Y)| \to |\mathcal{K}(X \times Y)|$ is well defined since $support(\sum\limits_{i,j} t_{ij}(x_i, y_j))$ is a chain and $\sum t_{ij} = \sum\limits_{i,j} length(U_i \cap V_j) = \sum\limits_{i} length(U_i) = 1$. Moreover, the compositions gf and fg are the corresponding identities. $\square$

If X is a finite T_0-space, and $A \subseteq X$ is a subspace, the quotient X/A need not be T_0. For example, if X is the chain of three elements $0 < 1 < 2$ and $A = \{0, 2\}$, X/A is the indiscrete space of two elements. We will exhibit a necessary and sufficient condition for X/A to be T_0. Recall that $\overline{A}$ denotes the closure of A.

Let X be a finite space and $A \subseteq X$ a subspace. We will denote by $q : X \to X/A$ the quotient map, qx the class in the quotient of an element $x \in X$ and denote by $\underline{A} = \{x \in X \mid \exists a \in A \text{ with } x \le a\} = \bigcup\limits_{a \in A} U_a \subseteq X$, the open hull of A.

Lemma 2.7.6. *Let $x \in X$. If $x \in \overline{A}$, $U_{qx} = q(U_x \cup \underline{A})$. If $x \notin \overline{A}$, $U_{qx} = q(U_x)$.*

Proof. Suppose $x \in \overline{A}$. Since $q^{-1}(q(U_x \cup \underline{A})) = U_x \cup \underline{A} \subseteq X$ is open, $q(U_x \cup \underline{A}) \subseteq X/A$ is open and contains qx. Therefore $U_{qx} \subseteq q(U_x \cup \underline{A})$. The other inclusion follows from the continuity of q since $x \in \overline{A}$.

If $x \notin \overline{A}$, $q^{-1}(q(U_x)) = U_x$, so $q(U_x)$ is open and therefore $U_{qx} \subseteq q(U_x)$. The other inclusion is trivial. $\square$

Proposition 2.7.7. *Let X be a finite space and $A \subseteq X$ a subspace. Let $x, y \in X$, then $qx \le qy$ in the quotient X/A if and only if $x \le y$ or there exist $a, b \in A$ such that $x \le a$ and $b \le y$.*

Proof. Assume $qx \leq qy$. If $y \in \overline{A}$, there exists $b \in A$ with $b \leq y$ and by the previous lemma $qx \in U_{qy} = q(U_y \cup \underline{A})$. Therefore $x \in U_y \cup \underline{A}$ and then $x \leq y$ or $x \leq a$ for some $a \in A$. If $y \notin \overline{A}$, $qx \in U_{qy} = q(U_y)$. Hence, $x \in U_y$.

Conversely if $x \leq y$ or there are some $a, b \in A$ such that $x \leq a$ and $b \leq y$, then $qx \leq qy$ or $qx \leq qa = qb \leq qy$. □

Proposition 2.7.8. *Let X be a finite T_0-space and $A \subseteq X$. The quotient X/A is not T_0 if and only if there exists a triple $a < x < b$ with $a, b \in A$ and $x \notin A$.*

Proof. Suppose there is not such triple and that $qx \leq qx'$, $qx' \leq qx$. Then $x \leq y$ or there exist $a, b \in A$ with $x \leq a$, $b \leq x'$, and, on the other hand, $y \leq x$ or there are some $a', b' \in A$ such that $x' \leq a'$, $b' \leq x$. If $x \leq x'$ and $x' \leq x$, then $x = x'$. In other case, both x and x' are in A. Therefore, $qx = qx'$. This proves that X/A is T_0. Conversely, if there exists a triple $a < x < b$ as above, $qa \leq qx \leq qb = qa$, but $qa \neq qx$. Therefore, X/A is not T_0. □

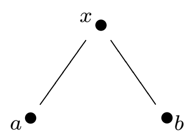
The non-existence of a triple as above is equivalent to say that $A = \overline{A} \cap \underline{A}$, i.e.:

$$X/A \text{ is } T_0 \text{ if and only if } A = \overline{A} \cap \underline{A}.$$

For example open or closed subsets satisfy this condition.

Now we want to study how the functors $\mathcal{X}$ and $\mathcal{K}$ behave with respect to quotients.

Example 2.7.9. Let $X = \mathbb{C}D_2 = \{x, a, b\}$ and let $A = \{a, b\}$ be the subspace of minimal elements.

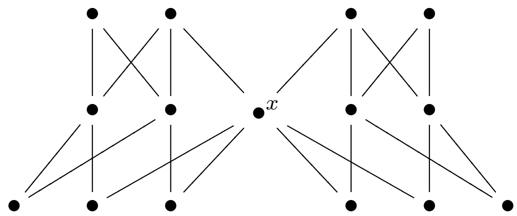


Then, X/A is the Sierpinski space and $|\mathcal{K}(X)|/|\mathcal{K}(A)|$ is homeomorphic to S^1. Therefore $|\mathcal{K}(X)|/|\mathcal{K}(A)|$ and $|\mathcal{K}(X/A)|$ are not homotopy equivalent. However $X'/A' = S^0 \oplus S^0$ and then $|\mathcal{K}(X')|/|\mathcal{K}(A')|$ and $|\mathcal{K}(X'/A')|$ are both homeomorphic to a circle. The application $\mathcal{K}$ does not preserve quotients in general. In 7.2.2 we prove that if A is a subspace of a finite T_0-space X, $|\mathcal{K}(X')|/|\mathcal{K}(A')|$ and $|\mathcal{K}(X'/A')|$ are homotopy equivalent.

A particular case of a quotient X/A is the one-point union or wedge. If X and Y are topological spaces with base points $x_0 \in X$, $y_0 \in Y$, then the wedge $X \vee Y$ is the quotient $X \sqcup Y/A$ with $A = \{x_0, y_0\}$. Clearly, if X and Y are finite T_0-spaces, $A = \{x_0, y_0\} \subseteq X \sqcup Y$ satisfies $A = \overline{A} \cap \underline{A}$ and then $X \vee Y$ is also T_0. Moreover, if $x, x' \in X$, then x covers x' in X if and only if x covers x' in $X \vee Y$. The same holds for Y, and if $x \in X \smallsetminus \{x_0\}$, $y \in Y \smallsetminus \{y_0\}$ then x does not cover y in $X \vee Y$ and y does not cover x. Thus, the Hasse diagram of $X \vee Y$ is the union of the Hasse diagrams of X and Y, identifying x_0 and y_0.

If $X \vee Y$ is contractible, then X and Y are contractible. This holds for general topological spaces. Let $i : X \to X \vee Y$ denote the canonical inclusion and $r : X \vee Y \to X$ the retraction which sends all of Y to x_0. If $H : (X \vee Y) \times I \to X \vee Y$ is a homotopy between the identity and a constant, then $rH(i \times 1_I) : X \times I \to X$ shows that X is contractible. The following example shows that the converse is not true for finite spaces.

Example 2.7.10. The space X of Example 2.2.6 is contractible, but the union at x of two copies of X is a minimal finite space, and in particular it is not contractible.



However, from Corollary 4.2.26 we will deduce that $X \vee X$ is homotopically trivial, or in other words, it is weak homotopy equivalent to a point. This is the first example we exhibit of a finite space which has trivial homotopy groups but which is not contractible. These spaces play a fundamental rol in the theory of finite spaces.

In Proposition 4.2.25 we will prove that if X and Y are finite T_0-spaces, there is a weak homotopy equivalence $|\mathcal{K}(X)| \vee |\mathcal{K}(Y)| \to X \vee Y$.

Chapter 3

Minimal finite models

In 2.3 we proved that in general, if K is a finite simplicial complex, there is no finite space with the homotopy type of $|K|$. However, the theory of McCord, shows how to use finite spaces to model compact polyhedra finding for each finite complex K a weak homotopy equivalent finite space $\mathcal{X}(K)$. In this Chapter we will study finite models of polyhedra in this sense and we will describe the *minimal finite models* of some known Hausdorff spaces, i.e. weak homotopy equivalent finite spaces of minimum cardinality.

Definition 3.0.1. Let X be a space. We say that a finite space Y is a *finite model* of X if it is weak homotopy equivalent to X. We say that Y is a *minimal finite model* if it is a finite model of minimum cardinality.

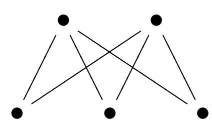
For example, the singleton is the unique minimal finite model of every contractible space. Moreover, it is the unique minimal finite model of every homotopically trivial space, i.e. with trivial homotopy groups.

Since every finite space is homotopy equivalent to its core, which is a smaller space, we have the following

Remark 3.0.2. Every minimal finite model is a minimal finite space.

Since $\mathcal{K}(X) = \mathcal{K}(X^{op})$, if X is a minimal finite model of a space Y, then so is X^{op}.

Example 3.0.3. The 5-point T_0-space X, whose Hasse diagram is



has an associated polyhedron $|\mathcal{K}(X)|$, which is homotopy equivalent to $S^1 \vee S^1$. Therefore, X is a finite model of $S^1 \vee S^1$. In fact, it is a minimal finite model since every space with less than 5 points is either contractible, or non connected or weak homotopy equivalent to S^1. However, this minimal finite model is not unique since X^{op} is another minimal finite model not homeomorphic to X.

We will generalize this result later, when we characterize the minimal finite models of graphs.

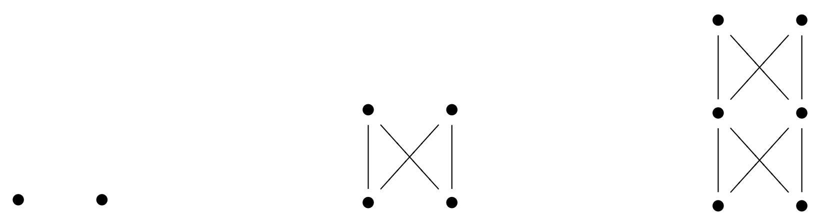
Note that, by Whitehead Theorem, if X is a finite model of a polyhedron Y, then Y is homotopy equivalent to $|\mathcal{K}(X)|$.

Generalizing the Definition made in Section 2.7, we define the *non-Hausdorff suspension* $\mathbb{S}(X)$ of a topological space X as the space $X \cup \{+, -\}$ whose open sets are those of X together with $X \cup \{+\}$, $X \cup \{-\}$ and $X \cup \{+, -\}$. If X is a finite space, the non-Hausddorff suspension of X is the join $\mathbb{S}(X) = X \oplus S^0$. The *non-Hausdorff suspension of order n* is defined recursively by $\mathbb{S}^n(X) = \mathbb{S}(\mathbb{S}^{n-1}(X))$. For convenience we define $\mathbb{S}^0(X) = X$.

The following result is due to McCord [26].

Proposition 3.0.4. *The finite space $\mathbb{S}^n(S^0)$ is a finite model of the n-dimensional sphere S^n for every $n \geq 0$.*

Proof. By 2.7.2, $|\mathcal{K}(\mathbb{S}^n(S^0))| = |\mathcal{K}(S^0 \oplus S^0 \oplus \ldots \oplus S^0)| = |\mathcal{K}(S^0)| * |\mathcal{K}(S^0)| * \ldots * |\mathcal{K}(S^0)| = S^0 * S^0 * \ldots * S^0 = S^n$. □



S^0 $\mathbb{S}(S^0)$ $\mathbb{S}^2(S^0)$

In [23] P. May conjectures that $\mathbb{S}^n(S^0)$ is a minimal finite model of S^n. We will show that this conjecture is true. In fact, we prove a stronger result. Namely, we will see that any space with the same homotopy groups as S^n has at least $2n+2$ points. Moreover, if it has exactly $2n+2$ points then it has to be homeomorphic to $\mathbb{S}^n S^0$.

3.1 Minimal finite models of the spheres

The *height* $h(X)$ of a finite poset X is one less that the maximum cardinality of a chain of X. Therefore $h(X)$ coincides with the dimension of the associated complex $\mathcal{K}(X)$.

Theorem 3.1.1. *Let $X \neq *$ be a minimal finite space. Then X has at least $2h(X)+2$ points. Moreover, if X has exactly $2h(X)+2$ points, then it is homeomorphic to $\mathbb{S}^{h(X)}(S^0)$.*

Proof. Let $x_0 < x_1 < \ldots < x_h$ be a chain in X of length $h = h(X)$. Since X is a minimal finite space, x_i is not an up beat point for any $0 \leq i < h$. Then, for every $0 \leq i < h$ there exists $y_{i+1} \in X$ such that $y_{i+1} > x_i$ and $y_{i+1} \not\geq x_{i+1}$. We assert that the points y_i (for $0 < i \leq h$) are all distinct from each other and also different from the x_j ($0 \leq j \leq h$).

Since $y_{i+1} > x_i$, it follows that $y_{i+1} \neq x_j$ for all $j \leq i$. But $y_{i+1} \neq x_j$ for all $j > i$ because $y_{i+1} \not\geq x_{i+1}$.

If $y_{i+1} = y_{j+1}$ for some $i < j$, then $y_{i+1} = y_{j+1} \geq x_j \geq x_{i+1}$, which is a contradiction.

Since finite spaces with minimum or maximum are contractible and $X \neq *$ is a minimal finite space, it cannot have a minimum. Then there exists $y_0 \in X$ such that $y_0 \ngeq x_0$. Therefore, y_0 must be distinct from the other $2h+1$ points and $\#X \geq 2h+2$.

Let us suppose now that X has exactly $2h+2$ points, i.e.

$$X = \{x_0, x_1, \dots, x_h, y_0, y_1, \dots, y_h\}.$$

Because of the maximality of the chain $x_0 < \dots < x_h$, we get that x_i and y_i are incomparable for all i.

We show that $y_i < x_j$ and $y_i < y_j$ for all $i < j$ by induction in j.

For $j = 0$ there is nothing to prove. Let $0 \leq k < h$ and assume the statement holds for $j = k$. As x_{k+1} is not a down beat point, there exists $z \in X$ such that $z < x_{k+1}$, and $z \nleq x_k$. Since x_{k+1} and y_{k+1} are incomparable, it follows that $z \neq y_{k+1}$. By induction we know that every point in X, with the exception of y_k and y_{k+1}, is greater than x_{k+1} or less than x_k. Then $z = y_k$ and so, $y_k < x_{k+1}$. Analogously, y_{k+1} is not a down beat point and there exists $w \in X$ such that $w < y_{k+1}$ and $w \nleq x_k$. Again by induction, and because $y_{k+1} \ngeq x_{k+1}$, we deduce that w must be y_k and then $y_k < y_{k+1}$. Furthermore, if $i < k$, then $y_i < x_k < x_{k+1}$ and $y_i < x_k < y_{k+1}$.

We proved that, for any $i < j$, we have that $y_i < x_j$, $y_i < y_j$, $x_i < x_j$ and $x_i < y_j$. Moreover, for any $0 \leq i \leq h$, x_i and y_i are incomparable.

This is exactly the order of $\mathbb{S}^h(S^0)$. Therefore X is homeomorphic to $\mathbb{S}^h(S^0)$. □

Theorem 3.1.2. *Any space with the same homotopy groups as S^n has at least $2n+2$ points. Moreover, $\mathbb{S}^n(S^0)$ is the unique space with $2n+2$ points with this property.*

Proof. The case $n = 1$ is trivial. In the other cases, let us suppose that X is a finite space with minimum cardinality such that $\pi_k(X, x) = \pi_k(S^n, s)$ for all $k \geq 0$. Then X must be a minimal finite space and so is T_0.

By the Hurewicz Theorem, $H_n(|\mathcal{K}(X)|) = \pi_n(|\mathcal{K}(X)|) = \pi_n(S^n) \neq 0$. This implies that the dimension of the simplicial complex $\mathcal{K}(X)$ must be at least n, which means that the height of X is at least n. The result now follows immediately from the previous theorem. □

Corollary 3.1.3. *The n-sphere has a unique minimal finite model and it has $2n+2$ points.*

Remark 3.1.4. After obtaining these results, we found another article of McCord (*Singular homology and homotopy groups of finite spaces*, Notices of the American Mathematical Society, vol. 12(1965)) with a result (Theorem 2) without proof, from which the first part of 3.1.2 could be deduced. McCord's result can be easily deduced from our stronger theorem 3.1.1 (which also implies the uniqueness of these minimal models).

Furthermore, we think that the proof of 3.1.1 itself is interesting because it relates the combinatorial methods of Stong's theory with McCord's point of view.

3.2 Minimal finite models of graphs

Remark 3.2.1. If X is a connected finite T_0-space of height one, $|\mathcal{K}(X)|$ is a connected graph, i.e. a CW complex of dimension one. Therefore, the weak homotopy type of X is completely determined by its Euler characteristic. More precisely, if $\chi(X) = \#X - \#\mathtt{E}(\mathcal{H}(X)) = n$, then X is a finite model of $\bigvee_{i=1}^{1-n} S^1$. Recall that $\mathtt{E}(\mathcal{H}(X))$ denotes the set of edges of the Hasse diagram of X.

Proposition 3.2.2. *Let X be a connected finite T_0-space and let $x_0, x \in X$, $x_0 \neq x$ such that x is neither maximal nor minimal in X. Then the inclusion map of the associated simplicial complexes $\mathcal{K}(X \smallsetminus \{x\}) \subseteq \mathcal{K}(X)$ induces an epimorphism*

$$i_* : E(\mathcal{K}(X \smallsetminus \{x\}), x_0) \to E(\mathcal{K}(X), x_0)$$

between their edge-path groups.

Proof. We have to check that every closed edge path in $\mathcal{K}(X)$ with base point x_0 is equivalent to another edge path that does not go through x. Let us suppose that $y \leq x$ and $(y,x)(x,z)$ is an edge path in $\mathcal{K}(X)$. If $x \leq z$ then $(y,x)(x,z) \equiv (y,z)$. In the case that $z < x$, since x is not maximal in X, there exists $w > x$. Therefore $(y,x)(x,z) \equiv (y,x)(x,w)(w,x)(x,z) \equiv (y,w)(w,z)$. The case $y \geq x$ is analogous.

In this way, one can eliminate x from the writing of any closed edge path with base point x_0. □

Note that the space $X \smallsetminus \{x\}$ of the previous proposition is also connected. An alternative proof of the previous proposition is given by the Van Kampen Theorem. Let $C_x = U_x \cup F_x$ be the star of x. Since x is not maximal or minimal, the link $\hat{C}_x = C_x \smallsetminus \{x\}$ is connected. Then Van Kampen gives an epimorphism $\pi_1(|\mathcal{K}(X \smallsetminus x)|) * \pi_1(|\mathcal{K}(C_x)|) \to \pi_1(|\mathcal{K}(X)|)$. But $\mathcal{K}(C_x) = x\mathcal{K}(\hat{C}_x)$ is a cone, and then $\pi_1(|\mathcal{K}(C_x)|) = 0$. Therefore, $i_* : \pi_1(|\mathcal{K}(X \smallsetminus x)|) \to \pi_1(|\mathcal{K}(X)|)$ is an epimorphism.

The result above shows one of the advantages of using finite spaces instead of simplicial complexes. The conditions of maximality or minimality of points in a finite space are hard to express in terms of simplicial complexes.

Remark 3.2.3. If X is a finite T_0-space, then $h(X) \leq 1$ if and only if every point in X is maximal or minimal.

Corollary 3.2.4. *Let X be a connected finite space. Then there exists a connected T_0-subspace $Y \subseteq X$ of height at most one such that the fundamental group of X is a quotient of the fundamental group of Y.*

Proof. We can assume that X is T_0 because X has a core. Now, the result follows immediately from the previous proposition. □

Remark 3.2.5. Note that the fundamental group of a connected finite T_0-space of height at most one is finitely generated by 3.2.1. Therefore, path-connected spaces whose fundamental group does not have a finite set of generators do not admit finite models.

Corollary 3.2.6. *Let $n \in \mathbb{N}$. If X is a minimal finite model of $\bigvee_{i=1}^{n} S^1$, then $h(X) = 1$.*

Proof. Let X be a minimal finite model of $\bigvee_{i=1}^{n} S^1$. Then there exists a connected T_0-subspace $Y \subseteq X$ of height one, $x \in Y$ and an epimorphism from $\pi_1(Y, x)$ to $\pi_1(X, x) = \mathop{*}_{i=1}^{n} \mathbb{Z}$.

Since $h(Y) = 1$, Y is a model of a graph, thus $\pi_1(Y, x) = \mathop{*}_{i=1}^{m} \mathbb{Z}$ for some integer m. Note that $m \geq n$.

There are m edges of $\mathcal{H}(Y)$ which are not in a maximal tree of the underlying non directed graph of $\mathcal{H}(Y)$ (i.e. $\mathcal{K}(Y)$). Therefore, we can remove $m - n$ edges from $\mathcal{H}(Y)$ in such a way that it remains connected and the new space Z obtained in this way is a model of $\bigvee_{i=1}^{n} S^1$.

Note that $\#Z = \#Y \leq \#X$, but since X is a minimal finite model, $\#X \leq \#Z$ and then $X = Y$ has height one. □

If X is a minimal finite model of $\bigvee_{i=1}^{n} S^1$ and we call $i = \#\{y \in X \mid y \text{ is maximal}\}$, $j = \#\{y \in X \mid y \text{ is minimal}\}$, then $\#X = i + j$ and $\#\mathtt{E}(\mathcal{H}(X)) \leq ij$. Since $\chi(X) = 1 - n$, we have that $n \leq ij - (i + j) + 1 = (i - 1)(j - 1)$.

We can now state the main result of this Section.

Theorem 3.2.7. *Let $n \in \mathbb{N}$. A finite T_0-space X is a minimal finite model of $\bigvee_{i=1}^{n} S^1$ if and only if $h(X) = 1$, $\#X = min\{i+j \mid (i-1)(j-1) \geq n\}$ and $\#\mathtt{E}(\mathcal{H}(X)) = \#X + n - 1$.*

Proof. We have already proved that if X is a minimal finite model of $\bigvee_{i=1}^{n} S^1$, then $h(X) = 1$ and $\#X \geq min\{i + j \mid (i - 1)(j - 1) \geq n\}$. If i and j are such that $n \leq (i - 1)(j - 1)$, we can consider $Y = \{x_1, x_2, \ldots, x_i, y_1, y_2, \ldots y_j\}$ with the order $y_k \leq x_l$ for all k, l, which is a model of $\bigvee_{k=1}^{(i-1)(j-1)} S^1$. Then we can remove $(i - 1)(j - 1) - n$ edges from $\mathcal{H}(X)$ to obtain a connected space of cardinality $i + j$ which is a finite model of $\bigvee_{k=1}^{n} S^1$. Therefore $\#X \leq \#Y = i + j$. This is true for any i, j with $n \leq (i - 1)(j - 1)$, then $\#X = min\{i + j \mid (i - 1)(j - 1) \geq n\}$. Moreover, $\#\mathtt{E}(\mathcal{H}(X)) = \#X + n - 1$ because $\chi(X) = 1 - n$.

In order to show the converse of the theorem we only need to prove that the conditions $h(X) = 1$, $\#X = min\{i + j \mid (i - 1)(j - 1) \geq n\}$ and $\#\mathtt{E}(\mathcal{H}(X)) = \#X + n - 1$ imply that X is connected, because in this case, by 3.2.1, the first and third conditions would say that X is a model of $\bigvee_{i=1}^{n} S^1$, and the second condition would say that it has the right cardinality.

Suppose X satisfies the conditions of above and let X_l, $1 \leq l \leq k$, be the connected components of X. Let us denote by M_l the set of maximal elements of X_l and let $m_l =$

$X_l \smallsetminus M_l$. Let $i = \sum\limits_{r=1}^{k} \# M_l$, $j = \sum\limits_{r=1}^{k} \# m_l$. Since $i+j = \# X = min\{s+t \mid (s-1)(t-1) \geq n\}$, it follows that $(i-2)(j-1) < n = \#\mathtt{E}(\mathcal{H}(X)) - \#X + 1 = \#\mathtt{E}(\mathcal{H}(X)) - (i+j) + 1$. Hence $ij - \#\mathtt{E}(\mathcal{H}(X)) < j - 1$. This means that $\mathcal{K}(X)$ differs from the complete bipartite graph $(\cup m_l, \cup M_l)$ in less than $j-1$ edges. Since there are no edges from m_r to M_l if $r \neq l$,

$$j - 1 > \sum_{l=1}^{k} \# M_l (j - \# m_l) \geq \sum_{l=1}^{k} (j - \# m_l) = (k-1)j.$$

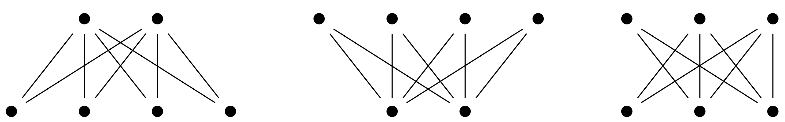
Therefore $k = 1$ and the proof is complete. □

Remark 3.2.8. Since the minimum $min\{i + j \mid (i-1)(j-1) \geq n\}$ is attained for $i = j$ or $i = j + 1$, the cardinality of a minimal finite model of $\bigvee\limits_{i=1}^{n} S^1$ is

$$min\{2\lceil \sqrt{n} + 1 \rceil, 2 \left\lceil \frac{1 + \sqrt{1 + 4n}}{2} \right\rceil + 1\}.$$

Note that a space may admit many minimal finite models as we can see in the following example.

Example 3.2.9. Any minimal finite model of $\bigvee\limits_{i=1}^{3} S^1$ has 6 points and 8 edges. So, they are, up to homeomorphism



In fact, using our characterization, it is not hard to prove the following

Proposition 3.2.10. $\bigvee\limits_{i=1}^{n} S^1$ *has a unique minimal finite model if and only if n is a square.*

Note that since any graph is a $K(G, 1)$, the minimal finite models of a graph X are, in fact, the smallest spaces with the same homotopy groups as X.

Chapter 4

Simple homotopy types and finite spaces

J.H.C. Whitehead's theory of simple homotopy types is inspired by Tietze's theorem in combinatorial group theory, which states that any finite presentation of a group could be deformed into any other by a finite sequence of elementary moves, which are now called Tietze transformations. Whitehead translated these algebraic moves into the well-known geometric moves of elementary collapses and expansions of finite simplicial complexes. His beautiful theory turned out to be fundamental for the development of piecewise-linear topology: The s-cobordism theorem, Zeeman's conjecture [45], the applications of the theory in surgery, Milnor's classical paper on Whitehead Torsion [28] and the topological invariance of torsion are some of its major uses and advances.

In this Chapter we show how to use finite topological spaces to study simple homotopy types using the relationship between finite spaces and simplicial complexes

We have seen that if two finite T_0-spaces X, Y are homotopy equivalent, their associated simplicial complexes $\mathcal{K}(X), \mathcal{K}(Y)$ are also homotopy equivalent. Furthermore, Osaki [31] showed that in this case, the latter have the same simple homotopy type. Nevertheless, we noticed that the converse of this result is not true in general: There are finite spaces with different homotopy types whose associated simplicial complexes have the same simple homotopy type. Starting from this point, we were looking for the relation that X and Y should satisfy for their associated complexes to be simple homotopy equivalent. More specifically, we wanted to find an elementary move in the setting of finite spaces (if it existed) which corresponds exactly to a simplicial collapse of the associated polyhedra.

We discovered this elementary move when we were looking for a homotopically trivial finite space (i.e. weak homotopy equivalent to a point) which was non-contractible. In order to construct such a space, we developed a method of reduction, i.e. a method that allows us to reduce a finite space to a smaller weak homotopy equivalent space. This method of reduction together with the homotopically trivial and non-contractible space (of 11 points) that we found are exhibited in Section 4.2. Suprisingly, this method, which consists of removing a *weak point* of the space (see Definition 4.2.2), turned out to be the key to solve the problem of translating simplicial collapses into this setting.

We will say that two finite spaces are *simple homotopy equivalent* if we can obtain one of them from the other by adding and removing weak points. If Y is obtained from X by only removing weak points, we say that X *collapses* to Y and write $X \searrow Y$. The first main result of this Chapter is the following

Theorem 4.2.12.

(a) Let X and Y be finite T_0-spaces. Then, X and Y are simple homotopy equivalent if and only if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type. Moreover, if $X \searrow Y$ then $\mathcal{K}(X) \searrow \mathcal{K}(Y)$.

(b) Let K and L be finite simplicial complexes. Then, K and L are simple homotopy equivalent if and only if $\mathcal{X}(K)$ and $\mathcal{X}(L)$ are simple homotopy equivalent. Moreover, if $K \searrow L$ then $\mathcal{X}(K) \searrow \mathcal{X}(L)$.

In particular, the functors $\mathcal{K}$ and $\mathcal{X}$ induce a one-to-one correspondence between simple equivalence classes of finite spaces and simple homotopy types:

$$\{Finite\ T_0 - Spaces\}\Big/\!\diagup\!\!\!\searrow \quad \underset{\mathcal{X}}{\overset{\mathcal{K}}{\rightleftarrows}} \quad \{Finite\ Simplicial\ Complexes\}\Big/\!\diagup\!\!\!\searrow$$

We are now able to study finite spaces using all the machinery of Whitehead's simple homotopy theory for CW-complexes. But also, what is more important, we can use finite spaces to strengthen the classical theory. The elementary move in this setting is much simpler to handle and describe because it consists of adding or removing just one single point. Applications of this theorem will appear constantly in the next Chapters.

In the third Section of this Chapter we investigate the class of maps between finite spaces which induce simple homotopy equivalences between their associated simplicial complexes. To this end, we introduce the notion of a *distinguished* map. Similarly to the classical case, the class of simple equivalences between finite spaces can be generated, in a certain way, by expansions and a kind of formal homotopy inverses of expansions. Remarkably this class, denoted by $\mathcal{S}$, is also generated by the distinguished maps. The second main result of this Chapter is the following

Theorem 4.3.12.

(a) Let $f : X \to Y$ be a map between finite T_0-spaces. Then f is a simple equivalence if and only if $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ is a simple homotopy equivalence.

(b) Let $\varphi : K \to L$ be a simplicial map between finite simplicial complexes. Then φ is a simple homotopy equivalence if and only if $\mathcal{X}(\varphi)$ is a simple equivalence.

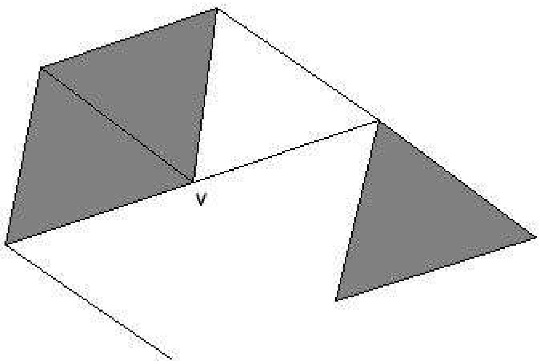
Many of the results of this Chapter were originally published in [8], but here we exhibit more applications and shorter proofs in some cases.

4.1 Whitehead's simple homotopy types

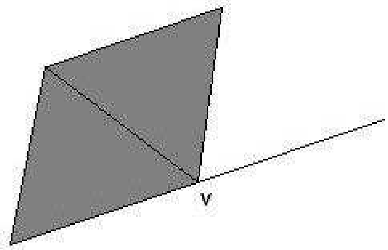
At the end of the 1930's, J.H.C. Whitehead, started to study homotopy types of polyhedra from a combinatorial point of view. An elementary simplicial collapse of a finite simplicial complex K is a fundamental move that transforms K into another complex L. This move leads to the notion of simple homotopy types of simplicial complexes. It is easy to prove that simple homotopy equivalent complexes have homotopy equivalent geometric realizations. In 1939, Whitehead asked whether the converse of this result held. It was Whitehead himself who, in 1950 proved that the answer of his question was negative. Moreover, he found the obstruction for this implication to hold, which is now called the Whitehead group of the complex. During the development of his theory, Whitehead had to overcome a lot of difficulties intrinsic from the rigid structure of simplicial complexes. These obstacles finally led him to the definition of CW-complexes.

In this Section we will recall some basic notions on simplicial complexes and simple homotopy theory for complexes and we will fix the notations that we will use henceforth. The standard references for this are Whitehead's papers [44, 42, 43], Milnor's article [28] and M.M.Cohen's book [14].

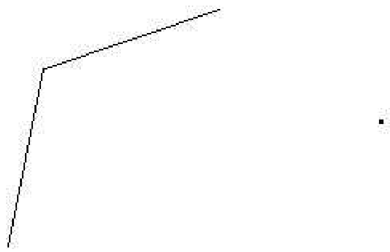
If K is a simplicial complex and v is a vertex of K, the *(simplicial) star* of v in K is the subcomplex $st(v) \subseteq K$ of simplices $\sigma \in K$ such that $v\sigma \in K$. The *link* of v in K is the subcomplex $lk(v) \subseteq st(x)$ of the simplices which do not contain v.



A complex K and a vertex $v \in K$.



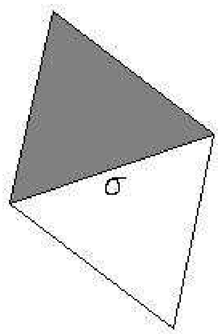
The star $st(v)$ of v.



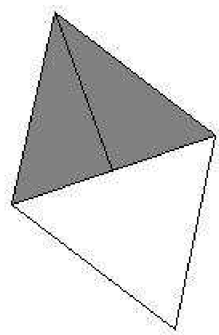
The link $lk(v)$ of v.

More generally, if σ is a simplex of K, its *star* $st(\sigma)$ is the subcomplex of K whose simplices are the simplices $\sigma' \in K$ such that $\sigma\sigma' \in K$. The *link* $lk(\sigma)$ is the subcomplex of $st(\sigma)$ of the simplices which are disjoint with σ.

If σ is a simplex of K, $\dot{\sigma}$ denotes its boundary and σ^c denotes the subcomplex of K of the simplices which do not contain σ. The *stellar subdivision* of K at the simplex σ is the complex $a\dot{\sigma}lk(\sigma) + \sigma^c$ where a is a vertex which is not in K. The first barycentric subdivision K' of K can be obtained from K by performing a sequence of stellar subdivisions (see [18]).



A complex K and a simplex $\sigma \in K$.



The stellar subdivision of K at σ.

Let L be a subcomplex of a finite simplicial complex K. There is an *elementary simplicial collapse* from K to L if there is a simplex σ of K and a vertex a of K not in σ such that $K = L \cup a\sigma$ and $L \cap a\sigma = a\dot{\sigma}$. This is equivalent to say that there are only two simplices σ, σ' of K which are not in L and such that σ' is the unique simplex containing σ properly. In this case we say that σ is a free face of σ'. Elementary collapses will be denoted, as usual, $K \overset{e}{\searrow} L$.

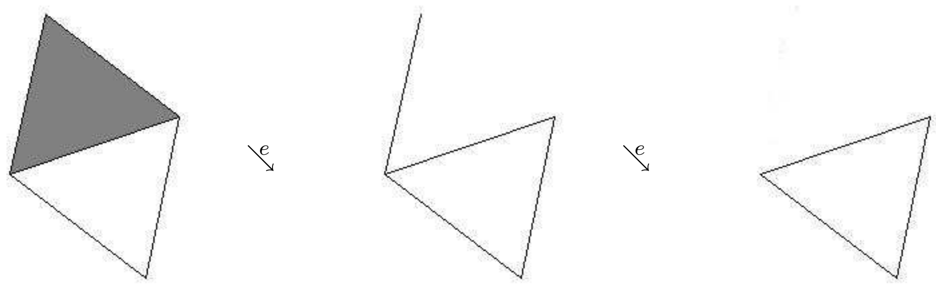


Figure 4.1: A complex which collapses to the boundary of a 2-simplex

We say that K *(simplicially) collapses* to L (or that L *expands* to K) if there exists a sequence $K = K_1, K_2, \dots, K_n = L$ of finite simplicial complexes such that $K_i \overset{e}{\searrow} K_{i+1}$ for all i. This is denoted by $K \searrow L$ or $L \nearrow K$. Two complexes K and L have the same *simple homotopy type* (or they are *simple homotopy equivalent*) if there is a sequence $K = K_1, K_2, \dots, K_n = L$ such that $K_i \searrow K_{i+1}$ or $K_i \nearrow K_{i+1}$ for all i. Following M.M. Cohen's notation, we denote this by $K \diagup\!\!\searrow L$.

The notion of simple homotopy types is extended to CW-complexes. The Whitehead group $Wh(G)$ of a group G is a quotient of the first group of K-theory $K_1(\mathbb{Z}(G))$ (see [14]). The Whitehead group $Wh(X)$ of a connected CW-complex X is the Whitehead group of its fundamental group $Wh(\pi_1(X))$. If two homotopy equivalent CW-complexes have trivial Whitehead group, then they are simple homotopy equivalent, but if $Wh(X) \neq 0$, there exists a CW-complex Y with the homotopy type of X and different simple homotopy type.

For example, if G is a free group, $Wh(G) = 0$. In particular, contractible CW-complexes are simple homotopy equivalent.

Remark 4.1.1. If K and L are subcomplexes of a finite simplicial complex, then $K \cup L \searrow K$ if and only if $L \searrow K \cap L$.

Remark 4.1.2. If K is a finite simplicial complex, then $K \diagup\!\!\!\searrow K'$. In fact we can perform all the collapses and expansions involving complexes of dimension at most $n+1$ where n is the dimension of K. In this case we say that K $(n+1)$-deforms to K'. Moreover, this is true not only for the barycentric subdivision, but for any stellar subdivision αK of K.

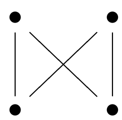
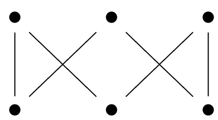
Suppose σ is a simplex of K and a is a vertex which is not in K. Then $a\dot{\sigma} \overset{e}{\nearrow} a\sigma \searrow \sigma$ (see 4.2.10). Therefore $a\dot{\sigma}lk(\sigma) \nearrow a\sigma lk(\sigma) \searrow \sigma lk(\sigma)$ and then

$$\alpha K = a\dot{\sigma}lk(\sigma) + \sigma^c \nearrow a\sigma lk(\sigma) + \sigma^c \searrow \sigma lk(\sigma) + \sigma^c = K$$

where αK is the stellar subdivision at the simplex σ.

4.2 Simple homotopy types: The first main Theorem

The first mathematician who investigated the relationship between finite spaces and simple homotopy types of polyhedra was T. Osaki [31]. He showed that if $x \in X$ is a beat point, $\mathcal{K}(X)$ collapses to $\mathcal{K}(X \smallsetminus \{x\})$. In particular, if two finite T_0-spaces, X and Y are homotopy equivalent, their associated simplicial complexes, $\mathcal{K}(X)$ and $\mathcal{K}(Y)$, have the same simple homotopy type. However, there exist finite spaces which are not homotopy equivalent but whose associated complexes have the same simple homotopy type. Consider, for instance, the spaces with the following Hasse diagrams.

They are not homotopy equivalent because they are non-homeomorphic minimal finite spaces. However their associated complexes are triangulations of S^1 and therefore, have the same simple homotopy type.

A more interesting example is the following.

Example 4.2.1 (The Wallet)**.** Let W be a finite T_0-space, whose Hasse diagram is the one of Figure 4.2 below.

This finite space is not contractible since it does not have beat points, but it is not hard to see that $|\mathcal{K}(W)|$ is contractible and therefore, it has the same simple homotopy type as a point. In fact we will deduce from Proposition 4.2.3 that W is a homotopically trivial space, i.e. all its homotopy groups are trivial. This example also shows that Whitehead Theorem does not hold in the context of finite spaces, not even for homotopically trivial spaces.

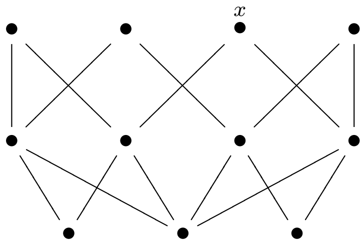


Figure 4.2: W.

We introduce now the notion of a *weak beat point* which generalizes Stong's definition of beat points.

Definition 4.2.2. Let X be a finite T_0-space. We will say that $x \in X$ is a *weak beat point of* X (or a *weak point*, for short) if either $\hat{U}_x$ is contractible or $\hat{F}_x$ is contractible. In the first case we say that x is a *down weak point* and in the second, that x is an *up weak point.*

Note that beat points are in particular weak points since spaces with maximum or minimum are contractible. Since the link $\hat{C}_x = \hat{U}_x \oplus \hat{F}_x$ is a join, we conclude from 2.7.3 that x is a weak point if and only if $\hat{C}_x$ is contractible.

When x is a beat point of X, we have seen that the inclusion $i : X \smallsetminus \{x\} \hookrightarrow X$ is a homotopy equivalence. This is not the case if x is just a weak point. However, a slightly weaker result holds.

Proposition 4.2.3. *Let x be a weak point of a finite T_0-space X. Then the inclusion map $i : X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence.*

Proof. We may suppose that x is a down weak point since the other case follows immediately from this one, considering X^{op} instead of X. Note that $\mathcal{K}(X^{op}) = \mathcal{K}(X)$.

Given $y \in X$, the set $i^{-1}(U_y) = U_y \smallsetminus \{x\}$ has a maximum if $y \neq x$ and is contractible if $y = x$. Therefore $i|_{i^{-1}(U_y)} : i^{-1}(U_y) \to U_y$ is a weak homotopy equivalence for every $y \in X$. Now the result follows from Theorem 1.4.2 applied to the basis-like cover given by the minimal basis of X. □

As an application of the last proposition, we verify that the space W defined above, is a non-contractible homotopically trivial space. As we pointed out in Example 4.2.1, W is not contractible since it is a minimal finite space with more than one point. However, it contains a weak point x (see Figure 4.2), since $\hat{U}_x$ is contractible (see Figure 4.3). Therefore W is weak homotopy equivalent to $W \smallsetminus \{x\}$. Now it is easy to see that this

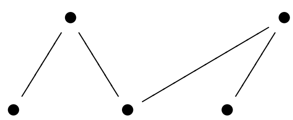


Figure 4.3: $\hat{U}_x$.

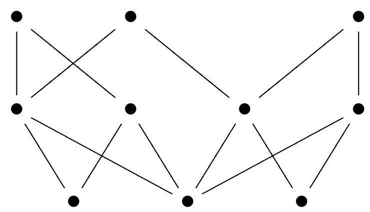


Figure 4.4: $W \smallsetminus \{x\}$.

subspace is contractible, because it does have beat points, and one can get rid of them one by one.

Definition 4.2.4. Let X be a finite T_0-space and let $Y \subsetneq X$. We say that X *collapses* to Y by an *elementary collapse* (or that Y *expands* to X by an *elementary expansion*) if Y is obtained from X by removing a weak point. We denote $X \searrow^e Y$ or $Y \nearrow^e X$. In general, given two finite T_0-spaces X and Y, we say that X *collapses* to Y (or Y *expands* to X) if there is a sequence $X = X_1, X_2, \ldots, X_n = Y$ of finite T_0-spaces such that for each $1 \leq i < n$, $X_i \searrow^e X_{i+1}$. In this case we write $X \searrow Y$ or $Y \nearrow X$. Two finite T_0-spaces X and Y are *simple homotopy equivalent* if there is a sequence $X = X_1, X_2, \ldots, X_n = Y$ of finite T_0-spaces such that for each $1 \leq i < n$, $X_i \searrow X_{i+1}$ or $X_i \nearrow X_{i+1}$. We denote in this case $X \diagup\!\!\searrow Y$, following the same notation that we adopted for simplicial complexes.

In contrast with the classical situation, where a simple homotopy equivalence is a special kind of homotopy equivalence, we will see that homotopy equivalent finite spaces are simple homotopy equivalent. In fact this follows almost immediately from the fact that beat points are weak points.

It follows from Proposition 4.2.3 that simple homotopy equivalent finite spaces are weak homotopy equivalent.

In order to prove Theorem 4.2.12, we need some previous results. The first one concerns the homotopy type of the associated finite space $\mathcal{X}(K)$ of a simplicial cone K. Suppose $K = aL$ is a cone, i.e. K is the join of a simplicial complex L with a vertex $a \notin L$. Since $|K|$ is contractible, it is clear that $\mathcal{X}(K)$ is homotopically trivial. The following lemma shows that $\mathcal{X}(K)$ is in fact contractible (compare with [33]).

Lemma 4.2.5. *Let $K = aL$ be a finite cone. Then $\mathcal{X}(K)$ is contractible.*

Proof. Define $f : \mathcal{X}(K) \to \mathcal{X}(K)$ by $f(\sigma) = \sigma \cup \{a\}$. This function is order-preserving and therefore continuous.

If we consider the constant map $g : \mathcal{X}(K) \to \mathcal{X}(K)$ that takes all $\mathcal{X}(K)$ into $\{a\}$, we have that $1_{\mathcal{X}(K)} \leq f \geq g$. This proves that the identity is homotopic to a constant map. □

The following construction is the analogue to the mapping cylinder of general spaces and the simplicial mapping cylinder of simplicial complexes.

Definition 4.2.6. Let $f : X \to Y$ be a map between finite T_0-spaces. We define the *non-Hausdorff mapping cylinder* $B(f)$ as the following finite T_0-space. The underlying set is the disjoint union $X \sqcup Y$. We keep the given ordering within X and Y and for $x \in X$, $y \in Y$ we set $x \leq y$ in $B(f)$ if $f(x) \leq y$ in Y.

Lemma 4.2.7. *Let $f : X \to Y$ be a map between finite T_0-spaces such that $f^{-1}(U_y)$ is contractible for every $y \in Y$. Then $B(f) \searrow i(X)$ and $B(f) \searrow j(Y)$, where $i : X \hookrightarrow B(f)$ and $j : Y \hookrightarrow B(f)$ are the canonical inclusions.*

Proof. Label all the elements $x_1, x_2, \dots, x_n$ of X in such a way that $x_r \leq x_s$ implies $r \leq s$ and define $Y_r = j(Y) \cup \{i(x_1), i(x_2), \dots, i(x_r)\} \subseteq B(f)$ for each $0 \leq r \leq n$. Then

$$\hat{F}^{Y_r}_{i(x_r)} = \{j(y) \mid y \geq f(x_r)\}$$

is homeomorphic to the contractible space $F^{Y}_{f(x_r)}$. It follows that $Y_r \overset{e}{\searrow} Y_{r-1}$ for $1 \leq r \leq n$, and then $B(f) = Y_n$ collapses to $j(Y) = Y_0$. Notice that we have not yet used the hypothesis on f.

Now order the elements $y_1, y_2, \dots, y_m$ of Y in such a way that $y_r \leq y_s$ implies $r \leq s$ and define $X_r = i(X) \cup \{j(y_{r+1}), j(y_{r+2}), \dots, j(y_m)\} \subseteq B(f)$ for every $0 \leq r \leq m$. Then

$$\hat{U}^{X_{r-1}}_{j(y_r)} = \{i(x) \mid f(x) \leq y_r\}$$

is homeomorphic to $f^{-1}(U_{y_r})$, which is contractible by hypothesis. Thus $X_{r-1} \overset{e}{\searrow} X_r$ for $1 \leq r \leq m$ and therefore $B(f) = X_0$ collapses to $i(X) = X_m$. $\square$

Notice that in Definition 4.2.4 it is not explicit that homeomorphic finite T_0-spaces are simple homotopy equivalent. One could have added that to the definition, but it is not needed since it can be deduced from it. If X and Y are disjoint homeomorphic finite T_0-spaces, then we can take a homeomorphism $f : X \to Y$ and the underlying set of $B(f)$ as the union of the dijoint sets X and Y. Then by Lemma 4.2.7, $X \nearrow B(f) \searrow Y$. In the case that X and Y are non-disjoint, one can choose a third space Z homeomorphic to X and Y and disjoint from both of them. Therefore $X \diagup\!\!\searrow Z \diagup\!\!\searrow Y$.

Now we can positively deduce the following

Remark 4.2.8. Homotopy equivalent finite T_0-spaces are simple homotopy equivalent. Suppose $X \overset{he}{\simeq} Y$ and that X_c and Y_c are cores of X and Y. Since beat points are weak points, $X \searrow X_c$ and $Y \searrow Y_c$. On the other hand, X_c and Y_c are homeomorphic and therefore, $X_c \diagup\!\!\searrow Y_c$.

As we pointed out above, any finite simplicial complex K has the same simple homotopy type of its barycentric subdivision K'. We prove next an analogous result for finite spaces. Following [20], the *barycentric subdivision* of a finite T_0-space X is defined by $X' = \mathcal{X}(\mathcal{K}(X))$. Explicitly, X' consists of the non-empty chains of X ordered by inclusion. It is shown in [20] that there is a weak homotopy equivalence $h : X' \to X$ which takes each chain C to its maximum $max(C)$.

Proposition 4.2.9. *Let X be a finite T_0-space. Then X and X' are simple homotopy equivalent.*

Proof. It suffices to show that the map $h : X' \to X$ satisfies the hypothesis of Lemma 4.2.7. This is clear since $h^{-1}(U_x) = \{C \mid max(C) \leq x\} = (U_x)' = \mathcal{X}(x\mathcal{K}(\hat{U}_x))$ is contractible by Lemma 4.2.5 (in fact, if a finite T_0-space Y is contractible, so is Y' (see Corollary 5.0.18)). □

Note also that the proof of Proposition 4.2.9 shows that h is a weak homotopy equivalence. Moreover, any map in the hypothesis of Lemma 4.2.7 is a weak homotopy equivalence by Theorem 1.4.2.

We will use the following easy lemma whose proof we omit.

Lemma 4.2.10. *Let aK be a simplicial cone of a finite complex K. Then, K is collapsible if and only if $aK \searrow K$.*

Lemma 4.2.11. *Let v be a vertex of a finite simplicial complex K. Then, $lk(v)$ is collapsible if and only if $K \searrow K \smallsetminus v$.*

Proof. By Lemma 4.2.10, $lk(v)$ is collapsible if and only if $st(v) = vlk(v) \searrow lk(v) = st(v) \cap (K \smallsetminus v)$ if and only if $K = st(v) \cup (K \smallsetminus v) \searrow K \smallsetminus v$. □

Theorem 4.2.12.

(a) Let X and Y be finite T_0-spaces. Then, X and Y are simple homotopy equivalent if and only if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same simple homotopy type. Moreover, if $X \searrow Y$ then $\mathcal{K}(X) \searrow \mathcal{K}(Y)$.

(b) Let K and L be finite simplicial complexes. Then, K and L are simple homotopy equivalent if and only if $\mathcal{X}(K)$ and $\mathcal{X}(L)$ are simple homotopy equivalent. Moreover, if $K \searrow L$ then $\mathcal{X}(K) \searrow \mathcal{X}(L)$.

Proof. Let X be a finite T_0-space and assume first that $x \in X$ is a beat point. Then, there exists $x' \in X$ and subspaces $Y, Z \subseteq X$ such that $\hat{C}_x = Y \oplus \{x'\} \oplus Z$. The link $lk(x)$ of the vertex x in $\mathcal{K}(X)$ is collapsible, since $lk(x) = \mathcal{K}(\hat{C}_x) = x'\mathcal{K}(Y \oplus Z)$. By Lemma 4.2.11, $\mathcal{K}(X) \searrow \mathcal{K}(X \smallsetminus \{x\})$. In particular, if X is contractible, $\mathcal{K}(X)$ is collapsible.

Now suppose $x \in X$ is a weak point. Then $\hat{C}_x$ is contractible and therefore $lk(x) = \mathcal{K}(\hat{C}_x)$ is collapsible. Again, by 4.2.11, $\mathcal{K}(X) \searrow \mathcal{K}(X \smallsetminus \{x\})$. We have then proved that $X \searrow Y$ implies $\mathcal{K}(X) \searrow \mathcal{K}(Y)$. In particular, $X \diagup\!\!\searrow Y$ implies $\mathcal{K}(X) \diagup\!\!\searrow \mathcal{K}(Y)$.

Suppose now that K and L are finite simplicial complexes such that $K \overset{e}{\searrow} L$. Then, there exist $\sigma \in K$ and a vertex a of K not in σ such that $a\sigma \in K$, $K = L \cup \{\sigma, a\sigma\}$ and $a\sigma \cap L = a\dot{\sigma}$. It follows that σ is an up beat point of $\mathcal{X}(K)$, and since $\hat{U}_{a\sigma}^{\mathcal{X}(K)\smallsetminus\{\sigma\}} = \mathcal{X}(a\dot{\sigma})$, by Lemma 4.2.5, $a\sigma$ is a down weak point of $\mathcal{X}(K) \smallsetminus \{\sigma\}$. Therefore $\mathcal{X}(K) \overset{e}{\searrow} \mathcal{X}(K) \smallsetminus \{\sigma\} \overset{e}{\searrow} \mathcal{X}(K) \smallsetminus \{\sigma, a\sigma\} = \mathcal{X}(L)$. This proves the first part of (b) and the "moreover" part.

Let X, Y be finite T_0-spaces such that $\mathcal{K}(X) \diagup\!\!\searrow \mathcal{K}(Y)$. Then $X' = \mathcal{X}(\mathcal{K}(X)) \diagup\!\!\searrow \mathcal{X}(\mathcal{K}(Y)) = Y'$ and by Proposition 4.2.9, $X \diagup\!\!\searrow Y$. Finally, if K, L are finite simplicial complexes such that $\mathcal{X}(K) \diagup\!\!\searrow \mathcal{X}(L)$, $K' = \mathcal{K}(\mathcal{X}(K)) \diagup\!\!\searrow \mathcal{K}(\mathcal{X}(L)) = L'$ and therefore $K \diagup\!\!\searrow L$. This completes the proof. □

Corollary 4.2.13. *The functors $\mathcal{K}$, $\mathcal{X}$ induce a one-to-one correspondence between simple equivalence classes of finite spaces and simple homotopy types of finite simplicial complexes*

$$\{Finite\ T_0 - Spaces\}\Big/\diagup\!\!\searrow \quad \underset{\mathcal{X}}{\overset{\mathcal{K}}{\rightleftarrows}} \quad \{Finite\ Simplicial\ Complexes\}\Big/\diagup\!\!\searrow$$

The following diagrams illustrate the whole situation.

$$\begin{array}{ccccccc}
X \overset{he}{\simeq} Y & \Longrightarrow & X \diagup\!\!\searrow Y & \Longrightarrow & X \overset{we}{\approx} Y & & \\
 & & \Updownarrow & & \Updownarrow & & \\
 & & \mathcal{K}(X)\diagup\!\!\searrow\mathcal{K}(Y) & \Longrightarrow & |\mathcal{K}(X)| \overset{we}{\approx} |\mathcal{K}(Y)| & \Longleftrightarrow & |\mathcal{K}(X)| \overset{he}{\simeq} |\mathcal{K}(Y)|
\end{array}$$

$$\begin{array}{ccccccc}
\mathcal{X}(K) \overset{he}{\simeq} \mathcal{X}(L) & \Longrightarrow & \mathcal{X}(K)\diagup\!\!\searrow\mathcal{X}(L) & \Longrightarrow & \mathcal{X}(K) \overset{we}{\approx} \mathcal{X}(L) & & \\
 & & \Updownarrow & & \Updownarrow & & \\
 & & K\diagup\!\!\searrow L & \Longrightarrow & |K| \overset{we}{\approx} |L| & \Longleftrightarrow & |K| \overset{he}{\simeq} |L|
\end{array}$$

The Wallet W satisfies $W \searrow *$, however $W \overset{he}{\not\simeq} *$. Therefore $X\diagup\!\!\searrow Y \not\Rightarrow X \overset{he}{\simeq} Y$. Since $|K| \overset{he}{\simeq} |L| \not\Rightarrow K\diagup\!\!\searrow L$, $X \overset{we}{\approx} Y \not\Rightarrow X\diagup\!\!\searrow Y$. Note that, if $X \overset{we}{\approx} Y$ and their Whitehead group $Wh(\pi_1(X))$ is trivial, then $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ are simple homotopy equivalent CW-complexes. It follows from Theorem 4.2.12 that $X\diagup\!\!\searrow Y$. Thus, we have proved

Corollary 4.2.14. *Let X, Y be weak homotopy equivalent finite T_0-spaces such that $Wh(\pi_1(X)) = 0$. Then $X\diagup\!\!\searrow Y$.*

Beat points defined by Stong provide an effective way of deciding whether two finite spaces are homotopy equivalent. The problem becomes much harder when one deals with weak homotopy types instead. There is no easy way to decide whether two finite spaces are weak homotopy equivalent or not. However if two finite T_0-spaces have trivial Whitehead group, then they are weak homotopy equivalent if and only we can obtain one from the other just by adding and removing weak points.

Another immediate consequence of the Theorem is the following

Corollary 4.2.15. *Let X, Y be finite T_0-spaces. If $X \searrow Y$, then $X' \searrow Y'$.*

Note that from Theorem 4.2.12 one also deduces the following well-known fact: If K and L are finite simplicial complexes such that $K \searrow L$, then $K' \searrow L'$.

4.2.1 Joins, products, wedges and collapsibility

The notion of collapsibility for finite spaces is closely related with the analogous notion for simplicial complexes: We say that a finite T_0-space is *collapsible* if it collapses to a point. Observe that every contractible finite T_0-space is collapsible, however the converse is not true. The Wallet W introduced in Example 4.2.1 is collapsible and non-contractible. Note that if a finite T_0-space X is collapsible, its associated simplicial complex $\mathcal{K}(X)$ is also collapsible. Moreover, if K is a collapsible complex, then $\mathcal{X}(K)$ is a collapsible finite space. Therefore, if X is a collapsible finite space, its subdivision X' is also collapsible.

Remark 4.2.16. Note that if the link $\hat{C}_x$ of a point $x \in X$ is collapsible, $\mathcal{K}(\hat{C}_x)$ is also collapsible and one has that $\mathcal{K}(X) \searrow \mathcal{K}(X \smallsetminus \{x\})$ by 4.2.11.

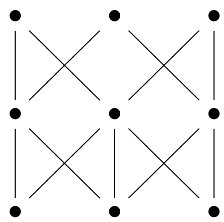
Example 4.2.17. Let W be the Wallet, and $\mathbb{C}(W)$ its non-Hausdorff cone. By Remark 4.2.16, $\mathcal{K}(\mathbb{C}(W)) \searrow \mathcal{K}(W)$ but $\mathbb{C}(W)$ does not collapse to W.

Let us consider now a compact contractible polyhedron X with the property that any triangulation of X is non-collapsible, for instance the Dunce Hat [45]. Let K be any triangulation of X. The associated finite space $\mathcal{X}(K)$ is homotopically trivial because X is contractible. However, $\mathcal{X}(K)$ is not collapsible since K' is not collapsible. In Figure 7.3 we exhibit a finite space of 15 points which is homotopically trivial and non-collapsible.

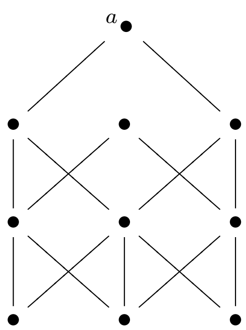
We have therefore the following strict implications in the context of finite spaces:

$$\text{contractible} \Rightarrow \text{collapsible} \Rightarrow \text{homotopically trivial}.$$

Example 4.2.18. The following space X is another example of a collapsible space which is not contractible.



The space $X \cup \{a\}$ is contractible and collapses to X. Therefore contractibility is not invariant under collapses.



It is known that if K and L are finite simplicial complexes and one of them is collapsible, then $K * L$ is also collapsible. As far as we know the converse of this result is an open problem (see [41, (4.1)]). In the setting of finite spaces, the analogous result and its converse hold.

Proposition 4.2.19. *Let X and Y be finite T_0-spaces. Then $X \oplus Y$ is collapsible if and only if X or Y is collapsible.*

Proof. We proceed as in Proposition 2.7.3, replacing beat points by weak points and deformation retractions by collapses. Note that if x_i is a weak point of X_i, then x_i is also a weak point of $X_i \oplus Y$, since $\hat{C}_{x_i}^{X_i \oplus Y} = \hat{C}_{x_i}^{X_i} \oplus Y$ is contractible by Proposition 2.7.3.

On the other hand, if z_i is a weak point of $X_i \oplus Y_i$ and $z_i \in X_i$, then by Proposition 2.7.3, z_i is a weak point of X_i or Y_i is contractible. □

By the proof of Proposition 4.2.19 one also has the following

Proposition 4.2.20. *Let X_1, X_2, Y_1, Y_2 be finite T_0-spaces. If $X_1 \diagup\!\!\searrow X_2$ and $Y_1 \diagup\!\!\searrow Y_2$, $X_1 \oplus Y_1 \diagup\!\!\searrow X_2 \oplus Y_2$.*

These are a similar results for products.

Lemma 4.2.21. *Let X and Y be finite T_0-spaces. If $X \searrow A$, $X \times Y \searrow A \times Y$.*

Proof. It suffices to show that if $x \in X$ is a weak point of X, $X \times Y \searrow (X \smallsetminus \{x\}) \times Y$. Suppose without loss of generality that x is a down weak point. If $y \in Y$,

$$\hat{U}_{(x,y)} = \hat{U}_x \times U_y \cup \{x\} \times \hat{U}_y.$$

Let $y_0 \in Y$ be a minimal point. Then $\hat{U}_{(x,y_0)} = \hat{U}_x \times U_{y_0}$ is contractible since each factor is contractible. Therefore, (x, y_0) is a down weak point of $X \times Y$. Now, let y_1 be minimal in $Y \smallsetminus \{y_0\}$. Then $\hat{U}_{(x,y_1)}^{X \times Y \smallsetminus \{(x,y_0)\}} = \hat{U}_x \times U_{y_1}^Y \cup \{x\} \times \hat{U}_{y_1}^Y \smallsetminus \{(x, y_0)\} = \hat{U}_x \times U_{y_1}^Y \cup \{x\} \times \hat{U}_{y_1}^{Y \smallsetminus \{y_0\}} = \hat{U}_x \times U_{y_1}^Y$ which again is contractible. Therefore (x, y_1) is a weak point in $X \times Y \smallsetminus \{(x, y_0)\}$. Following this reasoning we remove from $X \times Y$ all the points of the form (x, y) with $y \in Y$. □

In particular we deduce the following two results.

Proposition 4.2.22. *Let X_1, X_2, Y_1, Y_2 be finite T_0-spaces. If $X_1 \diagup\!\!\searrow X_2$ and $Y_1 \diagup\!\!\searrow Y_2$, $X_1 \times Y_1 \diagup\!\!\searrow X_2 \times Y_2$.*

Proposition 4.2.23. *Let X and Y be collapsible finite T_0-spaces. Then $X \times Y$ is collapsible.*

There is an analogous result of Proposition 4.2.23 for the associated complexes, which relates the collapsibility of $\mathcal{K}(X \times Y)$ with the collapsibility of $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ (see [41]).

The following lemma, was used in the original proof of Theorem 4.2.12 in [8]. The shorter proof we exhibit here does not use this result, but we will need it for the proof of Proposition 4.2.25

Lemma 4.2.24. *Let L be a subcomplex of a finite simplicial complex K. Let T be a set of simplices of K which are not in L, and let a be a vertex of K which is contained in no simplex of T, but such that $a\sigma$ is a simplex of K for every $\sigma \in T$. Finally, suppose that $K = L \cup \bigcup_{\sigma\in T} \{\sigma, a\sigma\}$ (i.e. the simplices of K are those of L together with the simplices σ and $a\sigma$ for every σ in T). Then $L \nearrow K$.*

Proof. Number the elements $\sigma_1, \sigma_2, \dots, \sigma_n$ of T in such a way that for every i, j with $i \leq j$, $\#\sigma_i \leq \#\sigma_j$. Here $\#\sigma_k$ denotes the cardinality of σ_k. Define $K_i = L \cup \bigcup_{j=1}^{i} \{\sigma_j, a\sigma_j\}$ for $0 \leq i \leq n$. Let $\sigma \subsetneq \sigma_i$. If $\sigma \in T$, then $\sigma, a\sigma \in K_{i-1}$, since $\#\sigma < \#\sigma_i$. If $\sigma \notin T$, then $\sigma, a\sigma \in L \subseteq K_{i-1}$. This proves that $a\sigma_i \cap K_{i-1} = a\dot{\sigma_i}$.

By induction, K_i is a simplicial complex for every i, and $K_{i-1} \overset{e}{\nearrow} K_i$. Therefore $L = K_0 \nearrow K_n = K$. □

Proposition 4.2.25. *Let (X, x_0) and (Y, y_0) be finite T_0-pointed spaces. Then, there exists a weak homotopy equivalence $|\mathcal{K}(X)| \vee |\mathcal{K}(Y)| \to X \vee Y$.*

Proof. Let $\mathcal{K}(X) \vee \mathcal{K}(Y) \subseteq \mathcal{K}(X \vee Y)$ be the simplicial complex which is the union of the complexes $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ identifying the vertices x_0 and y_0. Then $|\mathcal{K}(X)| \vee |\mathcal{K}(Y)|$ is homeomorphic to $|\mathcal{K}(X) \vee \mathcal{K}(Y)|$. The McCord map $\mu_{X\vee Y} : |\mathcal{K}(X \vee Y)| \to X \vee Y$ induces a map $f = \mu_{X\vee Y} i : |\mathcal{K}(X)| \vee |\mathcal{K}(Y)| \to X \vee Y$, where $i : |\mathcal{K}(X)| \vee |\mathcal{K}(Y)| \hookrightarrow |\mathcal{K}(X \vee Y)|$ is the canonical inclusion. In order to prove that f is a weak homotopy equivalence, we only need to prove that i is a homotopy equivalence. We show something stronger: there is a simplicial expansion from $\mathcal{K}(X) \vee \mathcal{K}(Y)$ to $\mathcal{K}(X \vee Y)$.

Take $K = \mathcal{K}(X \vee Y)$ and $L = \mathcal{K}(X) \vee \mathcal{K}(Y)$. Let $a = x_0 = y_0$ and let $T = \{\sigma \in K \mid \sigma \notin L \text{ and } a \notin \sigma\}$. If $\sigma \in T$, then every point of σ is comparable with a, and therefore $a\sigma \in K$. By Lemma 4.2.24, $L \nearrow K$. □

Corollary 4.2.26. *Let X and Y be finite T_0-spaces. Then $X \vee Y$ is homotopically trivial if and only if both X and Y are.*

Proof. If X and Y are homotopically trivial, the polyhedra $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ are contractible and therefore $|\mathcal{K}(X)| \vee |\mathcal{K}(Y)|$ is contractible. Thus, $X \vee Y$ is homotopically trivial by Proposition 4.2.25. Conversely, if $X \vee Y$ is homotopically trivial, $|\mathcal{K}(X)| \vee |\mathcal{K}(Y)|$ is contractible and then $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ are contractible. Therefore, X and Y are homotopically trivial. □

Suppose that X and Y are finite T_0-spaces and $x_0 \in X$, $y_0 \in Y$ are minimal points. If $X \vee Y$ is collapsible it can be proved by induction that both X and Y are collapsible. If $z \in X \vee Y$ is a weak point, $z \neq \overline{x}_0$ (the class of x_0 in $X \vee Y$) unless $X = *$ or $Y = *$. But the distinguished point $\overline{x}_0 \in X \vee Y$ could be a weak point with $X \neq * \neq Y$ if $x_0 \in X$ or $y_0 \in Y$ is not minimal. It is not known in the general case whether $X \vee Y$ collapsible implies that X and Y are collapsible. However, the converse is false as the next example shows.

Example 4.2.27. The simplicial complex K of Example 9.1.8 is collapsible and therefore, $\mathcal{X}(K)$ is collapsible. The space $\mathcal{X}(K)$ has a unique weak point σ corresponding to the unique free face of K. Then, the union $X = \mathcal{X}(K) \vee \mathcal{X}(K)$ of two copies of $\mathcal{X}(K)$ at $x_0 = \sigma$ is homotopically trivial, but it has no weak points and then it is not collapsible. If $x \in \mathcal{X}(K)$ is distinct from $\overline{x}_0$, $\hat{C}_x^X$ deformation retracts into $\hat{C}_x^{\mathcal{X}(K)}$ which is not contractible. The point $\overline{x}_0 \in X$ is not a weak point either, since its link $\hat{C}_{\overline{x}_0}^X$ is a join of non-connected spaces.

4.3 Simple homotopy equivalences: The second main Theorem

In this Section we prove the second main result of the Chapter, which relates simple homotopy equivalences of complexes with *simple equivalences* between finite spaces. Like in the classical setting, the class of simple equivalences is generated by the elementary expansions. However, in the context of finite spaces this class is also generated by the *distinguished* maps, which play a key role in this theory.

Recall that a homotopy equivalence $f : |K| \to |L|$ between compact polyhedra is a simple homotopy equivalence if it is homotopic to a composition of a finite sequence of maps $|K| \to |K_1| \to \ldots \to |K_n| \to |L|$, each of them an expansion or a homotopy inverse of one [14, 35].

We prove first that homotopy equivalences between finite spaces induce simple homotopy equivalences between the associated polyhedra.

Theorem 4.3.1. *If $f : X \to Y$ is a homotopy equivalence between finite T_0-spaces, then $|\mathcal{K}(f)| : |\mathcal{K}(X)| \to |\mathcal{K}(Y)|$ is a simple homotopy equivalence.*

Proof. Let X_c and Y_c be cores of X and Y. Let $i_X : X_c \to X$ and $i_Y : Y_c \to Y$ be the inclusions and $r_X : X \to X_c$, $r_Y : Y \to Y_c$ retractions of i_X and i_Y such that $i_X r_X \simeq 1_X$ and $i_Y r_Y \simeq 1_Y$.

Since $r_Y f i_X : X_c \to Y_c$ is a homotopy equivalence between minimal finite spaces, it is a homeomorphism. Therefore $\mathcal{K}(r_Y f i_X) : \mathcal{K}(X_c) \to \mathcal{K}(Y_c)$ is an isomorphism and then $|\mathcal{K}(r_Y f i_X)|$ is a simple homotopy equivalence. Since $\mathcal{K}(X) \searrow \mathcal{K}(X_c)$, $|\mathcal{K}(i_X)|$ is a simple homotopy equivalence, and then the homotopy inverse $|\mathcal{K}(r_X)|$ is also a simple homotopy equivalence. Analogously $|\mathcal{K}(i_Y)|$ is a simple homotopy equivalence.

Finally, since $f \simeq i_Y r_Y f i_X r_X$, it follows that $|\mathcal{K}(f)| \simeq |\mathcal{K}(i_Y)||\mathcal{K}(r_Y f i_X)||\mathcal{K}(r_X)|$ is a simple homotopy equivalence. □

In order to describe the class of simple equivalences, we will use a kind of maps that was already studied in Lemma 4.2.7.

Definition 4.3.2. A map $f : X \to Y$ between finite T_0-spaces is *distinguished* if $f^{-1}(U_y)$ is contractible for each $y \in Y$. We denote by $\mathcal{D}$ the class of distinguished maps.

Note that by the Theorem of McCord 1.4.2, every distinguished map is a weak homotopy equivalence and therefore induces a homotopy equivalence between the associated complexes. We will prove in Theorem 4.3.4 that in fact the induced map is a simple homotopy equivalence. From the proof of Proposition 4.2.3, it is clear that if $x \in X$ is a down weak point, the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is distinguished.

Remark 4.3.3. The map $h : X' \to X$ defined by $h(C) = max(C)$, is distinguished by the proof of Proposition 4.2.9.

Clearly, homeomorphisms are distinguished. However it is not difficult to show that homotopy equivalences are not distinguished in general.

Theorem 4.3.4. *Every distinguished map induces a simple homotopy equivalence.*

Proof. Suppose $f : X \to Y$ is distinguished. Consider the non-Hausdorff mapping cylinder $B(f)$ and the canonical inclusions $i : X \hookrightarrow B(f)$, $j : Y \hookrightarrow B(f)$.

The following diagram

$$\begin{array}{ccc} & B(f) & \\ {}^{i}\nearrow & & \nwarrow^{j} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

does not commute, but $i \leq jf$ and then $i \simeq jf$. Therefore $|\mathcal{K}(i)| \simeq |\mathcal{K}(j)||\mathcal{K}(f)|$. By Lemma 4.2.7 and Theorem 4.2.12, $|\mathcal{K}(i)|$ and $|\mathcal{K}(j)|$ are expansions (composed with isomorphisms) and then, $|\mathcal{K}(f)|$ is a simple homotopy equivalence. □

In Proposition 6.2.9 we will prove that Theorem 4.3.4 also holds for a weaker notion of distinguished map: if $f : X \to Y$ is a map between finite T_0 spaces such that $f^{-1}(U_y)$ is homotopically trivial for every $y \in Y$, then f induces a simple homotopy equivalence.

We have already shown that expansions, homotopy equivalences and distinguished maps induce simple homotopy equivalences at the level of complexes. Note that if f, g, h are three maps between finite T_0-spaces such that $fg \simeq h$ and two of them induce simple homotopy equivalences, then so does the third.

Definition 4.3.5. Let $\mathcal{C}$ be a class of continuous maps between topological spaces. We say that $\mathcal{C}$ is *closed* if it satisfies the following homotopy 2-out-of-3 property: For any f, g, h with $fg \simeq h$, if two of the three maps are in $\mathcal{C}$, then so is the third.

Definition 4.3.6. Let $\mathcal{C}$ be a class of continuous maps. The class $\overline{\mathcal{C}}$ *generated* by $\mathcal{C}$ is the smallest closed class containing $\mathcal{C}$.

It is clear that $\overline{\mathcal{C}}$ is always closed under composition and homotopy. The class of simple homotopy equivalences between CW-complexes is closed and it is generated by the elementary expansions. Note that every map in the class $\mathcal{E}$ of elementary expansions between finite spaces induces a simple homotopy equivalence at the level of complexes and therefore the same holds for the maps of $\overline{\mathcal{E}}$. Contrary to the case of CW-complexes, a map between finite spaces which induces a simple homotopy equivalence, need not have a homotopy inverse. This is the reason why the definition of $\overline{\mathcal{E}}$ is not as simple as in the setting of complexes. We will prove that $\overline{\mathcal{E}} = \overline{\mathcal{D}}$, the class generated by the distinguished maps.

A map $f : X \to Y$ such that $f^{-1}(F_y)$ is contractible for every y, need not be distinguished. However we will show that $f \in \overline{\mathcal{D}}$. We denote by $f^{op} : X^{op} \to Y^{op}$ the map that coincides with f in the underlying sets, and let $\mathcal{D}^{op} = \{f \mid f^{op} \in \mathcal{D}\}$.

Lemma 4.3.7. $\overline{\mathcal{D}^{op}} = \overline{\mathcal{D}}$.

Proof. Suppose that $f : X \to Y$ lies in $\mathcal{D}^{op}$. Consider the following commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{h_X} & X' = (X^{op})' & \xrightarrow{h_{X^{op}}} & X^{op} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f^{op}} \\ Y & \xleftarrow{h_Y} & Y' = (Y^{op})' & \xrightarrow{h_{Y^{op}}} & Y^{op}. \end{array}$$

Here, f' denotes the map $\mathcal{X}(\mathcal{K}(f))$. Since $\overline{\mathcal{D}}$ satisfies the 2-out-of-3 property and $h_{X^{op}}$, $h_{Y^{op}}$, f^{op} are distinguished by Remark 4.3.3, $f' \in \overline{\mathcal{D}}$. And since h_X, h_Y are distinguished, $f \in \overline{\mathcal{D}}$. This proves that $\overline{\mathcal{D}^{op}} \subseteq \overline{\mathcal{D}}$. The other inclusion follows analogously from the opposite diagram. □

Proposition 4.3.8. *$\overline{\mathcal{E}} = \overline{\mathcal{D}}$, and this class contains all homotopy equivalences between finite T_0-spaces.*

Proof. Every expansion of finite spaces is in $\overline{\mathcal{E}}$ because it is a composition of maps in $\mathcal{E}$.

Let $f : X \to Y$ be distinguished. By the proof of Theorem 4.3.4 there exist expansions (eventually composed with homeomorphisms) i, j, such that $i \simeq jf$. Therefore $f \in \overline{\mathcal{E}}$.

If $x \in X$ is a down weak point, the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is distinguished. If x is an up weak point, $X \smallsetminus \{x\} \hookrightarrow X$ lies in $\overline{\mathcal{D}}$ by the previous lemma and therefore $\overline{\mathcal{E}} \subseteq \overline{\mathcal{D}}$.

Suppose now that $f : X \to Y$ is a homotopy equivalence. From the proof of Theorem 4.3.1, $fi_X \simeq i_Y r_Y f i_X$ where i_X, i_Y are expansions and $r_Y f i_X$ is a homeomorphism. This implies that $f \in \overline{\mathcal{E}} = \overline{\mathcal{D}}$. □

We denote by $\mathcal{S} = \overline{\mathcal{E}} = \overline{\mathcal{D}}$ the class of *simple equivalences* between finite spaces. In the rest of the paper we study the relationship between simple equivalences of finite spaces and simple homotopy equivalences of polyhedra.

Given $n \in \mathbb{N}$ we denote by K^n the n-th barycentric subdivision of K.

Lemma 4.3.9. *Let $\lambda : K^n \to K$ be a simplicial approximation to the identity. Then $\mathcal{X}(\lambda) \in \mathcal{S}$.*

Proof. It suffices to prove the case $n = 1$. Suppose $\lambda : K' \to K$ is a simplicial approximation of $1_{|K|}$. Then $\mathcal{X}(\lambda) : \mathcal{X}(K)' \to \mathcal{X}(K)$ is homotopic to $h_{\mathcal{X}(K)}$, for if $\sigma_1 \subsetneq \sigma_2 \subsetneq \ldots \subsetneq \sigma_m$ is a chain of simplices of K, then $\mathcal{X}(\lambda)(\{\sigma_1, \sigma_2, \ldots, \sigma_m\}) = \{\lambda(\sigma_1), \lambda(\sigma_2), \ldots, \lambda(\sigma_m)\} \subseteq \sigma_m = h_{\mathcal{X}(K)}(\{\sigma_1, \sigma_2, \ldots, \sigma_m\})$. By Remark 4.3.3, it follows that $\mathcal{X}(\lambda) \in \mathcal{S}$. □

Lemma 4.3.10. *Let $\varphi, \psi : K \to L$ be simplicial maps such that $|\varphi| \simeq |\psi|$. If $\mathcal{X}(\varphi) \in \mathcal{S}$, then $\mathcal{X}(\psi)$ also lies in $\mathcal{S}$.*

Proof. There exists an approximation to the identity $\lambda : K^n \to K$ for some $n \geq 1$, such that $\varphi\lambda$ and $\psi\lambda$ lie in the same contiguity class. By Proposition 2.1.3, $\mathcal{X}(\varphi)\mathcal{X}(\lambda) = \mathcal{X}(\varphi\lambda) \simeq \mathcal{X}(\psi\lambda) = \mathcal{X}(\psi)\mathcal{X}(\lambda)$. By Lemma 4.3.9, $\mathcal{X}(\lambda) \in \mathcal{S}$ and since $\mathcal{X}(\varphi) \in \mathcal{S}$, it follows that $\mathcal{X}(\psi) \in \mathcal{S}$. □

Theorem 4.3.11. *Let $K_0, K_1, \ldots, K_n$ be finite simplicial complexes and let*

$$|K_0| \xrightarrow{f_0} |K_1| \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} |K_n|$$

be a sequence of continuous maps such that for each $0 \leq i < n$ either

(1) $f_i = |\varphi_i|$ *where* $\varphi_i : K_i \to K_{i+1}$ *is a simplicial map such that* $\mathcal{X}(\varphi_i) \in \mathcal{S}$ *or*

(2) f_i *is a homotopy inverse of a map* $|\varphi_i|$ *with* $\varphi_i : K_{i+1} \to K_i$ *a simplicial map such that* $\mathcal{X}(\varphi_i) \in \mathcal{S}$.

If $\varphi : K_0 \to K_n$ *is a simplicial map such that* $|\varphi| \simeq f_{n-1}f_{n-2}\dots f_0$, *then* $\mathcal{X}(\varphi) \in \mathcal{S}$.

Proof. We may assume that f_0 satisfies condition (1). Otherwise we define $\widetilde{K_0} = K_0$, $\widetilde{f_0} = |1_{K_0}| : |\widetilde{K_0}| \to |K_0|$ and then $|\varphi| \simeq f_{n-1}f_{n-2}\dots f_0\widetilde{f_0}$.

We proceed by induction on n. If $n = 1$, $|\varphi| \simeq |\varphi_0|$ where $\mathcal{X}(\varphi_0) \in \mathcal{S}$ and the result follows from Lemma 4.3.10. Suppose now that $n \geq 1$ and let $K_0, K_1, \dots, K_n, K_{n+1}$ be finite simplicial complexes and $f_i : |K_i| \to |K_{i+1}|$ maps satisfying conditions (1) or (2), f_0 satisfying condition (1). Let $\varphi : K_0 \to K_{n+1}$ be a simplicial map such that $|\varphi| \simeq f_n f_{n-1}\dots f_0$. We consider two cases: f_n satisfies condition (1) or f_n satisfies condition (2).

In the first case we define $g : |K_0| \to |K_n|$ by $g = f_{n-1}f_{n-2}\dots f_0$. Let $\widetilde{g} : K_0^m \to K_n$ be a simplicial approximation to g and let $\lambda : K_0^m \to K_0$ be a simplicial approximation to the identity. Then $|\widetilde{g}| \simeq g|\lambda| = f_{n-1}f_{n-2}\dots f_1(f_0|\lambda|)$ where $f_0|\lambda| = |\varphi_0\lambda|$ and $\mathcal{X}(\varphi_0\lambda) = \mathcal{X}(\varphi_0)\mathcal{X}(\lambda) \in \mathcal{S}$ by Lemma 4.3.9. By induction, $\mathcal{X}(\widetilde{g}) \in \mathcal{S}$, and then $\mathcal{X}(\varphi_n\widetilde{g}) \in \mathcal{S}$. Since $|\varphi\lambda| \simeq f_n g|\lambda| \simeq f_n|\widetilde{g}| = |\varphi_n\widetilde{g}|$, by Lemma 4.3.10, $\mathcal{X}(\varphi\lambda)$ lies in $\mathcal{S}$. Therefore $\mathcal{X}(\varphi) \in \mathcal{S}$.

In the other case, $|\varphi_n\varphi| \simeq f_{n-1}f_{n-2}\dots f_0$ and by induction, $\mathcal{X}(\varphi_n\varphi) \in \mathcal{S}$. Therefore $\mathcal{X}(\varphi)$ also lies in $\mathcal{S}$. □

Theorem 4.3.12.

(a) Let $f : X \to Y$ *be a map between finite* T_0*-spaces. Then* f *is a simple equivalence if and only if* $|\mathcal{K}(f)| : |\mathcal{K}(X)| \to |\mathcal{K}(Y)|$ *is a simple homotopy equivalence.*

(b) Let $\varphi : K \to L$ *be a simplicial map between finite simplicial complexes. Then* $|\varphi|$ *is a simple homotopy equivalence if and only if* $\mathcal{X}(\varphi)$ *is a simple equivalence.*

Proof. By definition, if $f \in \mathcal{S}$, $|\mathcal{K}(f)|$ is a simple homotopy equivalence.

Let $\varphi : K \to L$ be a simplicial map such that $|\varphi|$ is a simple homotopy equivalence. Then there exist finite complexes $K = K_0, K_1, \dots, K_n = L$ and maps $f_i : |K_i| \to |K_{i+1}|$, which are simplicial expansions or homotopy inverses of simplicial expansions, and such that $|\varphi| \simeq f_{n-1}f_{n-2}\dots f_0$. By Theorem 4.2.12, simplicial expansions between complexes induce expansions between the associated finite spaces and therefore, by Theorem 4.3.11, $\mathcal{X}(\varphi) \in \mathcal{S}$.

Suppose now that $f : X \to Y$ is a map such that $|\mathcal{K}(f)|$ is a simple homotopy equivalence. Then, $f' = \mathcal{X}(\mathcal{K}(f)) : X' \to Y'$ lies in $\mathcal{S}$. Since $fh_X = h_Y f'$, $f \in \mathcal{S}$.

Finally, if $\varphi : K \to L$ is a simplicial map such that $\mathcal{X}(\varphi) \in \mathcal{S}$, $|\varphi'| : |K'| \to |L'|$ is a simple homotopy equivalence. Here $\varphi' = \mathcal{K}(\mathcal{X}(\varphi))$ is the barycentric subdivision of φ. Let $\lambda_K : K' \to K$ and $\lambda_L : L' \to L$ be simplicial approximations to the identities. Then $\lambda_L\varphi'$ and $\varphi\lambda_K$ are contiguous. In particular $|\lambda_L||\varphi'| \simeq |\varphi||\lambda_K|$ and then $|\varphi|$ is a simple homotopy equivalence. □

In the setting of finite spaces one has the following strict inclusions

$$\{homotopy\ equivalences\} \subsetneq \mathcal{S} \subsetneq \{weak\ equivalences\}.$$

Clearly, if $f : X \to Y$ is a weak homotopy equivalence between finite T_0-spaces with trivial Whitehead group, $f \in \mathcal{S}$.

4.3.1 Simple homotopy version of Quillen's Theorem A

Results which carry local information to global information appear frequently in Algebraic Topology. The Theorem of McCord 1.4.2 roughly states that if a map is locally a weak homotopy equivalence, then it is also a weak homotopy equivalence (globally). In the following we prove a result of this kind for simplicial maps and simple homotopy equivalences.

Let K and L be finite simplicial complexes and let $\varphi : K \to L$ be a simplicial map. Given a simplex $\sigma \in L$, we denote by $\varphi^{-1}(\sigma)$ the full subcomplex of K spanned by the vertices $v \in K$ such that $\varphi(v) \in \sigma$.

Recall that the simplicial version of Quillen's Theorem A, states that if $\varphi : K \to L$ is a simplicial map and $|\varphi|^{-1}(\overline{\sigma})$ is contractible for every simplex $\sigma \in L$, then $|\varphi|$ is a homotopy equivalence (see [32], page 93). Note that $|\varphi^{-1}(\sigma)| = |\varphi|^{-1}(\overline{\sigma})$. In particular, if $\varphi^{-1}(\sigma)$ is collapsible for every $\sigma \in L$, $|\varphi|$ is a homotopy equivalence. We prove that under these hypothesis, $|\varphi|$ is a simple homotopy equivalence.

First, we need to state a stronger version of Lemma 4.2.7. We keep the notation we use there.

Lemma 4.3.13. *Let $f : X \to Y$ be a map between finite T_0-spaces such that $f^{-1}(U_y)$ is collapsible for every $y \in Y$. Then $\mathcal{K}(B(f)) \searrow \mathcal{K}(i(X))$.*

Proof. We follow the proof and notation of Lemma 4.2.7. The set $\hat{U}^{X_{r-1}}_{j(y_r)} = \{i(x) \mid f(x) \leq y_r\}$ is homeomorphic to $f^{-1}(U_{y_r})$, which is collapsible by hypothesis. Therefore, $\hat{C}^{X_{r-1}}_{j(y_r)}$ is collapsible by 4.2.19 and, from Remark 4.2.16, $\mathcal{K}(X_{r-1}) \searrow \mathcal{K}(X_r)$. Thus, $\mathcal{K}(B(f)) = \mathcal{K}(X_0)$ collapses to $\mathcal{K}(i(X)) = \mathcal{K}(X_m)$. □

Theorem 4.3.14. *Let $\varphi : K \to L$ be a simplicial map between finite simplicial complexes. If $\varphi^{-1}(\sigma)$ is collapsible for every simplex σ of L, then $|\varphi|$ is a simple homotopy equivalence.*

Proof. Let $\sigma \in L$. We show first that $\mathcal{X}(\varphi)^{-1}(U_\sigma) = \mathcal{X}(\varphi^{-1}(\sigma))$. Let $\tau \in K$. Then, $\tau \in \mathcal{X}(\varphi^{-1}(\sigma))$ if and only if τ is a simplex of $\varphi^{-1}(\sigma)$. But this is equivalent to say that for every vertex v of τ, $\varphi(v) \in \sigma$ or, in other words, that $\varphi(\tau) \subseteq \sigma$ which means that $\mathcal{X}(\varphi)(\tau) \leq \sigma$. By Theorem 4.2.12, $\mathcal{X}(\varphi)^{-1}(U_\sigma)$ is collapsible.

By Lemma 4.3.13, $|\mathcal{K}(i)| : |K'| \to |\mathcal{K}(B(\mathcal{X}(\varphi)))|$ is a simple homotopy equivalence, and so is $|\mathcal{K}(j)| : |L'| \to |\mathcal{K}(B(\mathcal{X}(\varphi)))|$, where $i : \mathcal{X}(K) \hookrightarrow B(\mathcal{X}(\varphi))$ and $j : \mathcal{X}(L) \hookrightarrow B(\mathcal{X}(\varphi))$ are the inclusions. Since $|\mathcal{K}(i)| \simeq |\mathcal{K}(j)||\varphi'|$, $|\varphi'|$ is a simple homotopy equivalence and then, so is $|\varphi|$. □

4.4 The multiple non-Hausdorff mapping cylinder

It is easy to prove that if K_1 and K_2 are simple homotopy equivalent finite CW-complexes, there exists a third complex L such that $K_1 \nearrow L \searrow K_2$. When CW-complexes are changed by simplicial complexes or finite spaces, the structure becomes much more rigid, and the result is not so trivial. In this Section we will prove that if X and Y are finite T_0-spaces, there exists a finite T_0-space Z which collapses to both of them. One such space is the *multiple non-Hausdorff mapping cylinder* of some maps which is a generalization of the non-Hausdorff mapping cylinder defined in Definition 4.2.6.

Definition 4.4.1. Let $X_0, X_1, \dots, X_n$ be a sequence of finite T_0-spaces and let $f_0, f_1, \dots, f_{n-1}$ be a sequence of maps such that $f_i : X_i \to X_{i+1}$ or $f_i : X_{i+1} \to X_i$. If $f_i : X_i \to X_{i+1}$ we say that f_i *goes right*, and in other case we say that it *goes left*. We define the *multiple non-Hausdorff mapping cylinder* $B(f_0, f_1, \dots, f_{n-1}; X_0, X_1, \dots, X_n)$ as follows. The underlying set is the disjoint union $\bigsqcup_{i=0}^{n} X_i$. We keep the given ordering in each copy X_i and for x and y in different copies, we set $x < y$ in any of the following cases:

- If $x \in X_{2i}$, $y \in X_{2i+1}$ and $f_{2i}(x) \leq y$ or $x \leq f_{2i}(y)$.
- If $x \in X_{2i}$, $y \in X_{2i-1}$ and $f_{2i-1}(x) \leq y$ or $x \leq f_{2i-1}(y)$.

Note that the multiple non-Hausdorff mapping cylinder coincides with the ordinary non-Hausdorff mapping cylinder (Definition 4.2.6) when $n = 1$ and the unique map goes right.

Lemma 4.4.2. *Let $B = B(f_0, f_1, \dots, f_{n-1}, X_0, X_1, \dots, X_n)$. If f_0 goes right or if f_0 goes left and it lies in $\mathcal{D}^{op}$, then $B \searrow B \smallsetminus X_0$.*

Proof. If f_0 goes right, $B(f_0)$ collapses to X_1 (see the proof of Lemma 4.2.7). In fact, the collapse $B(f_0) \searrow X_1$ is a strong collapse since the points removed are not only weak points, but beat points. Since the points of X_0 are not comparable with the points of $X_2, X_3, \dots X_n$, the same elementary collapses can be performed in B. Then $B \searrow B \smallsetminus X_0$.

Now, if f_0 goes left and $f_0^{op} \in \mathcal{D}$, then $B(f_0^{op}) \searrow X_1^{op}$ by 4.2.7. Thus, $B(f_0^{op})^{op} \searrow X_1$. On the other hand, $B(f_0^{op}) = B(f_0^{op}; X_1^{op}, X_0^{op}) = B(f_0; X_0, X_1)^{op}$ and then $B(f_0; X_0, X_1) \searrow X_1$. By the same argument as before, $B \searrow B \smallsetminus X_0$. $\square$

The following remark is an easy consecuence of the definition.

Remark 4.4.3.

$$B(f_1, f_2, \dots, f_{n-1}; X_1, X_2, \dots, X_n)^{op} = B(f_0^{op}, f_1^{op}, \dots, f_{n-1}^{op}; X_0^{op}, X_1^{op}, \dots X_n^{op}) \smallsetminus X_0^{op}.$$

Lemma 4.4.4. *Let $B = B(f_0, f_1, \dots, f_{n-1}, X_0, X_1, \dots, X_n)$. Suppose that*

$f_{2i} \in \mathcal{D}$ if f_{2i} goes right.
$f_{2i} \in \mathcal{D}^{op}$ if f_{2i} goes left.
$f_{2i+1} \in \mathcal{D}^{op}$ if f_{2i+1} goes right.
$f_{2i+1} \in \mathcal{D}$ if f_{2i+1} goes left.
Then $B \searrow X_n$. If in addition n is even, $B \searrow X_0$.

Proof. By 4.4.2, $B \searrow B \smallsetminus X_0$. By the previous remark,

$$B \smallsetminus X_0 = B(f_1^{op}, f_2^{op}, \dots, f_{n-1}^{op}; X_1^{op}, X_2^{op}, \dots, X_n^{op})^{op}.$$

By induction $B(f_1^{op}, f_2^{op}, \dots, f_{n-1}^{op}; X_1^{op}, X_2^{op}, X_n^{op}) \searrow X_n^{op}$. Therefore $B \searrow B \smallsetminus X_0 \searrow X_n$.

If n is even, $B = B(f_{n-1}, f_{n-2}, \dots, f_0; X_n, X_{n-1}, \dots, X_0) \searrow X_0$. □

Theorem 4.4.5. *Let X and Y be simple equivalent finite T_0-spaces. Then, there exists a finite T_0-space Z such that $X \nearrow Z \searrow \widetilde{Y}$, where $\widetilde{Y}$ is homeomorphic to Y.*

Proof. If $X \diagup\!\!\searrow Y$, there exists a sequence of elementary expansions and collapses from X to Y. An elementary expansion $X_i \overset{e}{\nearrow} X_{i+1}$ induces an inclusion map $X_i \hookrightarrow X_{i+1}$ which lies in $\mathcal{D}$ or $\mathcal{D}^{op}$ depending on if the weak point removed is a down weak point or an up weak point. In particular, there exists a sequence $X = X_0, X_1, X_2, \dots, X_n = Y$ of finite T_0-spaces and a sequence $f_0, f_1, \dots, f_{n-1}$ of maps such that $f_i : X_i \to X_{i+1}$ or $f_i : X_{i+1} \to X_i$ and $f_i \in \mathcal{D} \cup \mathcal{D}^{op}$ for every $0 \leq i \leq n-1$. Adding identities if needed, we can assume that the maps are in the hypothesis of Lemma 4.4.4, and the result follows. □

Chapter 5

Strong homotopy types

The notion of collapse of finite spaces is directly connected with the concept of simplicial collapse. In Chapter 2 we have studied the notion of elementary strong collapse which is the fundamental move that describes homotopy types of finite spaces. In this Chapter we will define the notion of *strong collapse* of simplicial complexes which leads to *strong homotopy types* of complexes. This notion corresponds to the homotopy types of the associated finite spaces, but we shall see that it also arises naturally from the concept of contiguity classes.

Strong homotopy types of simplicial complexes have a beautiful caracterization which is similar to the description of homotopy types of finite spaces given by Stong.

Definition 5.0.1. Let K be a finite simplicial complex and let $v \in K$ be a vertex. We denote by $K \smallsetminus v$ the full subcomplex of K spanned by the vertices different from v. We say that there is an *elementary strong collapse* from K to $K \smallsetminus v$ if $lk(v)$ is a simplicial cone $v'L$. In this case we say that v is *dominated* (by v') and we note $K \searrow\!\!\searrow^{e} K \smallsetminus v$. There is a *strong collapse* from a complex K to a subcomplex L if there exists a sequence of elementary strong collapses that starts in K and ends in L. In this case we write $K \searrow\!\!\searrow L$. The inverse of a strong collapse is a *strong expansion* and two finite complexes K and L have the same *strong homotopy type* if there is a sequence of strong collapses and strong expansions that starts in K and ends in L.

Remark 5.0.2. Isomorphic complexes have the same strong homotopy type. Let K be a finite simplicial complex and let $v \in K$ be a vertex. Let v' be a vertex which is not in K and consider the complex $L = K + v'st_K(v) = K \smallsetminus v + v'vlk_K(v)$. Since $lk_L(v') = vlk_K(v)$, $L \searrow\!\!\searrow K$. Moreover, by symmetry $L \searrow\!\!\searrow L \smallsetminus v = \widetilde{K}$. Clearly, there is an isomorphism $K \to \widetilde{K}$ which sends v to v' and fixes the other vertices. Thus, if K_1 and K_2 are isomorphic simplicial complexes, we can obtain a third complex K_3 whose vertices are different from the vertices of K_1 and K_2 and such that K_i and K_3 have the same strong homotopy type for $i = 1, 2$.

If $v \in K$ is dominated, $lk(v)$ is collapsible and therefore $st(v) = v(lk(v)) \searrow lk(v) = st(v) \cap (K \smallsetminus v)$. Then $K = st(v) \cup (K \smallsetminus v) \searrow K \smallsetminus v$. Thus, the usual notion of collapse is weaker than the notion of strong collapse.

Remark 5.0.3. A vertex v is dominated by another v' if and only if every maximal simplex that contains v also contains v'.

We will prove that this notion of collapse corresponds exactly to the notion of strong collapse of finite spaces (i.e., strong deformation retracts).

If two simplicial maps $\varphi, \psi : K \to L$ lie in the same contiguity class, we will write $\varphi \sim \psi$. It is easy to see that if $\varphi_1, \varphi_2 : K \to L$, $\psi_1, \psi_2 : L \to M$ are simplicial maps such that $\varphi_1 \sim \varphi_2$ and $\psi_1 \sim \psi_2$, then $\psi_1\varphi_1 \sim \psi_2\varphi_2$.

Definition 5.0.4. A simplicial map $\varphi : K \to L$ is a *strong equivalence* if there exists $\psi : L \to K$ such that $\psi\varphi \sim 1_K$ and $\varphi\psi \sim 1_L$. If there is a strong equivalence $\varphi : K \to L$ we write $K \sim L$.

The relation $\sim$ is clearly an equivalence.

Definition 5.0.5. A finite simplicial complex K is a *minimal complex* if it has no dominated vertices.

Proposition 5.0.6. *Let K be a minimal complex and let $\varphi : K \to K$ be simplicial map which lies in the same contiguity class as the identity. Then φ is the identity.*

Proof. We may assume that φ is contiguous to 1_K. Let $v \in K$ and let $\sigma \in K$ be a maximal simplex such that $v \in \sigma$. Then $\varphi(\sigma) \cup \sigma$ is a simplex, and by the maximality of σ, $\varphi(v) \in \varphi(\sigma) \cup \sigma = \sigma$. Therefore every maximal simplex which contains v, also contains $\varphi(v)$. Hence, $\varphi(v) = v$, since K is minimal. □

Corollary 5.0.7. *A strong equivalence between minimal complexes is an isomorphism.*

Proposition 5.0.8. *Let K be a finite simplicial complex and $v \in K$ a vertex dominated by v'. Then, the inclusion $i : K \smallsetminus v \hookrightarrow K$ is a strong equivalence. In particular, if two complexes K and L have the same strong homotopy type, then $K \sim L$.*

Proof. Define a vertex map $r : K \to K \smallsetminus v$ which is the identity on $K \smallsetminus v$ and such that $r(v) = v'$. If $\sigma \in K$ is a simplex with $v \in \sigma$, consider $\sigma' \supseteq \sigma$ a maximal simplex. Therefore $v' \in \sigma'$ and $r(\sigma) = \sigma \cup \{v'\} \smallsetminus \{v\} \subseteq \sigma'$ is a simplex of $K \smallsetminus v$. Moreover $ir(\sigma) \cup \sigma = \sigma \cup \{v'\} \subseteq \sigma'$ is a simplex of K. This proves that r is simplicial and that ir is contiguous to 1_K. Therefore, i is a strong equivalence. □

Definition 5.0.9. A *core* of a finite simplicial complex K is a minimal subcomplex $K_0 \subseteq K$ such that $K \searrow\!\!\searrow K_0$.

Theorem 5.0.10. *Every complex has a core and it is unique up to isomorphism. Two finite simplicial complexes have the same strong homotopy type if and only if their cores are isomorphic.*

Proof. A core of a complex can be obtained removing dominated points one at the time. If K_1 and K_2 are two cores of K, they have the same strong homotopy type and by Proposition 5.0.8, $K_1 \sim K_2$. Since they are minimal, by Corollary 5.0.7 they are isomorphic.

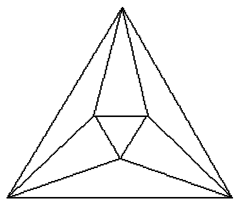
Let K, L be two finite complexes. If they have the same strong homotopy type, then also their cores K_0 and L_0 do. As above, we conclude that K_0 and L_0 are isomorphic.

Conversely, If K_0 and L_0 are isomorphic, then they have the same strong homotopy type by Remark 5.0.2 and then K and L have the same strong homotopy type. □

If K and L are two complexes such that $K \sim L$ and $K_0 \subseteq K$, $L_0 \subseteq L$ are their cores, then $K_0 \sim L_0$ and therefore K_0 and L_0 are homeomorphic. Hence, we deduce the following

Corollary 5.0.11. *Two finite simplicial complexes K and L have the same strong homotopy type if and only if $K \sim L$.*

Example 5.0.12. The following homogeneous 2-complex is collapsible (moreover it is non-evasive [41]). However, it is a minimal complex and therefore it does not have the strong homotopy type of a point.



Example 5.0.13. In contrast to the case of simple homotopy types, a complex and its barycentric subdivision need not have the same strong homotopy type. The boundary of a 2-simplex and its barycentric subdivision are minimal non-isomorphic complexes, therefore they do not have the same strong homotopy type.

Proposition 5.0.14. *Strong equivalences are simple homotopy equivalences.*

Proof. Let $\varphi : K \to L$ be a strong equivalence. Let K_0 be a core of K and L_0 a core of L. Then, the inclusion $i : K_0 \hookrightarrow K$ is a strong equivalence and there exists a strong equivalence $r : L \to L_0$ which is a homotopy inverse of the inclusion $L_0 \hookrightarrow L$. Since K_0 and L_0 are minimal complexes, the strong equivalence $r\varphi i$ is an isomorphism. Therefore, $|i|, |r|$ and $|r\varphi i|$ are simple homotopy equivalences, and then so is $|\varphi|$. □

Now we will study the relationship between strong homotopy types of simplicial complexes and homotopy types of finite spaces. The following result is a direct consequence of Propositions 2.1.2 and 2.1.3.

Theorem 5.0.15.

(a) If two finite T_0-spaces are homotopy equivalent, their associated complexes have the same strong homotopy type.

(b) If two finite complexes have the same strong homotopy type, the associated finite spaces are homotopy equivalent.

Proof. Suppose $f : X \to Y$ is a homotopy equivalence between finite T_0-spaces with homotopy inverse $g : Y \to X$. Then by Proposition 2.1.2, $\mathcal{K}(g)\mathcal{K}(f) \sim 1_{\mathcal{K}(X)}$ and $\mathcal{K}(f)\mathcal{K}(g) \sim 1_{\mathcal{K}(Y)}$. Thus, $\mathcal{K}(X) \sim \mathcal{K}(Y)$.

If K and L are complexes with the same strong homotopy type, there exist $\varphi : K \to L$ and $\psi : L \to K$ such that $\psi\varphi \sim 1_K$ and $\varphi\psi \sim 1_L$. By Proposition 2.1.3, $\mathcal{X}(\varphi) : \mathcal{X}(K) \to \mathcal{X}(L)$ is a homotopy equivalence with homotopy inverse $\mathcal{X}(\psi)$. □

In fact, we will give a more precise result:

Theorem 5.0.16.

(a) Let X be a finite T_0-space and let $Y \subseteq X$. If $X \searrow\searrow Y$, $\mathcal{K}(X) \searrow\searrow \mathcal{K}(Y)$.

(b) Let K be a finite simplicial complex and let $L \subseteq K$. If $K \searrow\searrow L$, $\mathcal{X}(K) \searrow\searrow \mathcal{X}(K)$.

Proof. If $x \in X$ is a beat point, there exist a point $x' \in X$ and subspaces A, B such that $\hat{C}_x = A \oplus \{x'\} \oplus B$. Then $lk(x) = \mathcal{K}(\hat{C}_x) = x'\mathcal{K}(A \oplus B)$ is a simplicial cone. Therefore, $\mathcal{K}(X) \searrow\searrow \mathcal{K}(X) \smallsetminus x = \mathcal{K}(X \smallsetminus \{x\})$.

If K is a finite complex and $v \in K$ is such that $lk(v) = aL$ is a simplicial cone, we define $r : \mathcal{X}(K) \to \mathcal{X}(K \smallsetminus v)$ as follows:

$$r(\sigma) = \begin{cases} a\sigma \smallsetminus \{v\} & \text{if } v \in \sigma \\ \sigma & \text{if } v \notin \sigma \end{cases}$$

Clearly r is a well defined order preserving map. Denote $i : \mathcal{X}(K \smallsetminus v) \hookrightarrow \mathcal{X}(K)$ the inclusion and define $f : \mathcal{X}(K) \to \mathcal{X}(K)$,

$$f(\sigma) = \begin{cases} a\sigma & \text{if } v \in \sigma \\ \sigma & \text{if } v \notin \sigma \end{cases}$$

Then $ir \leq f \geq 1_{\mathcal{X}(K)}$ and both ir and f are the identity on $\mathcal{X}(K \smallsetminus v)$. Therefore $ir \simeq 1_{\mathcal{X}(K)}$ rel $\mathcal{X}(K \smallsetminus v)$ and then $\mathcal{X}(K) \searrow\searrow \mathcal{X}(K \smallsetminus v)$ by 2.2.5. □

In particular, if two finite T_0-spaces are homotopy equivalent, the associated complexes have the same strong homotopy type. If two finite simplicial complexes have the same strong homotopy type, the associated finite spaces are homotopy equivalent.

Example 5.0.17. The complex $\mathcal{K}(W)$ associated to the Wallet is collapsible because W is collapsible, but it is not strong collapsible since W' is not contractible.

Corollary 5.0.18. *If X is a contractible finite T_0-space, so is X'.*

Proof. If X is contractible, $X \searrow\searrow *$, then $\mathcal{K}(X) \searrow\searrow *$ and therefore $X' = \mathcal{X}(\mathcal{K}(X)) \searrow\searrow *$. □

5.1 The $\mathfrak{m}$ construction

We introduce an application which transforms a simplicial complex in another complex with the same homotopy type. This construction is closely related to the Čech cohomology of finite spaces. We will prove that this application can be used to obtain the core of a simplicial complex.

Recall that if $\mathcal{U}$ is an open cover of a topological space X, the nerve of $\mathcal{U}$ is the simplicial complex $N(\mathcal{U})$ whose simplices are the finite subsets $\{U_1, U_2, \ldots, U_r\}$ of $\mathcal{U}$ such that $\bigcap\limits_{i=1}^{r} U_i$ is nonempty. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, there is a simplicial map $N(\mathcal{V}) \to N(\mathcal{U})$

which is uniquely determined up to homotopy, and sends any vertex $V \in N(\mathcal{V})$ to a vertex $U \in N(\mathcal{U})$ such that $V \subseteq U$. The Čech cohomology of X is the direct limit $\check{H}^n = colim\ H^n(N(\mathcal{U}))$ taken over the family of covers of X preordered by refinement.

It is well known that if X is a CW-complex, the Čech cohomology coincides with the singular cohomology of X. But this is not true in general. Given a finite space X, we denote by $\mathcal{U}_0$ the open cover given by the minimal open sets of the maximal points of X. Note that $\mathcal{U}_0$ refines every open cover of X. Therefore $\check{H}^n(X) = H^n(N(\mathcal{U}_0))$.

Example 5.1.1. If $X = \mathbb{S}(S^0)$ is the minimal finite model of S^1, $N(\mathcal{U}_0)$ is a 1-simplex and therefore $\check{H}^1(X) = 0$. On the other hand $H^1(X) = H^1(S^1) = \mathbb{Z}$.

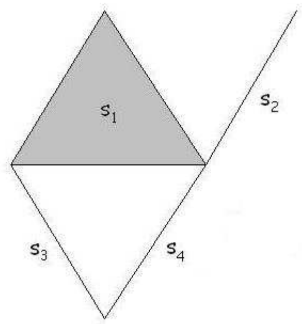
If K is a finite simplicial complex, the cover $\mathcal{U}_0$ of $\mathcal{X}(K)$ satisfies that arbitrary intersections of its elements is empty or homotopically trivial. Indeed, if $\sigma_1, \sigma_2, \dots, \sigma_r$ are maximal simplices of K, then $\cap U_{\sigma_i}$ is empty or it is $U_{\cap \sigma_i}$. By Theorem 2 of [25], there is a weak homotopy equivalence $|N(\mathcal{U}_0)| \to \mathcal{X}(K)$. Therefore $\check{H}^n(\mathcal{X}(K)) = H^n(|N(\mathcal{U}_0)|) = H^n(\mathcal{X}(K))$, so we proved

Proposition 5.1.2. *Let K be a finite simplicial complex. Then $\check{H}^n(\mathcal{X}(K)) = H^n(\mathcal{X}(K))$ for every $n \geq 0$.*

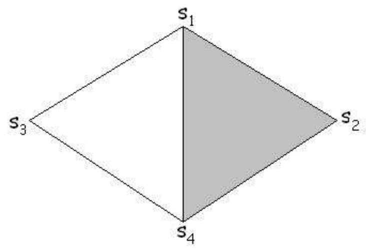
Another proof of the last result can be given invoking a Theorem of Dowker [15]. Let V be the set of vertices of K and S the set of its maximal simplices. Define the relation $R \subseteq V \times S$ by $vR\sigma$ if $v \in \sigma$. Dowker consider two simplicial complexes. The first has as simplices the finite subsets of V which are related with one element of S, this is the original complex K. The second complex has as simplices the finite subsets of S which are related with an element of V. This complex is isomorphic to $N(\mathcal{X}(\mathcal{U}_0))$. The Theorem of Dowker concludes that $|K|$ and $|N(\mathcal{X}(\mathcal{U}_0))|$ are homotopy equivalent. Therefore $H^n(\mathcal{X}(K)) = H^n(|K|) = H^n(|N(\mathcal{X}(\mathcal{U}_0))|) = \check{H}^n(\mathcal{X}(K))$.

We now let aside the Čech cohomology to center our attention in the construction which transforms K in the complex $N(\mathcal{X}(\mathcal{U}_0))$. We will denote $\mathfrak{m}(K) = N(\mathcal{X}(\mathcal{U}_0))$. Concretely, the vertices of $\mathfrak{m}(K)$ are the maximal simplices of K and the simplices of $\mathfrak{m}(K)$ are the sets of maximal simplices of K with nonempty intersection. The paragraph above shows that if K is a finite simplicial complex, $|K|$ and $|\mathfrak{m}(K)|$ have the same homotopy type. Given $n \geq 2$, we define recursively $\mathfrak{m}^n(K) = \mathfrak{m}(\mathfrak{m}^{n-1}(K))$.

Example 5.1.3. Let K be the following simplicial complex



Since K has four maximal simplices, $\mathfrak{m}(K)$ has four vertices, and it looks as follows



For $n \geq 2$, the complex $\mathfrak{m}^n(K)$ is the boundary of a 2-simplex.

If $\mathfrak{m}^r(K) = *$ for some $r \geq 1$, then $|K|$ is contractible. But there are contractible complexes such that $\mathfrak{m}^r(K)$ is not a point for every r. For instance, if K is the complex of Example 5.0.12, $\mathfrak{m}(K)$ has more vertices than K, but $\mathfrak{m}^2(K)$ is isomorphic to K. Therefore $\mathfrak{m}^r(K) \neq *$ for every r, although $|K|$ is contractible.

We will see that in fact, there is a strong collapse from K to a complex isomorphic to $\mathfrak{m}^2(K)$ and that there exists r such that $\mathfrak{m}^r(K) = *$ if and only if K is strong collapsible.

Lemma 5.1.4. *Let L be a full subcomplex of a finite simplicial complex K such that every vertex of K which is not in L is dominated by some vertex in L. Then $K \searrow\searrow L$.*

Proof. Let v be a vertex of K which is not in L. By hypothesis, v is dominated and then $K \searrow\searrow K \smallsetminus v$. Now suppose w is a vertex of $K \smallsetminus v$ which is not in L. Then, the link $lk_K(w)$ in K is a simplicial cone aM with $a \in L$. Therefore, the link $lk_{K \smallsetminus v}(w)$ in $K \smallsetminus v$ is $a(M \smallsetminus v)$. By induction $K \smallsetminus v \searrow\searrow L$ and then $K \searrow\searrow L$. □

Proposition 5.1.5. *Let K be a finite simplicial complex. Then, there exists a complex L isomorphic to $\mathfrak{m}^2(K)$ such that $K \searrow\searrow L$.*

Proof. A vertex of $\mathfrak{m}^2(K)$ is a maximal family $\Sigma = \{\sigma_0, \sigma_1, \dots, \sigma_r\}$ of maximal simplices of K with nonempty intersection. Consider a vertex map $\varphi : \mathfrak{m}^2(K) \to K$ such that $\varphi(\Sigma) \in \bigcap_{i=0}^{r} \sigma_i$. This is a simplicial map for if $\Sigma_0, \Sigma_1, \dots, \Sigma_r$ constitute a simplex of $\mathfrak{m}^2(K)$, then there is a common element σ in all of them, which is a maximal simplex of K. Therefore $\varphi(\Sigma_i) \in \sigma$ for every $0 \leq i \leq r$ and then $\{\varphi(\Sigma_1), \varphi(\Sigma_2), \dots, \varphi(\Sigma_r)\}$ is a simplex of K.

The vertex map φ is injective. If $\varphi(\Sigma_1) = v = \varphi(\Sigma_2)$ for $\Sigma_1 = \{\sigma_0, \sigma_1, \dots, \sigma_r\}$, $\Sigma_2 = \{\tau_0, \tau_1, \dots, \tau_t\}$, then $v \in \sigma_i$ for every $0 \leq i \leq r$ and $v \in \tau_i$ for every $0 \leq i \leq t$. Therefore $\Sigma_1 \cup \Sigma_2$ is a family of maximal simplices of K with nonempty intersection. By the maximality of Σ_1 and Σ_2, $\Sigma_1 = \Sigma_1 \cup \Sigma_2 = \Sigma_2$.

Suppose $\Sigma_0, \Sigma_1, \dots, \Sigma_r$ are vertices of $\mathfrak{m}^2(K)$ such that $v_0 = \varphi(\Sigma_0), v_1 = \varphi(\Sigma_1), \dots,$ $v_r = \varphi(\Sigma_r)$ constitute a simplex of K. Let σ by a maximal simplex of K which contains $v_0, v_1, \dots, v_r$. Then, by the maximality of the families Σ_i, $\sigma \in \Sigma_i$ for every i and therefore $\{\Sigma_0, \Sigma_1, \dots, \Sigma_r\}$ is a simplex of $\mathfrak{m}^2(K)$.

This proves that $L = \varphi(\mathfrak{m}^2(K))$ is a full subcomplex of K which is isomorphic to $\mathfrak{m}^2(K)$.

Now, suppose v is a vertex of K which is not in L. Let Σ be the set of maximal simplices of K which contain v. The intersection of the elements of Σ is nonempty, but Σ could be not maximal. Let $\Sigma' \supseteq \Sigma$ be a maximal family of maximal simplices of K with nonempty intersection. Then $v' = \varphi(\Sigma') \in L$ and if σ is a maximal simplex of K which contains v, then $\sigma \in \Sigma \subseteq \Sigma'$. Hence, $v' \in \sigma$. Therefore v is dominated by v'. By Lemma 5.1.4, $K \searrow\!\!\searrow L$. $\square$

Lemma 5.1.6. *A finite simplicial complex K is minimal if and only if $\mathfrak{m}^2(K)$ is isomorphic to K.*

Proof. By Proposition 5.1.5, there exists a complex L isomorphic to $\mathfrak{m}^2(K)$ such that $K \searrow\!\!\searrow L$. Therefore, if K is minimal, $L = K$.

If K is not minimal, there exists a vertex v dominated by other vertex v'. If v is contained in each element of a maximal family Σ of maximal simplices of K with nonempty intersection, then the same occur with v'. Therefore, we can define the map φ of the proof of Proposition 5.1.5 so that v is not in its image. Therefore, $L = \varphi(\mathfrak{m}^2(K))$ is isomorphic to $\mathfrak{m}^2(K)$ and has less vertices than K. Thus, $\mathfrak{m}^2(K)$ and K can not be isomorphic. $\square$

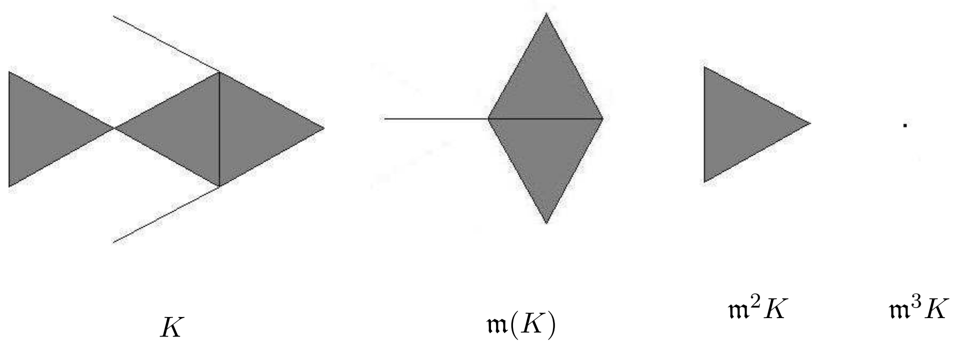
The sequence $K, \mathfrak{m}^2(K), \mathfrak{m}^4(K), \mathfrak{m}^6(K), \ldots$ is a decreasing sequence of subcomplexes of K (up to isomorphism). Therefore, there exists $n \geq 1$ such that $\mathfrak{m}^{2n}(K)$ and $\mathfrak{m}^{2n+2}(K)$ are isomorphic. Then K strongly collapses to a subcomplex L which is isomorphic to $\mathfrak{m}^{2n}(K)$ and which is minimal. Thus, we have proved the following

Proposition 5.1.7. *Given a finite simplicial complex K, there exists $n \geq 1$ such that $\mathfrak{m}^n(K)$ is isomorphic to the core of K.*

Theorem 5.1.8. *Let K be a finite simplicial complex. Then, K is strongly collapsible if and only if there exists $n \geq 1$ such that $\mathfrak{m}^n(K)$ is a point.*

Proof. If K is strongly collapsible, its core is a point and then, there exists n such that $\mathfrak{m}^n(K) = *$ by the previous proposition. If $\mathfrak{m}^n(K) = *$ for some n, then $\mathfrak{m}^{n+1}(K)$ is also a point. Therefore there exists an even positive integer r such that $\mathfrak{m}^r(K) = *$, and $K \searrow\!\!\searrow *$ by Proposition 5.1.5. $\square$

Example 5.1.9. The following complex K is strongly collapsible since $\mathfrak{m}^3(K) = *$.



Chapter 6

Methods of reduction

A method of reduction of finite spaces is a technique that allows one to reduce the number of points of a finite topological space preserving some properties of the original one.

An important example of reduction method is described by beat points defined by Stong. The property that is preserved when removing a beat point is the homotopy type. Stong method is effective in the sense that for any finite T_0-space X, one can obtain a space homotopy equivalent to X of minimum cardinality, by applying repeatedly the method of removing beat points.

Throghout our work, we have found other methods of reduction. The most important is probably the one described by weak points (see Chapter 4). However it is still an open problem to find an effective reduction method for the weak homotopy type and the simple homotopy type. This is a reason why it is so difficult to find minimal finite models.

6.1 Osaki's reduction methods

The first examples of reduction methods where introduced by T. Osaki [31]. In these cases, Osaki presents two methods that allow to find a quotient of a given finite space such that the quotient map is a weak homotopy equivalence.

Theorem 6.1.1. *(Osaki) Let X be a finite T_0-space. Suppose there exists $x \in X$ such that $U_x \cap U_y$ is either empty or homotopically trivial for all $y \in X$. Then the quotient map $q : X \to X/U_x$ is a weak homotopy equivalence.*

Proof. Let $y \in X$. If $U_x \cap U_y = \emptyset$, $q^{-1}(U_{qy}) = U_y$. In other case, $q^{-1}(U_{qy}) = U_x \cup U_y$ (see Lemma 2.7.6). In order to apply McCord Theorem 1.4.2 to the minimal basis of X/U_x, we only have to prove that if $U_x \cap U_y$ is homotopically trivial, then so is $U_x \cup U_y$. If $U_x \cap U_y$ is homotopically trivial, since U_x and U_y are contractible, we obtain from the Mayer-Vietoris sequence, that $\widetilde{H}_n(U_x \cup U_y) = 0$ for every $n \geq 0$ and from the Theorem of Van-Kampen, $U_x \cup U_y$ is simply connected. By Hurewicz's Theorem, it is homotopically trivial. Therefore, Theorem 1.4.2 applies and q is a weak homotopy equivalence. □

The process of obtaining X/U_x from X is called an *open reduction*. There is an analogous result for the *minimal closed sets* F_x, i.e. the closures of the one point spaces

$\{x\}$. This result follows from the previous one applied to the opposite X^{op}.

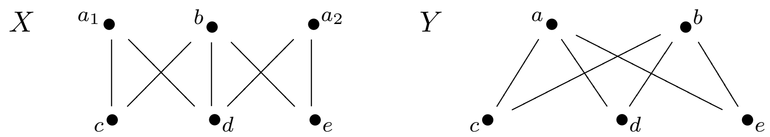
Theorem 6.1.2. *(Osaki) Let X be a finite T_0-space. Suppose there exists $x \in X$ such that $F_x \cap F_y$ is either empty or homotopically trivial for all $y \in X$. Then the quotient map $q : X \to X/F_x$ is a weak homotopy equivalence.*

The process of obtaining X/F_x from X is called a *closed reduction*.

Osaki asserts in [31] that he does not know whether by a sequence of reductions, each finite T_0-space can be reduced to the smallest space with the same homotopy groups.

We show with the following example that the answer to this question is negative.

Let $X = \{a_1, b, a_2, c, d, e\}$ be the 6-point T_0-space with the following order: $c, d < a_1$; $c, d, e < b$ and $d, e < a_2$. Let $D_3 = \{c, d, e\}$ be the 3-point discrete space and $Y = \mathbb{S}D_3 = \{a, b, c, d, e\}$ the non-Haussdorf suspension of D_3.



The function $f : X \to Y$ defined by $f(a_1) = f(a_2) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$ and $f(e) = e$ is continuous because it preserves the order.

In order to prove that f is a weak homotopy equivalence we use the Theorem of McCord 1.4.2.

The sets U_y form a basis-like cover of Y. It is easy to verify that $f^{-1}(U_y)$ is contractible for each $y \in Y$ and, since U_y is also contractible, the map $f|_{f^{-1}(U_y)} : f^{-1}(U_y) \to U_y$ is a weak homotopy equivalence for each $y \in Y$.

Applying Theorem 1.4.2, one proves that f is a weak homotopy equivalence. Therefore X and Y have the same homotopy groups.

Another way to show that X and Y are weak homotopy equivalent is considering the associated polyhedra $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ which are homotopy equivalent to $S^1 \vee S^1$.

On the other hand, it is easy to see that Osaki reduction methods cannot be applied to the space X. Therefore his methods are not effective in this case since we cannot obtain, by a sequence of reductions, the smallest space with the same homotopy groups as X.

6.2 γ-points and one-point reduction methods

In this Section we delve deeper into the study of one-point reductions of finite spaces, i.e. methods which consist on removing just one point of the space in such a way that it does not affect its homotopy, weak homotopy or simple homotopy type. Beat points and weak points provide two important examples of one-point reductions.

Recall that $x \in X$ is a weak point if and only if $\hat{C}_x$ is contractible. This motivates the following definition.

Definition 6.2.1. A point x of a finite T_0-space X is a γ-*point* if $\hat{C}_x$ is homotopically trivial.

Note that weak points are γ-points. It is not difficult to see that both notions coincide in spaces of height less than or equal to 2. This is because any space of height 1 is contractible if and only if it is homotopically trivial. However, this is false for spaces of height greater than 2.

Let x be a γ-point of a finite T_0-space X. We will show that the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence. Note that since $\hat{U}_x$ and $\hat{F}_x$ need not be homotopically trivial, we cannot proceed as we did in Proposition 4.2.3. However, in this case, one has the following pushout

$$\begin{array}{ccc} |\mathcal{K}(\hat{C}_x)| & \longrightarrow & |\mathcal{K}(C_x)| \\ \downarrow & & \downarrow \\ |\mathcal{K}(X \smallsetminus \{x\})| & \longrightarrow & |\mathcal{K}(X)| \end{array}$$

Where $|\mathcal{K}(\hat{C}_x)| \to |\mathcal{K}(C_x)|$ is a homotopy equivalence and $|\mathcal{K}(\hat{C}_x)| \to |\mathcal{K}(X \smallsetminus \{x\})|$ satisfies the homotopy extension property. Therefore $|\mathcal{K}(X \smallsetminus \{x\})| \to |\mathcal{K}(X)|$ is a homotopy equivalence. This proves the following

Proposition 6.2.2. *If $x \in X$ is a γ-point, the inclusion $i : X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence.*

As we mentioned in the introduction, this result improves an old result which appears for example in Walker's Thesis [40, Proposition 5.8], which asserts, in the language of finite spaces, that $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence provided $\hat{U}_x$ or $\hat{F}_x$ is homotopically trivial. By Proposition 6.2.11 below, it is clear that a point x is a γ-point if $\hat{U}_x$ or $\hat{F}_x$ is homotopically trivial, but the converse is false.

We will show that the converse of Proposition 6.2.2 is true in most cases. First, we need some results.

Proposition 6.2.3. *Let x be a point of a finite T_0-space X. The inclusion $i : X \smallsetminus \{x\} \hookrightarrow X$ induces isomorphisms in all homology groups if and only if the subspace $\hat{C}_x$ is acyclic.*

Proof. Apply the Mayer-Vietoris sequence to the triple $(\mathcal{K}(X); \mathcal{K}(C_x), \mathcal{K}(X \smallsetminus \{x\}))$. □

Remark 6.2.4. If X and Y are non-empty finite T_0-spaces with n and m connected components respectively, the fundamental group $\pi_1(X \oplus Y)$ is a free product of $(n-1)(m-1)$ copies of $\mathbb{Z}$. In particular if $x \in X$ is neither maximal nor minimal, the fundamental group of $\hat{C}_x = \hat{U}_x \oplus \hat{F}_x$ is a free group.

Theorem 6.2.5. *Let X be a finite T_0-space, and $x \in X$ a point which is neither maximal nor minimal and such that $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence. Then x is a γ-point.*

Proof. If $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence, $\hat{C}_x$ is acyclic by Proposition 6.2.3. Then $\pi_1(\hat{C}_x)$ is a perfect group and therefore trivial, by Remark 6.2.4. Now the result follows from the Hurewicz Theorem. □

The theorem fails if x is maximal or minimal as the next example shows.

Example 6.2.6. Let X be an acyclic finite T_0-space with nontrivial fundamental group. Let $\mathbb{S}(X) = X \cup \{-1, 1\}$ be its non-Hausdorff suspension. Then $\mathbb{S}(X)$ is also acyclic and $\pi_1(\mathbb{S}(X)) = 0$. Therefore it is homotopically trivial. Hence, $X \cup \{1\} \hookrightarrow \mathbb{S}(X)$ is a weak homotopy equivalence. However -1 is not a γ-point of $\mathbb{S}(X)$.

Using the relativity principle of simple homotopy theory [14, (5.3)] one can prove that if x is a γ-point, $|\mathcal{K}(X \smallsetminus \{x\})| \to |\mathcal{K}(X)|$ is a simple homotopy equivalence. In fact this holds whenever $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence.

Theorem 6.2.7. *Let X be a finite T_0-space and let $x \in X$. If the inclusion $i : X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence, it induces a simple homotopy equivalence $|\mathcal{K}(X \smallsetminus \{x\})| \to |\mathcal{K}(X)|$. In particular $X \smallsetminus \{x\} \diagup\!\!\searrow X$.*

Proof. Since $|\mathcal{K}(X \smallsetminus \{x\})|$ is a strong deformation retract of $|\mathcal{K}(X)|$ and the open star of x,

$$\overset{\circ}{st}(x) = |\mathcal{K}(X)| \smallsetminus |\mathcal{K}(X \smallsetminus \{x\})|$$

is contractible, then by [14, (20.1)], the Whitehead Torsion $\tau(|\mathcal{K}(X)|, |\mathcal{K}(X \smallsetminus \{x\})|) = 0$. □

This result essentially shows that one-point reductions are not sufficient to describe all weak homotopy types of finite spaces. Of course they are sufficient to reach all finite models of spaces with trivial Whitehead group. On the other hand, note that the fact that $X \smallsetminus \{x\}$ and X have the same weak homotopy type does not imply that the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence.

Definition 6.2.8. If $x \in X$ is a γ-point, we say that there is an *elementary γ-collapse* from X to $X \smallsetminus \{x\}$. A finite T_0-space X *γ-collapses* to Y if there is a sequence of elementary γ-collapses that starts in X and ends in Y. We denote this by $X \searrow^{\gamma} Y$. If X γ-collapses to a point, we say that it is γ-collapsible.

In contrast to collapses, a γ-collapse does not induce in general a collapse between the associated simplicial complexes. For example, if K is any triangulation of the Dunce hat, $\mathbb{C}(\mathcal{X}(K)) \searrow^{\gamma} \mathcal{X}(K)$, but $aK' \not\searrow K'$ since K' is not collapsible (see Lemma 4.2.10). However, if $X \searrow^{\gamma} Y$, then $X \diagup\!\!\searrow Y$ by Theorem 6.2.7 and then $\mathcal{K}(X)$ has the same simple homotopy type as $\mathcal{K}(Y)$.

Recall that $f : X \to Y$ is said to be distinguished if $f^{-1}(U_y)$ is contractible for every $y \in Y$. Distinguished maps are simple homotopy equivalences (see Section 4.3). The following result generalizes that fact.

Proposition 6.2.9. *Let $f : X \to Y$ be a map between finite T_0-spaces such that $f^{-1}(U_y)$ is homotopically trivial for every $y \in Y$. Then f is a simple homotopy equivalence.*

Proof. Consider the non-Hausdorff mapping cylinder $B(f)$ with the inclusions $i : X \hookrightarrow B(f)$, $j : Y \hookrightarrow B(f)$. Using the same proof of Lemma 4.2.7, one can show that $B(f) \searrow^{\gamma} i(X)$, while $B(f) \searrow\!\!\searrow j(Y)$ (the latter is true for every map f without more asumptions than its continuity). Then i and j are simple homotopy equivalences by Theorem 6.2.7, and since $jf = i$, so is f. □

Note that in the hypothesis of the last Proposition, every space Z with $f(X) \subseteq Z \subseteq Y$ has the simple homotopy type of Y, because in this case $f : X \to Z$ also satisfies the hypothesis of above.

Remark 6.2.10. The quotient maps of Theorems 6.1.1 and 6.1.2 are simple homotopy equivalences.

We finish this Section analyzing the relationship between γ-collapsibility and joins.

Proposition 6.2.11. *Let X and Y be finite T_0-spaces. Then*

(i) $X \oplus Y$ is homotopically trivial if X or Y is homotopically trivial.

(ii) $X \oplus Y$ is γ-collapsible if X or Y is γ-collapsible.

Proof. If X or Y is homotopically trivial, $|\mathcal{K}(X)|$ or $|\mathcal{K}(Y)|$ is contractible and then so is $|\mathcal{K}(X)| * |\mathcal{K}(Y)| = |\mathcal{K}(X \oplus Y)|$. Therefore $X \oplus Y$ is homotopically trivial.

The proof of (ii) follows as in Proposition 2.7.3. If $x_i \in X_i$ is a γ-point, $\hat{C}_{x_i}^{X_i \oplus Y} = \hat{C}_{x_i}^{X_i} \oplus Y$ is homotopically trivial by item (i) and then x_i is a γ-point of $X_i \oplus Y$. □

There is an analogous result for acyclic spaces that follows from the Künneth formula for joins [27].

Note that the converse of these results are false. To see this, consider two finite simply connected simplicial complexes K, L such that $H_2(|K|) = \mathbb{Z}_2$, $H_2(|L|) = \mathbb{Z}_3$ and $H_n(|K|) = H_n(|L|) = 0$ for every $n \geq 3$. Then $\mathcal{X}(K)$ and $\mathcal{X}(L)$ are not acyclic, but $\mathcal{X}(K) \oplus \mathcal{X}(L)$, which is weak homotopy equivalent to $|K| * |L|$, is acyclic by the Künneth formula and, since it is simply connected (see [27] or Remark 6.2.4), it is homotopically trivial.

A counterexample for the converse of item (ii) is the following.

Example 6.2.12. Let K be a triangulation of the Dunce hat. Then, $\mathcal{X}(K)$ is a homotopically trivial finite space of height 2. The non-Hausdorff suspension $\mathbb{S}(\mathcal{X}(K)) = \mathcal{X}(K) \cup \{-1, 1\}$ is γ-collapsible since 1 is a γ-point and $\mathbb{S}(\mathcal{X}(K)) \smallsetminus \{1\}$ has maximum. However $\mathcal{X}(K)$ is not collapsible and then $\mathbb{S}(\mathcal{X}(K))$ is not collapsible by Proposition 4.2.19. Moreover $\mathcal{X}(K)$ and S^0 are not γ-collapsible either because their heights are less than or equal to 2.

Chapter 7

H-regular complexes and quotients

7.1 H-regular CW-complexes and their associated finite spaces

Recall that a CW-complex K is regular if for each (open) cell e^n, the characteristic map $D^n \to \overline{e^n}$ is a homeomorphism, or equivalently, the attaching map $S^{n-1} \to K$ is a homeomorphism onto its image $\dot{e}^n$, the boundary of e^n. In this case, it can be proved that the closure $\overline{e^n}$ of each cell is a subcomplex, which is equivalent to say that $\dot{e}^n$ is a subcomplex.

A cell e of a regular complex K is a face of a cell e' if $e \subseteq \overline{e'}$. This will be denoted by $e \leq e'$. The barycentric subdivision K' is the simplicial complex whose vertices are the cells of K and whose simplices are the sets $\{e_1, e_2, \ldots, e_n\}$ such that e_i is a face of e_{i+1}.

We can define, as in the case of simplicial complexes, the face poset $\mathcal{X}(K)$ of a regular complex K, which is the set of cells ordered by $\leq$. Note that $\mathcal{K}(\mathcal{X}(K)) = K'$, which is homeomorphic to K and therefore $\mathcal{X}(K)$ is a finite model of K, i.e. it has the same weak homotopy type as K.

Example 7.1.1. The following figure (Figure 7.1) shows a regular structure for the real projective plane $\mathbb{R}P^2$. The edges are identified in the way indicated by the arrows. It has three 0-cells, six 1-cells and four 3-cells. Therefore its face poset has 13 points (Figure 7.2).

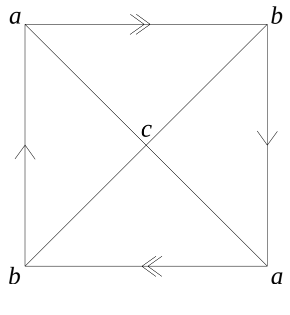


Figure 7.1: $\mathbb{R}P^2$.

In this Section we introduce the concept of *h-regular complex*, generalizing the notion of regular complex. Given an h-regular complex K, one can define $\mathcal{X}(K)$ as before. In

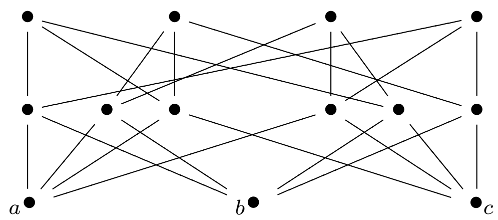


Figure 7.2: A finite model of $\mathbb{R}P^2$.

general, K and $\mathcal{K}(\mathcal{X}(K))$ are not homeomorphic. However we prove that $\mathcal{X}(K)$ is a finite model of K. We also study the relationship between collapses of h-regular complexes and of finite spaces.

Definition 7.1.2. A CW-complex K is *h-regular* if the attaching map of each cell is a homotopy equivalence with its image and the closed cells $\overline{e^n}$ are subcomplexes of K.

In particular, regular complexes are h-regular.

Proposition 7.1.3. *Let $K = L \cup e^n$ be a CW-complex such that $\dot{e}^n$ is a subcomplex of L. Then $\overline{e^n}$ is contractible if and only if the attaching map $\varphi : S^{n-1} \to \dot{e}^n$ of the cell e^n is a homotopy equivalence.*

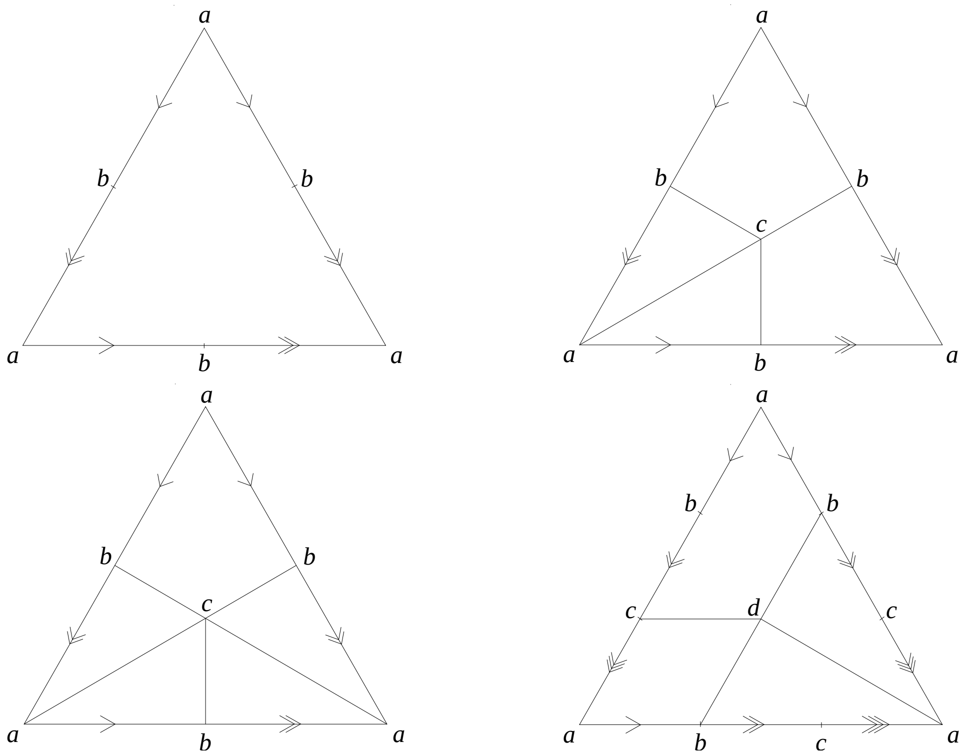
Proof. Suppose $\varphi : S^{n-1} \to \dot{e}^n$ is a homotopy equivalence. Since $S^{n-1} \hookrightarrow D^n$ has the homotopy extension property, the characteristic map $\psi : D^n \to \overline{e^n}$ is also a homotopy equivalence.

Suppose now that $\overline{e^n}$ is contractible. The map $\overline{\psi} : D^n/S^{n-1} \to \overline{e^n}/\dot{e}^n$ is a homeomorphism and therefore it induces isomorphisms in homology and, since $\overline{e^n}$ is contractible, by the long exact sequence of homology it follows that $\varphi_* : H_k(S^{n-1}) \to H_k(\dot{e}^n)$ is an isomorphism for every k.

If $n \geq 3$, $\pi_1(\dot{e}^n) = \pi_1(\overline{e^n}) = 0$ and by a theorem of Whitehead, φ is a homotopy equivalence. If $n = 2$, $\dot{e}^n$ is just a graph and since $\varphi_* : H_1(S^1) \to H_1(\dot{e}^n)$ is an isomorphism, the attaching map φ is a homotopy equivalence. Finally, if $n = 1$, since the cell is contractible, φ is one-to-one and therefore a homeomorphism. □

Corollary 7.1.4. *A CW-complex is h-regular if and only if the closed cells are contractible subcomplexes.*

Example 7.1.5. The following are four different h-regular structures for the Dunce hat which are not regular structures. In each example the edges are identified in the way indicated by the arrows.



For an h-regular complex K, we also define the *associated finite space* (or *face poset*) $\mathcal{X}(K)$ as the poset of cells of K ordered by the face relation $\leq$, like in the regular case. Note that since closed cells are subcomplexes, $e \leq e'$ if and only if $\overline{e} \subseteq \overline{e'}$.

The proof of the following lemma is standard.

Lemma 7.1.6. *Let $K \cup e$ be a CW-complex, let $\psi : D^n \to \overline{e}$ be the characteristic map of the cell e and let A be a subspace of $\dot{e}$. We denote $C_e(A) = \{\psi(x) \mid x \in D^n \smallsetminus \{0\},\ \psi(\frac{x}{\|x\|}) \in A\} \subseteq \overline{e}$. Then*

1. *If $A \subseteq \dot{e}$ is open, $C_e(A) \subseteq \overline{e}$ is open.*
2. *$A \subseteq C_e(A)$ is a strong deformation retract.*

Theorem 7.1.7. *If K is a finite h-regular complex, $\mathcal{X}(K)$ is a finite model of K.*

Proof. We define recursivelly a weak homotopy equivalence $f_K : K \to \mathcal{X}(K)$.

Assume $f_{K^{n-1}} : K^{n-1} \to \mathcal{X}(K^{n-1}) \subseteq \mathcal{X}(K)$ is already defined and let $x = \psi(a)$ be a point in an n-cell e^n with characteristic map $\psi : D^n \to \overline{e^n}$. If $a = 0 \in D^n$, define $f_K(x) = e^n$. Otherwise, define $f_K(x) = f_{K^{n-1}}(\psi(\frac{a}{\|a\|}))$.

In particular note that if $e^0 \in K$ is a 0-cell, $f_K(e^0) = e^0 \in \mathcal{X}(K)$. Notice also that if L is a subcomplex of K, $f_L = f_K|_L$.

We will show by induction on the number of cells of K, that for every cell $e \in K$, $f_K^{-1}(U_e)$ is open and contractible. This will prove that f_K is continuous and, by McCord's Theorem 1.4.2, a weak homotopy equivalence.

Let e be a cell of K. Suppose first that there exists a cell of K which is not contained in $\overline{e}$. Take a maximal cell e' (with respect to the face relation $\leq$) with this property.

Then $L = K \smallsetminus e'$ is a subcomplex and by induction, $f_L^{-1}(U_e)$ is open in L. It follows that $f_L^{-1}(U_e) \cap \dot{e}' \subseteq \dot{e}'$ is open and by the previous lemma, $C_{e'}(f_L^{-1}(U_e) \cap \dot{e}') \subseteq \overline{e'}$ is open. Therefore,

$$f_K^{-1}(U_e) = f_L^{-1}(U_e) \cup C_{e'}(f_L^{-1}(U_e) \cap \dot{e}')$$

is open in K.

Moreover, since $f_L^{-1}(U_e) \cap \dot{e}' \subseteq C_{e'}(f_L^{-1}(U_e) \cap \dot{e}')$ is a strong deformation retract, so is $f_L^{-1}(U_e) \subseteq f_K^{-1}(U_e)$. By induction, $f_K^{-1}(U_e)$ is contractible.

In the case that every cell of K is contained in $\overline{e}$, $f_K^{-1}(U_e) = \overline{e} = K$, which is open and contractible. □

As an application we deduce that the finite spaces associated to the h-regular structures of the Dunce hat considered in Example 7.1.5 are all homotopically trivial. The first one is a contractible space of 5 points, the second one is a collapsible and non-contractible space of 13 points and the last two are non-collapsible spaces of 15 points since they do not have weak points. Here we exhibit the Hasse diagram of the space associated to the third h-regular structure of the Dunce hat.

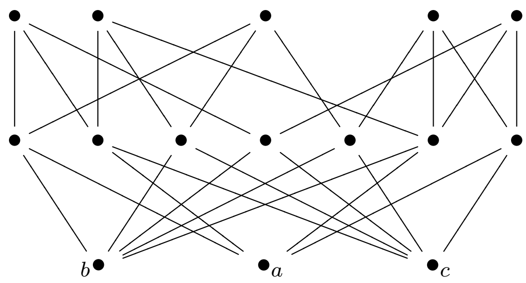
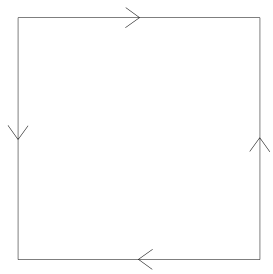
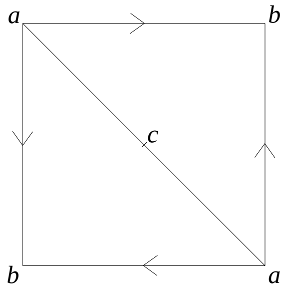


Figure 7.3: A homotopically trivial non-collapsible space of 15 points.

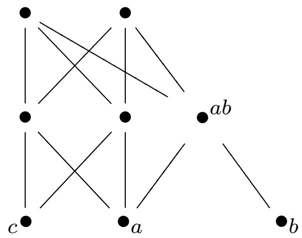
Example 7.1.8. Let K be the space which is obtained from a square by identifying all its edges as indicated.



We verify that K is homotopy equivalent to S^2 using techniques of finite spaces. Consider the following h-regular structure of K



which consists of three 0-cells, three 1-cells and two 2-cells. The Hasse diagram of the associated finite space $\mathcal{X}(K)$ is



The 0-cell b is an up beat point of $\mathcal{X}(K)$ and the 1-cell ab is a down beat point of $\mathcal{X}(K) \smallsetminus \{b\}$. Therefore K is weak homotopy equivalent to $\mathcal{X}(K) \smallsetminus \{b, ab\}$ which is a (minimal) finite model of S^2 (see Chapter 3). In fact $\mathcal{X}(K) \smallsetminus \{b, ab\} = S^0 \oplus S^0 \oplus S^0$ is weak homotopy equivalent to $S^0 * S^0 * S^0 = S^2$.

In Chapter 4 we proved that a collapse $K \searrow L$ of finite simplicial complexes induces a collapse $\mathcal{X}(K) \searrow \mathcal{X}(L)$ between the associated finite spaces. This is not true when K and L are regular complexes. Consider $L = \mathcal{K}(W)$ the associated simplicial complex to the Wallet W (see Figure 4.2), and K the CW-complex obtained from L by attaching a regular 2-cell e^2 with boundary $\mathcal{K}(\{a, b, c, d\})$ and a regular 3-cell e^3 with boundary $L \cup e^2$.

Note that the complex K is regular and collapses to L, but $\mathcal{X}(K) = \mathcal{X}(L) \cup \{e^2, e^3\}$ does not collapse to $\mathcal{X}(L)$ because $\hat{U}_{e^3}^{\mathcal{X}(K) \smallsetminus \{e^2\}} = \mathcal{X}(L) = W'$ is not contractible. However, one can prove that a collapse $K \searrow L$ between h-regular CW-complexes induces a γ-collapse $\mathcal{X}(K) \overset{\gamma}{\searrow} \mathcal{X}(L)$.

Theorem 7.1.9. *Let L be a subcomplex of an h-regular complex K. If $K \searrow L$, then $\mathcal{X}(K) \overset{\gamma}{\searrow} \mathcal{X}(L)$.*

Proof. Assume $K = L \cup e^n \cup e^{n+1}$. Then e^n is an up beat point of $\mathcal{X}(K)$. Since $K \searrow L$, $\overline{e^{n+1}} \searrow L \cap \overline{e^{n+1}} = \dot{e}^{n+1} \smallsetminus e^n$. In particular $\dot{e}^{n+1} \smallsetminus e^n$ is contractible and then

$$\hat{C}_{e^{n+1}}^{\mathcal{X}(K) \smallsetminus \{e^n\}} = \mathcal{X}(\dot{e}^{n+1} \smallsetminus e^n)$$

is homotopically trivial. Therefore

$$\mathcal{X}(K) \overset{e}{\searrow} \mathcal{X}(K) \smallsetminus \{e^n\} \overset{\gamma}{\searrow} \mathcal{X}(L).$$

□

We study the relatonship between the weak homotopy equivalence $f_K : |K| \to \mathcal{X}(K)$ defined in 7.1.7 and the McCord map $\mu_K : |K| \to \mathcal{X}(K)$. We will prove that both maps coincide if we take convenient characteristic maps for the cells of the polyhedron $|K|$.

Let σ be an n-simplex of the simplicial complex K. Let $\varphi : S^{n-1} \to \dot{\sigma}$ be a homeomorphism. Define the characteristic map $\overline{\varphi} : D^n \to \overline{\sigma}$ of the cell σ by

$$\overline{\varphi}(x) = \begin{cases} (1- \| x \|)b(\sigma) + \| x \| \varphi(\frac{x}{\|x\|}) & \text{if } x \neq 0 \\ b(\sigma) & \text{if } x = 0 \end{cases}$$

Here $b(\sigma) \in \overline{\sigma}$ denotes the barycenter of σ. Clearly $\overline{\varphi}$ is continuous and bijective and therefore a homeomorphism.

Definition 7.1.10. We say that the polyhedron $|K|$ has a *convenient cell structure* (as a CW-complex) if the characteristic maps of the cells are defined as above.

Proposition 7.1.11. *Let K be a finite simplicial complex. Consider a convenient cell structure for $|K|$. Then f_K and μ_K coincide.*

Proof. Let $x \in |K|$, contained in an open n-simplex σ. Let $\varphi : S^{n-1} \to |K|$ be the attaching map of the cell σ, and $\overline{\varphi} : D^n \to \overline{\sigma}$ its characteristic map. If x is the barycenter of σ, $f_K(x) = f_K(\overline{\varphi}(0)) = \sigma \in \mathcal{X}(K)$ and $\mu_K(x) = \mu_{\mathcal{X}(K)} s_K^{-1}(b(\sigma)) = \mu_{\mathcal{X}(K)}(\sigma) = \sigma$. Assume then that $x = \overline{\varphi}(y)$ with $y \neq 0$. Thus, $f_K(x) = f_K(\varphi(\frac{y}{\|y\|}))$. Then, by an inductive argument,

$$f_K(x) = \mu_K(\varphi(\frac{y}{\| y \|})) = \mu_{\mathcal{X}(K)}(s_K^{-1}\varphi(\frac{y}{\| y \|})).$$

On the other hand,

$$\mu_K(x) = \mu_{\mathcal{X}(K)} s_K^{-1}(\overline{\varphi}(y)) = \mu_{\mathcal{X}(K)} s_K^{-1}((1- \| y \|)b(\sigma) + \| y \| \varphi(\frac{y}{\| y \|})) =$$

$$= \mu_{\mathcal{X}(K)}((1- \| y \|)\sigma + \| y \| s_K^{-1}\varphi(\frac{y}{\| y \|})).$$

Finally, $s_K^{-1}\varphi(\frac{y}{\|y\|}) \in |(\dot{\sigma})'|$ and then,

$$\mu_{\mathcal{X}(K)}((1- \| y \|)\sigma + \| y \| s_K^{-1}\varphi(\frac{y}{\| y \|})) = min(support((1- \| y \|)\sigma + \| y \| s_K^{-1}\varphi(\frac{y}{\| y \|}))) =$$

$$= min(\{\sigma\} \cup support(s_K^{-1}\varphi(\frac{y}{\| y \|}))) = min(support(s_K^{-1}\varphi(\frac{y}{\| y \|}))) = \mu_{\mathcal{X}(K)}(s_K^{-1}\varphi(\frac{y}{\| y \|})).$$

Thus, $f_K(x) = \mu_K(X)$. □

7.2 Quotients of finite spaces: An exact sequence for homology groups

For CW-pairs, (Z, W) there exists a long exact sequence of reduced homology groups

$$\cdots \longrightarrow \widetilde{H}_n(W) \longrightarrow \widetilde{H}_n(Z) \longrightarrow \widetilde{H}_n(Z/W) \longrightarrow \widetilde{H}_{n-1}(W) \longrightarrow \cdots$$

More generally, this holds for any good pair (Z,W); i.e., a pair of topological spaces Z and W such that W is a closed subspace of Z which is a deformation retract of some neighborhood in Z. When Z and W are finite spaces, one does not have such a sequence in general. For a pair of finite spaces (X,A), $H_n(X,A)$ and $\widetilde{H}_n(X/A)$ need not be isomorphic (see Example 2.7.9) . However, we will prove that if A is a subspace of a finite T_0-space X, there is a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_n(A') \longrightarrow \widetilde{H}_n(X') \longrightarrow \widetilde{H}_n(X'/A') \longrightarrow \widetilde{H}_{n-1}(A') \longrightarrow \cdots$$

of the reduced homology groups of the subdivisions of X and A and their quotient. In fact, in this case we will prove that $\widetilde{H}_n(X'/A') = H_n(X,A) = H_n(X',A')$.

Recall that if W is a subcomplex of a CW-complex Z, Z/W is CW-complex with one n-cell for every n-cell of Z which is not a cell of W and an extra 0-cell. The n-squeleton $(Z/W)^n$ is the quotient Z^n/W^n. If $\overline{e^n}$ is a closed n-cell of Z which is not in W, there is a corresponding closed n-cell $q(\overline{e^n})$ in Z/W where $q: Z \to Z/W$ is the quotient map. If $\varphi : S^{n-1} \to Z^{n-1}$ is the attaching map of e^n and $\overline{\varphi} : D^n \to \overline{e^n}$ its characteristic map, $q\varphi : S^{n-1} \to Z^{n-1}/W^{n-1}$ and $q\overline{\varphi} : D^n \to q(e^n)$ are respectively, the attaching and caracteristic maps of the corresponding cell $\widetilde{e}^n$ in Z/W.

Theorem 7.2.1. *Let K be a finite simplicial complex and let $L \subseteq K$ be a full subcomplex. Then $|K|/|L|$ is an h-regular CW-complex and $\mathcal{X}(|K|/|L|)$ is homeomorphic to $\mathcal{X}(K)/\mathcal{X}(L)$.*

Proof. Let σ be an n-simplex of K which is not a simplex of L. If σ intersects L, then $\sigma \cap L = \tau$ is a proper face of σ. In particular $\overline{\tau}$ is contractible and therefore the corresponding closed cell $q(\overline{\sigma}) = \overline{\sigma}/\overline{\tau} \subseteq |K|/|L|$ is homotopy equivalent to $\overline{\sigma}$ which is contractible. Thus, closed cells of $|K|/|L|$ are contractible subcomplexes. By 7.1.4, $|K|/|L|$ is h-regular.

Now, if τ and σ are simplices of K which are not in L, then $\widetilde{\tau} \leq \widetilde{\sigma}$ in $\mathcal{X}(|K|/|L|)$ if and only if $q(\overline{\tau}) = \overline{\widetilde{\tau}} \subseteq \overline{\widetilde{\sigma}} = q(\overline{\sigma})$ if and only if τ is a face of σ in K if and only if $\tau \leq \sigma$ in $\mathcal{X}(K)/\mathcal{X}(L)$. Finally, if $\tau \in L$ and $\sigma \notin L$, $\widetilde{\tau} < \widetilde{\sigma}$ in $\mathcal{X}(|K|/|L|)$ if and only if $q(\overline{\tau}) \subset q(\overline{\sigma})$ if and only if $\sigma \cap L \neq \emptyset$ if and only if $\tau < \sigma$ in $\mathcal{X}(K)/\mathcal{X}(L)$. Therefore, $\mathcal{X}(|K|/|L|)$ and $\mathcal{X}(K)/\mathcal{X}(L)$ are homeomorphic. □

Corollary 7.2.2. *Let X be a finite T_0-space and $A \subseteq X$ a subspace. Then, the space $\mathcal{X}(|\mathcal{K}(X)|/|\mathcal{K}(A)|)$ is homeomorphic to X'/A'. In particular $|\mathcal{K}(X)|/|\mathcal{K}(A)|$ and $|\mathcal{K}(X'/A')|$ are homotopy equivalent.*

Proof. Apply 7.2.1 to $K = \mathcal{K}(X)$ and the full subcomplex $L = \mathcal{K}(A)$. □

Corollary 7.2.3. *If A is a subspace of a finite T_0-space X, $H_n(X,A) = \widetilde{H}_n(X'/A')$ for every $n \geq 0$.*

Proof. By the naturality of the long exact sequence of homology, the McCord map $\mu_X : |\mathcal{K}(X)| \to X$ induces isomorphisms $H_n(|\mathcal{K}(X)|, |\mathcal{K}(A)|) \to H_n(X,A)$. Thus,

$$H_n(X,A) = H_n(|\mathcal{K}(X)|, |\mathcal{K}(A)|) = \widetilde{H}_n(|\mathcal{K}(X)|/|\mathcal{K}(A)|) = \widetilde{H}_n(|\mathcal{K}(X'/A')|) = \widetilde{H}_n(X'/A').$$

□

Example 2.7.9 shows that $H_n(X,A)$ is not isomorphic to $\widetilde{H}_n(X/A)$ in general.

Proposition 7.2.4. *Let L be a full subcomplex of a finite simplicial complex K. Let $f_K : |K| \to \mathcal{X}(K)$, $f_{K/L} : |K|/|L| \to \mathcal{X}(|K|/|L|)$ be the weak homotopy equivalences constructed in Theorem 7.1.7 (for some characteristic maps of the cells of $|K|$). Let $q : |K| \to |K|/|L|$ and $\widetilde{q} : \mathcal{X}(K) \to \mathcal{X}(K)/\mathcal{X}(L)$ be the quotient maps and let $h : \mathcal{X}(|K|/|L|) \to \mathcal{X}(K)/\mathcal{X}(L)$ be the homeomorphism defined by $h(\widetilde{\sigma}) = \widetilde{q}(\sigma)$. Then, the following diagram commutes*

$$\begin{array}{ccc} |K| & \xrightarrow{q} & |K|/|L| \\ \downarrow{\scriptstyle f_K} & & \downarrow{\scriptstyle hf_{K/L}} \\ \mathcal{X}(K) & \xrightarrow{\widetilde{q}} & \mathcal{X}(K)/\mathcal{X}(L). \end{array}$$

Proof. Let $x \in |K|$, $x \in e^n$, an open n-simplex. We prove that $\widetilde{q}f_K(x) = hf_{K/L}q(x)$ by induction in n. Note that this is clear if $x \in |L|$, so we suppose $x \notin |L|$. If $n = 0$, $hf_{K/L}q(e^0) = hf_{K/L}(\widetilde{e}^0) = h(\widetilde{e}^0) = \widetilde{q}(e^0) = \widetilde{q}f_K(e^0)$. Assume then that $n > 0$, $x \in e^n$. Let $\varphi : S^{n-1} \to |K|$ and $\overline{\varphi} : D^n \to \overline{e^n}$ be the attaching and characteristic maps of e^n. Since e^n is not a simplex of L, $\overline{e}^n$ is a cell of $|K|/|L|$ with attaching map $q\varphi : S^{n-1} \to |K|/|L|$ and characteristic map $q\overline{\varphi} : D^n \to q(e^n)$. Let y in the interior of the disc D^n such that $x = \overline{\varphi}(y)$. By definition of $f_{K/L}$,

$$f_{K/L}(q(x)) = f_{K/L}((q\overline{\varphi})(y))) = \begin{cases} f_{K/L}((q\varphi)(\frac{y}{\|y\|})) & \text{if } y \neq 0 \\ \widetilde{e^n} & \text{if } y = 0 \end{cases}$$

If $y \neq 0$, $hf_{K/L}(q(x)) = hf_{K/L}q(\varphi(\frac{y}{\|y\|})) = \widetilde{q}f_K(\varphi(\frac{y}{\|y\|})) = \widetilde{q}f_K(x)$ by induction. If $y = 0$, $hf_{K/L}(x) = h(\widetilde{e^n}) = \widetilde{q}(e^n) = \widetilde{q}f_K(x)$. This proves that $\widetilde{q}f_K(x) = hf_{K/L}q(x)$. □

Let $\partial : \widetilde{H}_n(|K|/|L|) \to \widetilde{H}_{n-1}(|L|)$ be the connecting homomorphism of the long exact sequence of reduced homology. Define $\widetilde{\partial} = f_{L*}\partial((hf_{K/L})_*)^{-1} : \widetilde{H}_n(\mathcal{X}(K)/\mathcal{X}(L)) \to \widetilde{H}_n(\mathcal{X}(L))$. By the previous results, there exists a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_n(\mathcal{X}(L)) \xrightarrow{i_*} \widetilde{H}_n(\mathcal{X}(K)) \xrightarrow{\widetilde{q}_*} \widetilde{H}_n(\mathcal{X}(K)/\mathcal{X}(L)) \xrightarrow{\widetilde{\partial}} \widetilde{H}_{n-1}(\mathcal{X}(L)) \longrightarrow \cdots \tag{7.1}$$

Corollary 7.2.5. *Let A be a subspace of a finite T_0-space X. There exists a long exact sequence*

$$\cdots \longrightarrow \widetilde{H}_n(A') \xrightarrow{i_*} \widetilde{H}_n(X') \xrightarrow{\widetilde{q}_*} \widetilde{H}_n(X'/A') \xrightarrow{\widetilde{\partial}} \widetilde{H}_{n-1}(A') \longrightarrow \cdots \tag{7.2}$$

which is natural in the following sense: if $g : (X,A) \to (Y,B)$ is a map of pairs, there is a commutative diagram

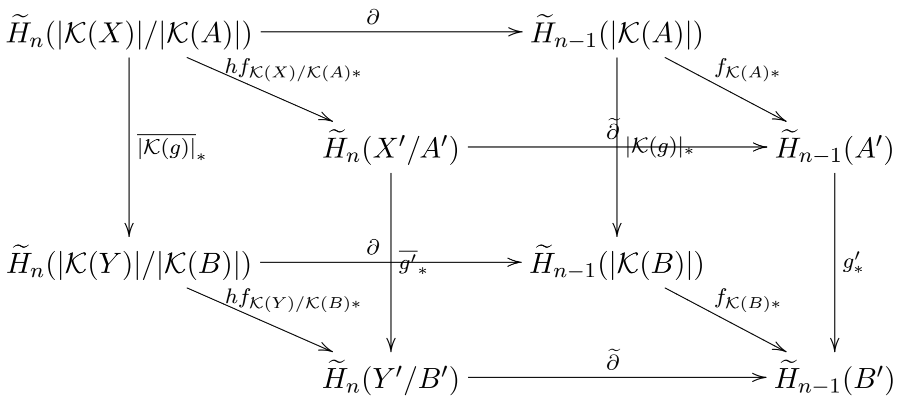
$$\begin{array}{ccccccccc} \cdots \longrightarrow & \widetilde{H}_n(A') & \xrightarrow{i_*} & \widetilde{H}_n(X') & \xrightarrow{\widetilde{q}_*} & \widetilde{H}_n(X'/A') & \xrightarrow{\widetilde{\partial}} & \widetilde{H}_{n-1}(A') & \longrightarrow \cdots \\ & \downarrow{\scriptstyle g'_*} & & \downarrow{\scriptstyle g'_*} & & \downarrow{\scriptstyle \overline{g'}_*} & & \downarrow{\scriptstyle g'_*} & \\ \cdots \longrightarrow & \widetilde{H}_n(B') & \xrightarrow{i_*} & \widetilde{H}_n(Y') & \xrightarrow{\widetilde{q}_*} & \widetilde{H}_n(Y'/B') & \xrightarrow{\widetilde{\partial}} & \widetilde{H}_{n-1}(B') & \longrightarrow \cdots \end{array} \tag{7.3}$$

where $g' = \mathcal{X}(\mathcal{K}(g))$ *is the induced map in the subdivisions.*

Proof. Consider a convenient cell structure for $|\mathcal{K}(X)|$. Taking $K = \mathcal{K}(X)$ and $L = \mathcal{K}(A)$ in 7.1 one obtains the long exact sequence 7.2 with the connecting morphism $\widetilde{\partial}$ defined as above for the maps f_K and $f_{K/L}$ induced by the cell structure of $|\mathcal{K}(X)|$.

The first two squares of 7.3 commute before taking homology. We only have to prove the commutativity of the third square.

Consider the following cube,



(7.4)

The top and bottom faces of the cube commute by definition of $\widetilde{\partial}$. The back face commute by the naturality of the long exact sequence for CW-complexes. Therefore, to prove that the front face commutes, we only have to check that the left and right faces do. To achieve this, we prove that these two squares commute up to homotopy:

$$\begin{array}{ccc} |\mathcal{K}(A)| & \xrightarrow{f_{\mathcal{K}(A)}} & A' \\ \downarrow{\scriptstyle |\mathcal{K}(g)|} & & \downarrow{\scriptstyle g'} \\ |\mathcal{K}(B)| & \xrightarrow{f_{\mathcal{K}(B)}} & B' \end{array} \qquad \begin{array}{ccc} |\mathcal{K}(X)|/|\mathcal{K}(A)| & \xrightarrow{hf_{\mathcal{K}(X)/\mathcal{K}(A)}} & X'/A' \\ \downarrow{\scriptstyle \overline{|\mathcal{K}(g)|}} & & \downarrow{\scriptstyle \overline{g'}} \\ |\mathcal{K}(Y)|/|\mathcal{K}(B)| & \xrightarrow{hf_{\mathcal{K}(Y)/\mathcal{K}(B)}} & Y'/B' \end{array}$$

For the first square this is clear, since the convenient cell structures for $|\mathcal{K}(X)|$ and $|\mathcal{K}(Y)|$ induce convenient cell structures for the subcomplexes $|\mathcal{K}(A)|$ and $|\mathcal{K}(B)|$ and in this case $f_{\mathcal{K}(A)} = \mu_{\mathcal{K}(A)}$ and $f_{\mathcal{K}(B)} = \mu_{\mathcal{K}(B)}$ by 7.1.11. For the second square we just have to remember that there exists a homotopy $H : \mu_{\mathcal{K}(Y)}|\mathcal{K}(g)| \simeq g'\mu_{\mathcal{K}(X)}$ such that $H(|\mathcal{K}(A)| \times I) \subseteq B'$ by 1.4.14 and this induces a homotopy $\overline{H} : |\mathcal{K}(X)|/|\mathcal{K}(A)| \times I \to Y'/B'$ which is the homotopy between $hf_{\mathcal{K}(Y)/\mathcal{K}(B)}\overline{|\mathcal{K}(g)|}$ and $\overline{g'}hf_{\mathcal{K}(X)/\mathcal{K}(A)}$ by 7.1.11 and 7.2.4. □

Remark 7.2.6. There is an alternative way to prove the existence of the sequence 7.1 and Corollary 7.2.5 above, which is in fact simpler than what we exhibit here. This proof does not use the fact that $\mathcal{X}(K)/\mathcal{X}(L)$ is a finite model of $|K|/|L|$ when L is a full subcomplex of K. However we chose that proof because Theorem 7.2.1 and Proposition 7.2.4 are applicatons of the first Section of this Chapter which give stronger results and provide an explicit formula for the weak homotopy equivalence $|K|/|L| \to \mathcal{X}(K)/\mathcal{X}(L)$.

The idea of the alternative proof is as follows: if L is a full subcomplex of K, $\mathcal{X}(L)^{op}$ is a closed subspace of $\mathcal{X}(K)^{op}$ which is a deformation retract of the nighborhood $\overline{\mathcal{X}(L)^{op}} \subseteq \mathcal{X}(K)^{op}$. Therefore, there is a long exact sequence as in Proposition 7.1 but for the opposite spaces $\mathcal{X}(L)^{op}, \mathcal{X}(K)^{op}$ and $\mathcal{X}(K)^{op}/\mathcal{X}(L)^{op}$. Using the associated complexes of these spaces we obtain the long exact sequence of Proposition 7.1 and the naturality of Corollary 7.2.5 follows from the naturality of the sequence for the opposite spaces.

Chapter 8

Actions, fixed points and a conjecture of Quillen

8.1 Equivariant homotopy theory and the poset of nontrivial p-subgroups of a group

Given a finite group G and a prime integer p, we denote by $S_p(G)$ the poset of nontrivial p-subgroups of G ordered by inclusion. In 1975, K. Brown [13] studies the relationship between the topological properties of the simplicial complex $\mathcal{K}(S_p(G))$ and the algebraic properties of G, and proves a very interesting variation of Sylow's Theorems for the Euler characteristic of $\mathcal{K}(S_p(G))$. In 1978, Daniel Quillen [33] investigates in depth topological properties of this complex [33]. One of his results claims that if G has a nontrivial normal p-subgroup, $|\mathcal{K}(S_p(G))|$ is contractible. He proves that the converse of this statement is true for solvable groups and conjectures that it is true for all finite groups. This conjecture is still open.

Apparently, Quillen was not aware of the theory of finite spaces by the time he wrote [33]. In fact, he works with the associated complex $\mathcal{K}(S_p)$ without considering the intrinsic topology of the poset $S_p(G)$.

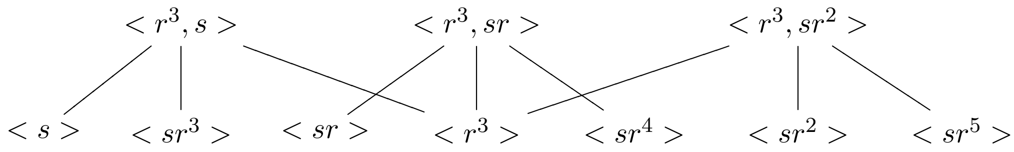
In 1984, Stong [38] investigates the equivariant homotopy theory of finite spaces and its relationship with Quillen's conjecture. He shows that G has a nontrivial p-subgroup if and only if $S_p(G)$ is a contractible finite space. Therefore, the conjecture can be restated as:

$$S_p(G) \text{ is contractible if and only if it is homotopically trivial.}$$

In this Section, we recall the basic results on equivariant homotopy theory of finite spaces due to Stong [38] and their applications to the study of the poset $S_p(G)$. At the end of the Section, we exhibit an alternative proof of K. Brown's result on the Euler characteristic of $S_p(G)$.

In the following, G will denote a finite group and p, a prime integer dividing the order of G.

Example 8.1.1. For $G = D_6 =< s, r \mid s^2 = r^6 = srsr = 1 >$, the dihedral group of order 12, and $p = 2$, the poset $S_2(D_6)$ looks as follows



Theorem 8.1.2 (Quillen)**.** *If G has a nontrivial normal p-subgroup, $|\mathcal{K}(S_p(G))|$ is contractible.*

Proof. Suppose N is a nontrivial normal p-subgroup of G. Define $f : S_p(G) \to S_p(G)$ by $f(H) = NH = \{nh \mid n \in N,\ h \in H\}$. NH is a subgroup of G since $N \lhd G$. Moreover, NH is a quotient of the semiderct product $N \rtimes H$, where $(n_1, h_1)(n_2, h_2) = (n_1 h_1 n_2 h_1^{-1}, h_1 h_2)$. Since N and H are p-groups, so is NH. Therefore, f is well defined. Clearly f is order preserving, and if c_N denotes the constant map N, $c_N \leq f \geq 1_{S_p(G)}$. Thus $1_{S_p(G)}$ is homotopic to a constant and then, $S_p(G)$ is contractible. In particular, its associated complex is contractible. □

Note that in fact Quillen proved that if G has a nontrivial normal p-subgroup, $S_p(G)$ is a contractible finite space.

Conjecture 8.1.3 (Quillen). If $|\mathcal{K}(S_p(G))|$ is contractible, G has a nontrivial normal p-subgroup.

By a G-space we will mean a topological space X with an action of G such that the maps $m_g : X \to X$ defined by $m_g(x) = gx$ are continuous for every $g \in G$. A G-map (or equivariant map) between G-spaces X and Y is a continuous map $f : X \to Y$ such that $f(gx) = gf(x)$ for every $g \in G$ and $x \in X$. A homotopy $H : X \times I \to Y$ is a G-homotopy (or equivariant homotopy) if $H(gx, t) = gH(x, t)$ for every $g \in G, x \in X, t \in I$. A G-map $f : X \to Y$ is a G-homotopy equivalence if there exists a G-map $h : Y \to X$ such that there exists G-homotopies between hf and 1_X and between fh and 1_Y. A subspace A of a G-space X is said to be G-invariant if $ga \in A$ for every $g \in G, a \in A$. A G-invariant subspace $A \subseteq X$ is an equivariant strong deformation retract if there is an equivariant retraction $r : X \to A$ such that ir is homotopic to 1_X via a G-homotopy which is stationary at A.

If x is a point of a G-space X, $Gx = \{gx\}_{g \in G}$ denotes the orbit of x. The set of fixed points by the action is denoted by $X^G = \{x \in X \mid gx = x\ \forall g \in G\}$.

A finite T_0-space which is a G-space will be called a finite T_0-G-space.

Proposition 8.1.4. *Let X be a finite T_0-space, $x \in X$ and let $f : X \to X$ be an automorphism. If x and $f(x)$ are comparable, $x = f(x)$.*

Proof. Assume without loss of generality that $x \leq f(x)$. Then, $f^i(x) \leq f^{i+1}(x)$ for every $i \geq 0$. By the finitness of X, the equality must hold for some i and since f is a homeomorphism $x = f(x)$. □

Lemma 8.1.5. *Let X be a finite T_0-G-space. Then, there exists a core of X which is G-invariant and an equivariant strong deformation retract of X.*

Proof. Suppose X is not minimal. Then, there exists a beat point $x \in X$. Without loss of generality suppose x is a down beat point. Let y be the maximum of $\hat{U}_x$. Since $m_g : X \to X$ is a homeomorphism, gy is the maximum of $\hat{U}_{gx}$ for every $g \in G$. Define the retraction $r : X \to X \smallsetminus Gx$ by $r(gx) = gy$. This map is well defined since $gy \notin Gx$ by Proposition 8.1.4 and since $gx = hx$ implies $gy = max\ \hat{U}_{gx} = max\ \hat{U}_{hx} = hy$. Moreover, r is a continuous G-map. The homotopy $X \times I \to X$ corresponding to the path $\alpha : I \to X^X$ given by $\alpha(t) = ir$ if $0 \leq t < 1$ and $\alpha(1) = 1_X$ is a G-homotopy between ir and 1_X relative to $X \smallsetminus Gx$. Therefore $X \smallsetminus Gx$ is an equivariant strong deformation retract of X. The proof is concluded by an inductive argument. □

Proposition 8.1.6. *A contractible finite T_0-G-space has a point which is fixed by the action of G.*

Proof. By Lemma 8.1.5 there is a core, i.e. a point, which is G-invariant. □

Proposition 8.1.7. *Let X and Y be finite T_0-G-spaces and let $f : X \to Y$ be a G-map which is a homotopy equivalence. Then f is an equivariant homotopy equivalence.*

Proof. Let X_c and Y_c be cores of X and Y which are equivariant strong deformation retracts. Denote i_X, i_Y and r_X, r_Y the inclusions and equivariant strong deformation retractions. Since f is a homotopy equivalence and a G-map, so is $r_Y f i_X : X_c \to Y_c$. Therefore, $r_y f i_X$ is a G-isomorphism. Define the G-map $g = i_X(r_Y f i_X)^{-1} r_Y : Y \to X$, then

$$fg = f i_X (r_Y f i_X)^{-1} r_Y \simeq i_Y r_Y f i_X (r_Y f i_X)^{-1} r_Y = i_Y r_Y \simeq 1_Y,$$
$$gf = i_X (r_Y f i_X)^{-1} r_Y f \simeq i_X (r_Y f i_X)^{-1} r_Y f i_X r_X = i_X r_X \simeq 1_X.$$

All the homotopies being equivariant. Therefore f is an equivariant homotopy equivalence with homotopy inverse g. □

Remark 8.1.8. Two finite T_0-G-spaces which are homotopy equivalent, need not have the same equivariant homotopy type. Let $X = \mathbb{S}(S^0)$. The group of automorphisms $Aut(X)$ acts on X in the usual way by $f \cdot x = f(x)$ and in the trivial way by $f \circ x = x$. Denote by X_0 the $Aut(X)$-space with the first action and by X_1, the second. Suppose there exists an equivariant homotopy equivalence $g : X_0 \to X_1$. Since X is minimal, g is a homeomorphism. Let $f : X \to X$ be an automorphism distinct from the identity. Then, $gf(x) = g(f \cdot x) = f \circ g(x) = g(x)$ for every $x \in X$. Thus, $f = 1_X$, which is a contradiction.

Theorem 8.1.9 (Stong)**.** *Let G be a finite group and let p be a prime integer. Then $S_p(G)$ is contractible if and only if G has a nontrivial normal p-subgroup.*

Proof. The poset $S_p(G)$ is a G-space with the action given by conjugation, $g \cdot H = gHg^{-1}$. If $S_p(G)$ is contractible, by Proposition 8.1.6, there exists $N \in S_p(G)$ such that $gNg^{-1} = N$ for every $g \in G$, i.e., N is a normal subgroup of G.

The converse can be deduced from the proof of Theorem 8.1.2. □

In the light of Theorem 8.1.9, the conjecture may be restated as follows:

Restatement of Quillen's conjecture (Stong): if $S_p(G)$ is homotopically trivial, it is contractible.

In [33], Quillen shows that his conjecture 8.1.3 is true for solvable groups. To do this, Quillen works with another poset $A_p(G)$ which is weak homotopy equivalent to $S_p(G)$, and proves that if G does not have nontrivial normal p-subgroups, then $A_p(G)$ has a nonvanishing homology group. The finite space $A_p(G)$ is the subspace of $S_p(G)$ consisting of the elementary abelian p-subgroups, i.e. abelian subgroups whose elements have all order 1 or p.

Proposition 8.1.10. *The inclusion $A_p(G) \hookrightarrow S_p(G)$ is a weak homotopy equivalence.*

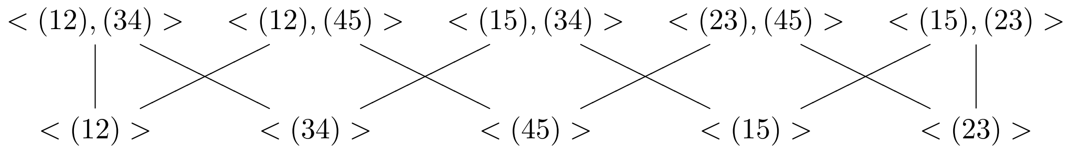
Proof. By Theorem 1.4.2, it suffices to show that $i^{-1}(U_H) = A_p(H)$ is contractible for every $H \in S_p(G)$. Since H is a nontrivial p-subgroup, its center Z is not trivial. Let $N \subseteq H$ be the subgroup of elements of order 1 or p. If $T \in A_p(H)$, $TN \in A_p(H)$ and $T \leq TN \geq N$. Therefore, $A_p(H)$ is contractible. □

In [38], Stong shows that in general $A_p(G)$ and $S_p(G)$ are not homotopy equivalent, however, if $A_p(G)$ is contractible, there is a fixed point by the action of G and then $S_p(G)$ is contractible. Apparently it is unknown whether the converse of this results holds.

Example 8.1.11. Let Σ_5 be the symmetric group on five letters. We give an alternative proof of the well known fact that Σ_5 has no nontrivial normal 2-subgroups.

The subgroup $< (1234), (13) > \subseteq \Sigma_5$ has order 8 and it is not abelian. All the other subgroups of order 8 are isomorphic to this Sylow 2-subgroup and therefore, Σ_5 has no elementary abelian subgroups of order 8. Thus, the height of the poset $A_2(\Sigma_5)$ is at most one.

On the other hand, there is a subspace of $A_2(\Sigma_5)$ with the following Hasse diagram



Then the graph $\mathcal{K}(A_2(\Sigma_5))$ has a cycle and therefore it is not contractible. Hence, $A_2(\Sigma_5)$ is not homotopically trivial and then neither is $S_2(\Sigma_5)$. In particular, $S_2(\Sigma_5)$ is not contractible and then Σ_5 does not have normal 2-subgroups which are nontrivial.

Now we exhibit an alternative proof of K. Brown's result on Euler characteristic.

Proposition 8.1.12. *Let H be a subgroup of G. Then, $S_p(G)^H$ is contractible.*

Proof. If $T \in S_p(G)^H$, $TH \in S_p(G)^H$. Since $T \leq TH \geq H$, the constant map $c_H : S_p(G)^H \to S_p(G)^H$ is homotopic to the identity. □

Note that if X is a finite T_0-G-space, the subdivision X' is also a G-space with the action given by $g \cdot \{x_0, x_1, \ldots, x_n\} = \{gx_0, gx_1, \ldots, gx_n\}$.

Let P be a Sylow p-subgroup of G. The action of P on $S_p(G)$ by conjugation induces an action of P on $S_p(G)'$. Given $c \in S_p(G)'$, let $P_c = \{g \in P \mid gc = c\}$ denote the isotropy group (or stabilizer) of c. Define $Y = \{c \in S_p(G)' \mid P_c \neq 0\}$.

Lemma 8.1.13. $\chi(S_p(G)', Y) \equiv 0 \ mod(\#P)$.

Proof. Let $C = \{c_0 < c_1 < \ldots < c_n\} \in S_p(G)''$ be a chain of $S_p(G)'$ which is not a chain of Y. Then, there exists $0 \leq i \leq n$ such that $c_i \notin Y$. Therefore, if g and h are two diferent elements of P, $gc_i \neq hc_i$. In other words, the orbit of c_i under the action of P has $\#P$ elements. Thus, the orbit of C also has $\#P$ elements. In particular, $\#P$ divides $\chi(S_p(G)', Y) = \sum\limits_{i \geq 0} (-1)^i \alpha_i$, where α_i is the number of chains of $(i+1)$-elements of $S_p(G)'$ which are not chains of Y. □

Lemma 8.1.14. *Y is homotopically trivial.*

Proof. Let $f : Y \to S_p(P)^{op}$ defined by $f(c) = P_c$, the isotropy group of c. By definition of Y, P_c is a nontrivial subgroup of P and then f is a well defined function. If $c_0 \leq c_1$, $P_{c_0} \supseteq P_{c_1}$. Thus, f is continuous. If $0 \neq H \subseteq P$, $f^{-1}(U_H) = \{c \in Y \mid H \subseteq P_c\} = (S_p(G)^H)'$, which is contractible by Proposition 8.1.12. From Theorem 1.4.2 we deduce that f is a weak homotopy equivalence. Since $S_p(P)^{op}$ has minimum, Y is homotopically trivial. □

In [33], Quillen proves that Y is homotopically trivial finding a third space Z which is weak homotopy equivalent to Y and $S_p(P)$. Our proof is somewhat more direct.

Theorem 8.1.15 (K. Brown)**.** $\chi(S_p(G)) \equiv 1 \ mod(\#P)$.

Proof. Since $\chi(Y) = 1$ by Lemma 8.1.14, $\chi(S_p(G)) = \chi(S_p(G)') = \chi(Y) + \chi(S_p(G)', Y) \equiv 1 \ mod(\#P)$. □

8.2 Equivariant simple homotopy types and Quillen's conjecture

Stong's result 8.1.6 says that if X is a finite T_0-G-space which is contractible, then there is a point which is fixed by G. This is not true if we change X by a polyhedron (see [30]). There exists a contractible finite G-simplicial complex K with no fixed points. Therefore, considering the associated finite space with the induced action of G, we obtain a finite T_0-G-space which is homotopically trivial and which has no fixed points. To prove the Quillen's conjecture, we need to show that if $S_p(G)$ is homotopically trivial, then the action of G by conjugation has a fixed point.

The proof of Proposition 8.1.6 and the previous results suggest that the hypothesis of contractibilty can be replaced by a weaker notion. Combining these ideas with the simple homotopy theory of finite spaces, we introduce the notion of G-collapse of finite spaces and of simplicial complexes. These two concepts are strongly related similarly as in the nonequivariant case.

Equivariant simple homotopy types of finite spaces allow us to attack the conjecture of Quillen and to deepen into equivariant homotopy theory of finite spaces originally studied by Stong. We obtain new formulations of the conjecture using these concepts, trying to get closer to its proof.

As in the previous Section, G will denote a finite group.

Recall that there is a strong collapse from a finite T_0-space X to a subspace Y if the second one is obtained from the first by removing beat points. By our results on minimal pairs, this is equivalent to say that $Y \subseteq X$ is a strong deformation retract. We denote this situation by $X \searrow\!\!\searrow Y$.

If x is a beat point of a finite T_0-G-space X, $gx \in X$ is a beat point for every $g \in G$. In this case we say that there is an elementary strong G-collapse from X to $X \smallsetminus Gx$. Note that elementary strong G-collapses are strong collapses. A sequence of elementary strong G-collapses is called a strong G-collapse and it is denoted by $X \searrow\!\!\searrow^G Y$. Strong G-expansions are defined dually.

Proposition 8.2.1. *Let X be a finite T_0-G-space and $Y \subseteq X$ a G-invariant subspace. The following are equivalent:*

i. $X \searrow\!\!\searrow^G Y$.

ii. $Y \subseteq X$ *is an equivariant strong deformation retract.*

iii. $Y \subseteq X$ *is a strong deformation retract.*

Proof. If there is an elementary strong G-collapse from X to Y, then by the proof of Lemma 8.1.5, Y is an equivariant strong deformation retract of X.

If $Y \subseteq X$ is a strong deformation retract and $x \in X \smallsetminus Y$ is a beat point of X, $X \searrow\!\!\searrow^G X \smallsetminus Gx = X_1$. In particular $X_1 \subseteq X$ is a strong deformation retract, and then, so is $Y \subseteq X_1$. By induction, $X_1 \searrow\!\!\searrow^G Y$ and then $X \searrow\!\!\searrow^G Y$. □

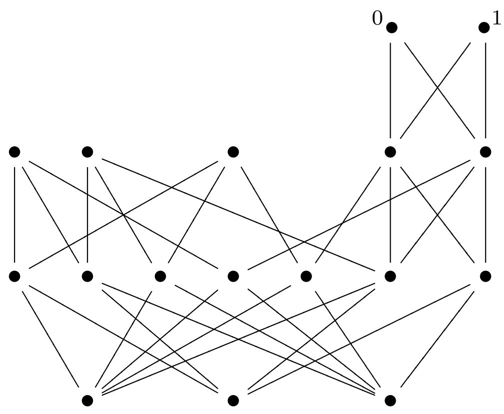
Let X be a finite T_0-G-space. A core of X which is G-invariant is called a *G-core*. From Stong's results (Lemma 8.1.5), it follows that every finite T_0-G-space has a G-core.

Definition 8.2.2. Let X be a finite T_0-G-space. If $x \in X$ is a weak point, $gx \in X$ is also a weak point for every $g \in G$ and we say that there is an *elementary G-collapse* from X to $X \smallsetminus Gx$. Note that the resulting subspace $X \smallsetminus Gx$ is G-invariant. A sequence of elementary G-collapses is called a *G-collapse* and it is denoted $X \searrow^G Y$. *G-expansions* are defined dually. X is *G-collapsible* if it G-collapses to a point.

Note that strong G-collapses are G-collapses and that G-collapses are collapses. If the action is trivial, G-collapses and collapses coincide.

A finite T_0-G-space is strongly collapsible if and only if it is G-strongly collapsible. However, this is not true for collapsibility and G-collapsibility as the nex example shows.

Example 8.2.3. Let X be the following finite space (cf. Figure 7.3 above Example 7.1.8)



Consider the action of the two-element group $\mathbb{Z}_2$ over X that permutes 0 and 1 and fixes every other element. The unique weak points of X are 0 and 1. $X \smallsetminus \{0\}$ is collapsible but $X \smallsetminus \{0,1\}$ is not. Therefore X is a collapsible finite space which is not G-collapsible.

The notion of G-collapse can be studied also in the setting of simplicial complexes.

Suppose K is a finite G-simplicial complex and $\sigma \in K$ is a free face of $\sigma' \in K$. Then for every $g \in G$, $g\sigma$ is a free face of $g\sigma'$, however it is not necessarily true that K collapses to $K \smallsetminus \bigcup\limits_{g\in G} \{g\sigma, g\sigma'\}$.

Example 8.2.4. Let σ' be a 2-simplex and $\sigma \subsetneq \sigma'$ a 1-face of σ. Consider the action of $\mathbb{Z}_3$ by rotation over $K = \overline{\sigma'}$. Then σ is a free face of σ', but $\overline{\sigma'}$ does not collapse to $\overline{\sigma'} \smallsetminus \bigcup\limits_{g\in \mathbb{Z}_3} \{g\sigma, g\sigma'\}$ which is the discrete complex with 3 vertices.

If σ is a free face of σ' in the G-complex K, and $g \in G$ is such that $g\sigma = \sigma$, then $\sigma \subsetneq g\sigma'$ and therefore $g\sigma' = \sigma'$. In other words, the isotropy group G_σ of σ is contained in the isotropy group $G_{\sigma'}$ of σ'. The other inclusion does not hold in general as the previous example shows.

Definition 8.2.5. Let K be a finite G-simplicial complex and let $\sigma \in K$ be a free face of $\sigma' \in K$ ($\sigma \subsetneq \sigma'$ is a *collapsible pair*). Consider the G-invariant subcomplex $L = K \smallsetminus \bigcup\limits_{g\in G} \{g\sigma, g\sigma'\}$. We say that there is an *elementary G-collapse* $K \searrow^{Ge} L$ from K to L, or that $\sigma \subsetneq \sigma'$ is a *G-collapsible pair*, if $G_\sigma = G_{\sigma'}$. A sequence of elementary G-collapses is called a *G-collapse* and denoted by $K \searrow^{G} L$. A G-complex K is *G-collapsible* if it G-collapses to a vertex.

Proposition 8.2.6. *Let K be a finite G-simplicial complex and let $\sigma \subsetneq \sigma'$ be a collapsible pair. The following are equivalent:*

1. *$\sigma \subsetneq \sigma'$ is a G-collapsible pair.*
2. *$K \searrow L = K \smallsetminus \bigcup\limits_{g\in G} \{g\sigma, g\sigma'\}$.*

Proof. Suppose σ is an n-simplex and that $K \searrow L$. Then, the set $\bigcup\limits_{g\in G} \{g\sigma, g\sigma'\}$ contains as many n-simplices as $(n+1)$-simplices i.e., the sets $G \cdot \sigma = \{g\sigma\}_{g\in G}$ and $G \cdot \sigma' = \{g\sigma'\}_{g\in G}$ have the same cardinality. Therefore

$$\#G_\sigma = \#G/\#G\cdot\sigma = \#G/\#G\cdot\sigma' = \#G_{\sigma'}.$$

Since $G_\sigma \subseteq G_{\sigma'}$, the equality holds.

Conversely, suppose $\sigma \subsetneq \sigma'$ is a G-collapsible pair. Then, the pairs $g\sigma \subsetneq g\sigma'$ can be collapsed one at the time. □

Therefore, G-collapses are collapses. The following is an extension of the classical result of Whitehead (see [44] for example) which says that if $K_1, K_2 \subseteq K$ are finite simplicial complexes, then $K_1 \cup K_2 \searrow K_1$ if and only if $K_2 \searrow K_1 \cap K_2$ (with the same sequence of collapses). The proof is straightforward.

Remark 8.2.7. Let K be a finite G-simplicial complex and let $K_1, K_2 \subseteq K$ be two G-invariant subcomplexes such that $K_1 \cup K_2 = K$. Then, $K \searrow^G K_1$ if and only if $K_2 \searrow^G K_1 \cap K_2$.

Remark 8.2.8. Let X be a finite T_0-G-space. If X is G-collapsible, it collapses to a G-invariant one-point subspace. In particular, the fixed point set X^G is non-empty.

The following result is a direct consequence of Remark 8.2.8 and Theorem 8.1.9.

Proposition 8.2.9. *For a finite group G and a prime number p, we have the following equivalences:*

1. *G has a nontrivial normal p-subgroup.*
2. *$S_p(G)$ is contractible.*
3. *$S_p(G)$ is G-collapsible.*

Now we will study the relationship between G-collapses of finite spaces and simplicial G-collapses.

If X is a finite T_0-G-space, there is a natural induced action on $\mathcal{K}(X)$. If we consider G both as a discrete topological group and a discrete simplicial complex, there is a natural isomorphism $\mathcal{K}(G \times X) = G \times \mathcal{K}(X)$ and an action $\theta : G \times X \to X$ induces an action $\mathcal{K}(\theta) : G \times \mathcal{K}(X) = \mathcal{K}(G \times X) \to \mathcal{K}(X)$ Analogously, an action $\theta : G \times K \to K$ over a finite simplicial complex K induces an action $\mathcal{X}(\theta) : G \times \mathcal{X}(K) = \mathcal{X}(G \times K) \to \mathcal{X}(K)$.

Unless we say the opposite, if X is a finite T_0-G-space and K a finite G-simplicial complex, we will assume the actions over $\mathcal{K}(X)$ and $\mathcal{X}(K)$ are the induced ones.

The main aim of this Section is to prove the equivariant version of Theorem 4.2.12. The proof will be similar to the proof of the nonequivariant case.

Lemma 8.2.10. *Let aK be a finite simplicial cone and suppose G acts on aK fixing the vertex a. Then $aK \searrow^G a$.*

Proof. Let σ be a maximal simplex of K. Then $\sigma \subsetneq a\sigma$ is a G-collapsible pair since $g \cdot a\sigma = a\sigma$ implies $g\sigma = \sigma$. Therefore $aK \searrow^G aK \smallsetminus \bigcup\limits_{g \in G} \{g\sigma, g \cdot a\sigma\} = a(K \smallsetminus \bigcup\limits_{g \in G} \{g\sigma\})$. The lemma follows from an inductive argument. □

Lemma 8.2.11. *Let X be a finite T_0-G-space and let $x \in X$. The stabilizer G_x of x acts on $\hat{C}_x$ and then on $\mathcal{K}(\hat{C}_x)$. If $\mathcal{K}(\hat{C}_x)$ is G_x-collapsible, $\mathcal{K}(X) \searrow^G \mathcal{K}(X \smallsetminus Gx)$.*

Proof. If $\sigma \subsetneq \sigma'$ is a G_x-collapsible pair in $\mathcal{K}(\hat{C}_x)$, $x\sigma \subsetneq x\sigma'$ is G_x-collapsible in $x\mathcal{K}(\hat{C}_x)$. In this way, copying the elementary G_x-collapses of $\mathcal{K}(\hat{C}_x) \searrow^{G_x} *$, one obtains that $\mathcal{K}(C_x) = x\mathcal{K}(\hat{C}_x) \searrow^{G_x} \mathcal{K}(\hat{C}_x) \cup \{x, x*\} \searrow^{G_x} \mathcal{K}(\hat{C}_x)$. Now we will show that since $\mathcal{K}(C_x) \searrow^{G_x} \mathcal{K}(\hat{C}_x)$,

$$\bigcup_{g \in G} g\mathcal{K}(C_x) \searrow^G \bigcup_{g \in G} g\mathcal{K}(\hat{C}_x). \tag{8.1}$$

Suppose $\mathcal{K}(C_x) = K_0 \searrow^{G_x e} K_1 \searrow^{G_x e} K_2 \searrow^{G_x e} \ldots \searrow^{G_x e} K_r = \mathcal{K}(\hat{C}_x)$. Notice that all the simplices removed in these collapses contain the vertex x. If $\sigma \subsetneq \sigma'$ is the G_x-collapsible pair collapsed in $K_i \searrow^{G_x e} K_{i+1}$ (along with the other simplices in the orbits of σ and σ'), we afirm that $\sigma \subsetneq \sigma'$ is G-collapsible in $\bigcup_{g\in G} gK_i$. Suppose $\sigma \subsetneq g\widetilde{\sigma}$ with $g \in G$, $\widetilde{\sigma} \in K_i$. Since $x \in \sigma \subsetneq g\widetilde{\sigma}$, $g^{-1}x \in \widetilde{\sigma}$ and then x and $g^{-1}x$ are comparable. By Proposition 8.1.4 $x = g^{-1}x$ and therefore $g \in G_x$. Since K_i is G_x-invariant and σ is a free face of σ' in K_i, $g\widetilde{\sigma} = \sigma'$. Therefore, $\sigma \subsetneq \sigma'$ is a collapsible pair in $\bigcup_{g\in G} gK_i$.

Let $g \in G$ be such that $g\sigma' = \sigma'$. By the same argument as above, $x, gx \in \sigma'$ and then $g \in G_x$. Since $\sigma \subsetneq \sigma'$ is G_x-collapsible in K_i, $g\sigma = \sigma$, which proves that it is also G-collapsible in $\bigcup_{g\in G} gK_i$. Thus,

$$\bigcup_{g\in G} gK_i \searrow^{Ge} \bigcup_{g\in G} gK_i \smallsetminus \bigcup_{g\in G}\{g\sigma, g\sigma'\} = \bigcup_{g\in G}(gK_i \smallsetminus \bigcup_{h\in G}\{gh\sigma, gh\sigma'\}) = \bigcup_{g\in G} g(K_i \smallsetminus \bigcup_{h\in G}\{h\sigma, h\sigma'\})$$

But $h\sigma$ and $h\sigma'$ are simplices of K_i if and only if $h \in G_x$, then

$$\bigcup_{g\in G} g(K_i \smallsetminus \bigcup_{h\in G}\{h\sigma, h\sigma'\}) = \bigcup_{g\in G} g(K_i \smallsetminus \bigcup_{h\in G_x}\{h\sigma, h\sigma'\}) = \bigcup_{g\in G} gK_{i+1}.$$

So 8.1 is proved, i.e.,

$$\bigcup_{g\in G} g\mathcal{K}(C_x) \searrow^{G} \bigcup_{g\in G} g\mathcal{K}(\hat{C}_x) = (\bigcup_{g\in G} g\mathcal{K}(C_x)) \cap \mathcal{K}(X \smallsetminus Gx).$$

By 8.2.7,

$$\mathcal{K}(X) = (\bigcup_{g\in G} g\mathcal{K}(C_x)) \cup \mathcal{K}(X \smallsetminus Gx) \searrow^{G} \mathcal{K}(X \smallsetminus Gx).$$

□

Theorem 8.2.12.

(a) Let X be a finite T_0-G-space and $Y \subseteq X$ a G-invariant subspace. If $X \searrow^{G} Y$, $\mathcal{K}(X) \searrow^{G} \mathcal{K}(Y)$.

(b) Let K be a finite G-simplicial complex and $L \subseteq K$ a G-invariant subcomplex. If $K \searrow^{G} L$, $\mathcal{X}(K) \searrow^{G} \mathcal{X}(K)$.

Proof. Suppose first that $x \in X$ is a beat point. Then there exists $y \in X$, $y \neq x$ such that $C_x \subseteq C_y$. Therefore $G_x \subseteq G_y$ by Proposition 8.1.4 and $\mathcal{K}(\hat{C}_x) = y\mathcal{K}(\hat{C}_x \smallsetminus \{y\})$. The stabilizer G_x of x acts on $\hat{C}_x$, and therefore on $\mathcal{K}(\hat{C}_x)$, and fixes y. By Lemma 8.2.10, $\mathcal{K}(\hat{C}_x) \searrow^{G_x} y$. By Lemma 8.2.11, $\mathcal{K}(X) \searrow^{G} \mathcal{K}(X \smallsetminus Gx)$. In particular if X is contractible, this says that $\mathcal{K}(X)$ is G-collapsible.

Suppose now that $x \in X$ is a weak point. Then C_x is contractible and $\mathcal{K}(C_x)$ is G_x-collapsible. Again from Lemma 8.2.11, we obtain that $\mathcal{K}(X) \searrow^{G} \mathcal{K}(X \smallsetminus Gx)$. This proves the first part of the theorem for elementary G-collapses. The general case follows immediately from this one.

To prove the second part of the theorem we can suppose that K elementary G-collapses to L. Let $\sigma \subsetneq \sigma'$ be a G-collapsible pair in K such that $L = K \smallsetminus \{g\sigma, g\sigma'\}_{g\in G}$. Then,

$\sigma \in \mathcal{X}(K)$ is an up beat point and therefore $\mathcal{X}(K) \searrow^{Ge} \mathcal{X}(K) \smallsetminus \{g\sigma\}_{g\in G}$. Now, $\sigma' \in \mathcal{X}(K) \smallsetminus \{gS\}_{g\in G}$ is a down weak point since $\overline{\sigma'} \smallsetminus \{\sigma, \sigma'\}$ is a simplicial cone and then $\hat{U}_{\sigma'}^{\mathcal{X}(K)\smallsetminus\{g\sigma\}_{g\in G}} = \hat{U}_{\sigma'}^{\mathcal{X}(K)\smallsetminus\{\sigma\}} = \mathcal{X}(\overline{\sigma'} \smallsetminus \{\sigma, \sigma'\})$ is contractible by Lemma 4.2.5. Therefore, $\mathcal{X}(K) \smallsetminus \{g\sigma\}_{g\in G} \searrow^{Ge} \mathcal{X}(K) \smallsetminus \{g\sigma, g\sigma'\}_{g\in G} = \mathcal{X}(L)$ and $\mathcal{X}(K) \searrow^{G} \mathcal{X}(L)$. □

Corollary 8.2.13. *G has a nontrivial normal p-subgroup if and only if $\mathcal{K}(S_p(G))$ is G-collapsible.*

Proof. If G has nontrivial normal p-subgroup, $S_p(G)$ is G-collapsible by Proposition 8.2.9 and then $\mathcal{K}(S_p(G))$ is G-collapsible by Theorem 8.2.12. Conversely, if $\mathcal{K}(S_p(G))$ is G-collapsible, there is a vertex of $\mathcal{K}(S_p(G))$ fixed by G, i.e. a nontrivial p-subgroup of G, which is fixed by the interior automorphisms of G. □

Therefore Quillen's conjecture is equivalent to the following statement: $|\mathcal{K}(S_p(G))|$ is contractible if and only if $\mathcal{K}(S_p(G))$ is G-collapsible.

The equivalence classes of the equivalence relations $\diagup\!\!\searrow^{G}$ generated by the G-collapses are called *equivariant simple homotopy types* in the setting of finite spaces and of simplicial complexes. An easy modification of Proposition 4.2.9 shows that if X is a finite T_0-G-space, X and X' are equivariantly simple homotopy equivalent (see Proposition 8.2.24). Therefore, we have the following Corollary of Theorem 8.2.12.

Corollary 8.2.14. *Let X and Y be finite T_0-G-spaces. Then X and Y have the same equivariant simple homotopy type if and only if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same equivariant simple homotopy type.*

However, the analogous result for the functor $\mathcal{X}$ is not true (see Example 8.2.22).

Remark 8.2.15. Let X be a finite G-space. Then $\overline{y} \leq \overline{x}$ in the quotient space X/G if and only if there exists $g \in G$ such that $y \leq gx$. In particular if X is T_0, so is X/G.

The quotient map $q : X \to X/G$ is open, moreover $q^{-1}(q(U_x)) = \bigcup\limits_{g\in G} gU_x = \bigcup\limits_{g\in G} U_{gx}$. Since $q(U_x) \ni \overline{x}$ is an open set, $U_{\overline{x}} \subseteq q(U_x)$. The other inclusion follows from the continuity of q. Therefore $U_{\overline{x}} = q(U_x)$. Now, $\overline{y} \leq \overline{x}$ if and only if $y \in q^{-1}(U_{\overline{x}}) = \bigcup\limits_{g\in G} U_{gx}$ if and only if there exists $g \in G$ with $y \leq gx$.

Suppose X is T_0, $\overline{y} \leq \overline{x}$ and $\overline{x} \leq \overline{y}$. Then there exist $g, h \in G$ such that $y \leq gx$ and $x \leq hy$. Hence, $y \leq gx \leq ghy$. By Proposition 8.1.4, $y = gx = ghy$ and then $\overline{y} = \overline{x}$.

Proposition 8.2.16. *Let X be a finite T_0-G-space which strongly G-collapses to an invariant subspace Y. Then X/G strongly collapses to Y/G and X^G strongly collapses to Y^G. In particular, if X is contractible, so are X/G and X^G.*

Proof. We can assume there is an elementary strong G-collapse from X to $Y = X \smallsetminus Gx$ where $x \in X$ is a beat point. Suppose $x \in X$ is a down beat point and let $y \prec x$. Then $\overline{y} < \overline{x}$ in X/G and if $\overline{z} < \overline{x}$ there exists g such that $gz < x$. Therefore $gz \leq y$ and $\overline{z} \leq \overline{y}$. This proves that $\overline{x} \in X/G$ is a down beat point and X/G strongly collapses to $X/G \smallsetminus \{\overline{x}\} = Y/G$.

If x is not fixed by G, $Y^G = X^G$. If $x \in X^G$, and $g \in G$, then $gy < gx = x$ and therefore $gy \leq y$. Thus, $gy = y$. This proves that y is also fixed by G and then x is a down beat point of X^G. In particular, $X^G \searrow\!\!\searrow Y^G$.

If in addition X is contractible, X strongly G-collapses to a G-core which is a point and then X/G and X^G are contractible. □

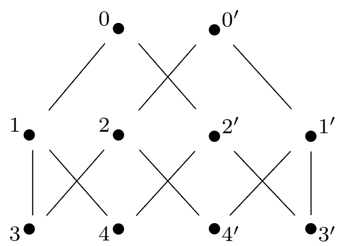
In fact, the first part of the previous result holds for general spaces. If X is a G-topological space and $Y \subseteq X$ is an equivariant strong deformation retract, Y/G is a strong deformation retract of X/G and so is $Y^G \subseteq X^G$. However if X is a G-topological space which is contractible, X^G need not be contractible. R. Oliver [30] proved that there are groups which act on disks without fixed points.

Proposition 8.2.17. *Let X be a finite T_0-G-space which G-collapses to Y. Then X^G collapses to Y^G. In particular, if X is G-collapsible, X^G is collapsible.*

Proof. Suppose $X \searrow^{Ge} Y = X \smallsetminus Gx$. If $x \notin X^G$, $Y^G = X^G$. If $x \in X^G$, $\hat{C}_x^X$ is G-invariant and contractible. By Proposition 8.2.16, $\hat{C}_x^{X^G} = (\hat{C}_x^X)^G$ is contractible and then x is a weak point of X^G, which means that $X^G \searrow Y^G$. □

The analogous for quotients is not true. There exist finite T_0-G-spaces such that $X \searrow^G Y$ but X/G does not collapse to Y/G, as the next example shows.

Example 8.2.18. Let X be the following $\mathbb{Z}_2$-space



where $\mathbb{Z}_2$ acts by symmetry, $1 \cdot i = i'$ for every $0 \leq i \leq 4$. Since $0 \in X$ is a weak point, X $\mathbb{Z}_2$-collapses to $Y = X \smallsetminus \{0, 0'\}$. However $X/\mathbb{Z}_2$ does not collapse to $Y/\mathbb{Z}_2$. Moreover, $X/\mathbb{Z}_2$ is contractible while $Y/\mathbb{Z}_2$ is the minimal finite model of the circle.

From Proposition 8.2.17 one easily deduces the next

Corollary 8.2.19. *Let X and Y be equivariantly simple homotopy equivalent finite T_0-G-spaces. Then X^G and Y^G have the same simple homotopy type.*

There is an analogous result of Proposition 8.2.17 for complexes.

Proposition 8.2.20. *Let K be a finite G-simplicial complex which G-collapses to a subcomplex L. Then K^G collapses to L^G. In particular, if K is G-collapsible, K^G is collapsible.*

Proof. Suppose that $K \searrow^{Ge} L = K \smallsetminus \bigcup_{g \in G} \{g\sigma, g\sigma'\}$, where $\sigma \subsetneq \sigma'$ is a G-collapsible pair. If $\sigma \notin K^G$, $L^G = K^G$. If $\sigma \in K^G$, then $\sigma' \in K^G$, because σ is a free face of σ'. Then $L = K \smallsetminus \{\sigma, \sigma'\}$ and $L^G = K^G \smallsetminus \{\sigma, \sigma'\}$. Since $\sigma \subsetneq \sigma'$ is a collapsible pair in K^G, $K^G \searrow L^G$. □

Corollary 8.2.21. *If K and L are two finite G-simplicial complexes with the same equivariant simple homotopy type, K^G and L^G have the same simple homotopy type. In particular K has a vertex which is fixed by the action of G if and only if L has a vertex fixed by G.*

Example 8.2.22. Let K be a 1-simplex with the unique nontrivial action of $\mathbb{Z}_2$. The barycentric subdivision K' has a vertex fixed by $\mathbb{Z}_2$ but $K^{\mathbb{Z}_2} = \emptyset$ therefore K and K' do not have the same equivariant simple homotopy type. On the other hand, $\mathcal{X}(K)$ and $\mathcal{X}(K')$ are contractible, and therefore they have the same equivariant simple homotopy type.

Corollary 8.2.23. *A finite group G has a nontrivial normal p-subgroup if and only if $\mathcal{K}(S_p(G))$ is equivariantly simple homotopy equivalent to a point.*

Proof. If G has a nontrivial normal p-subgroup, $\mathcal{K}(S_p(G))$ is G-collapsible by Corollary 8.2.13. If $\mathcal{K}(S_p(G))$ has trivial equivariant simple homotopy type, there is a vertex of $\mathcal{K}(S_p(G))$ fixed by the action of G, i.e. a normal p-subgroup of G. □

Now we turn our attention to the simpler poset $A_p(G)$.

Proposition 8.2.24. *Let $f : X \to Y$ be a G-map between finite T_0-G-spaces which is distinguished. Then X and Y have the same equivariant simple homotopy type.*

Proof. The non-Hausdorff mapping cylinder $B(f)$ is a G-space with the action induced by X and Y since if $x < y$, then $f(x) \leq y$ and therefore $f(gx) = gf(x) \leq gy$ for every $g \in G$. Moreover, Y is a G-invariant strong deformation retract of $B(f)$ and then $B(f) \searrow\searrow^G Y$. On the other hand, $B(f) \searrow^G X$. This follows from the proof of Lemma 4.2.7. Notice that we can remove orbits of minimal points of Y in $B(f)$ and collapse all $B(f)$ into X. □

Corollary 8.2.25. *$A_p(G)$ and $S_p(G)$ have the same equivariant simple homotopy type.*

Proof. The proof of Proposition 8.1.10 shows that the inclusion $A_p(G) \hookrightarrow S_p(G)$ is a distinguished map. □

Corollary 8.2.26. *If G has a nontrivial normal p-subgroup then it has a nontrivial normal elementary abelian p-subgroup.*

Proof. There is a simple algebraic proof of this fact, but we show a shorter one, using the last result. Since $S_p(G) \diagup\!\!\searrow^G A_p(G)$, by, Corollary 8.2.19, $S_p(G)^G \diagup\!\!\searrow A_p(G)^G$. Therefore, if $S_p(G)^G \neq \emptyset$, $A_p(G)^G$ is also nonempty. □

If $S_p(G)$ is contractible, $\mathcal{K}(S_p(G))$ has the strong homotopy type of a point by Theorem 5.0.16. Conversely, if $\mathcal{K}(S_p(G))$ is strongly collapsible, $S_p(G)'$ is contractible by Theorem 5.0.16 and then there is a point of $S_p(G)'$ which is fixed by the action of G, i.e. a chain of nontrivial normal p-subgroups of G. We summarize the results on the poset $S_p(G)$ in the following

Theorem 8.2.27. *Let G be a finite group and p a prime integer. The following are equivalent*

1. *G has a nontrivial normal p-subgroup.*
2. *$S_p(G)$ is a contractible finite space.*
3. *$S_p(G)$ is G-collapsible.*
4. *$S_p(G)$ has the equivariant simple homotopy type of a point.*
5. *$\mathcal{K}(S_p(G))$ is G-collapsible.*
6. *$\mathcal{K}(S_p(G))$ has the equivariant simple homotopy type of a point.*
7. *$\mathcal{K}(S_p(G))$ has the strong homotopy type of a point.*
8. *$A_p(G)$ has the equivariant simple homotopy type of a point.*
9. *$\mathcal{K}(A_p(G))$ has the equivariant simple homotopy type of a point.*

As a consequence of these equivalences, we obtain nine different formulations of Quillen's conjecture.

To finish this Section we prove a result on groups which at first sight seems to have no connection with finite spaces.

Proposition 8.2.28. *Let G be a finite group and suppose there exists a proper subgroup $H \subsetneq G$ such that for every nontrivial subgroup S of G, $S \cap H$ is nontrivial. Then G is not a simple group.*

Proof. Since H is a proper subgroup of G, G is nontrivial and therefore $H = G \cap H$ is nontrivial. Consider the poset $S(G)$ of nontrivial proper subgroups of G. Let $c_H : S(G) \to S(G)$ be the constant map H and define $f : S(G) \to S(G)$ by $f(S) = S \cap H$. The map f is well defined by hypothesis and it is clearly continuous. Moreover, $1_{S(G)} \geq f \leq c_H$ and then $S(G)$ is contractible.

On the other hand, G acts on $S(G)$ by conjugation. Then, by Proposition 8.1.6, G has a nontrivial proper normal subgroup. □

Example 8.2.29. Let $\mathcal{Q} = \{1, -1, i, -i, j, -j, k, -k\}$ be the quaternion group, where $(-1)^2 = 1, (-1)i = i(-1) = -i, (-1)j = j(-1) = -j, (-1)k = k(-1) = -k, i^2 = j^2 = k^2 = ijk = -1$. Let $H = \{1, -1\}$. Then H is in the hypothesis of Proposition 8.2.28 since -1 is a power of every nontrivial element of $\mathcal{Q}$. Therefore, $\mathcal{Q}$ is not simple.

Remark 8.2.30. There are also purely algebraic proofs of Proposition 8.2.28. In fact is easy to see that in the hypothesis of above, $\bigcap\limits_{g \in G} gHg^{-1}$ is a nontrivial normal subgroup of G. However, our topological proof is also very simple.

8.3 Reduced lattices

Recall that a poset P is said to be a lattice if every two points a and b have a join (or supremum) $a \vee b$ (i.e. $F_a \cap F_b$ has a minimum) and a meet (or infimum) $a \wedge b$ (i.e. $U_a \cap U_b$ has maximum). If X is a finite lattice it has maximum and minimum, and therefore they are not very interesting from the topological point of view. In this Section we will study the spaces obtained by removing from a lattice its maximum and minimum.

Definition 8.3.1. A finite poset X is called a *reduced lattice* if $\hat{X} = D^0 \oplus X \oplus D^0$ is a lattice.

For example, if G is a finite group and p is a prime integer, $S_p(G)$ is a reduced lattice. The finite space $S(G)$ defined in the proof of Proposition 8.2.28 is also a reduced lattice. However, the minimal finite model of S^1 is not.

A subset A of a poset P is lower bounded if there exists $x \in P$ such that $x \leq a$ for every $a \in A$. In that case x is called a lower bound of A. If the set of lower bounds has a maximum x, we say that x is the infimum of A. The notions of upper bound and supremum are defined dually.

Proposition 8.3.2. *Let P be a finite poset. The following are equivalent:*

1. *P is a reduced lattice.*
2. *Every lower bounded set of P has an infimum and every upper bounded set has a supremum.*
3. *Every lower bounded set $\{x, y\}$ has infimum.*
4. *Every upper bounded set $\{x, y\}$ has supremum.*

Proof. Straightforward. □

For instance, the associated space of a simplicial complex is a reduced lattice. If K is a finite simplicial complex, and $\{\sigma, \sigma'\}$ is lower bounded in $\mathcal{X}(K)$, the simplex $\sigma \cap \sigma'$ is the infimum of $\{\sigma, \sigma'\}$. It can be proved that if X is a finite T_0-space, then there exists a finite simplicial complex K such that $\mathcal{X}(K) = X$ if and only if X is a reduced lattice and every element of X is the supremum of a unique set of minimal elements.

Proposition 8.3.3. *If X is a reduced lattice and $Y \subseteq X$ is a strong deformation retract, Y is also a reduced lattice. In particular, if X is a reduced lattice, so is its core.*

Proof. It suffices to consider the case that $Y = X \smallsetminus \{x\}$, where $x \in X$ is a down beat point. Let $y \prec x$ and let $A = \{a, b\}$ be an upper bounded subset of Y. Then A has a supremum z in X. If x is an upper bound of A in X, $a < x$ and $b < x$ and then $a \leq y$, $b \leq y$. Therefore $z \neq x$ and then z is the supremum of A in Y. By Proposition 8.3.2, Y is a reduced lattice. □

However the fact of being a reduced lattice is not a homotopy type invariant. It is easy to find contractible spaces which are not reduced lattices. Reduced lattices do not describe all homotopy types of finite spaces. For example, since $\mathbb{S}(S^0)$ is minimal and it is not a reduced lattice, no reduced lattice is homotopy equivalent to $\mathbb{S}(S^0)$. On the other hand every finite space X has the weak homotopy type of a reduced lattice, e.g. X'.

The following result is due to Stong.

Proposition 8.3.4 (Stong). *Let G be a finite group and let p be a prime integer. Let A be the set of nontrivial intersections of Sylow p-subgroups of G. Then A is G-invariant and it is an equivariant strong deformation retract of $S_p(G)$.*

Proof. It is clear that A is G-invariant. Define the retraction $r : S_p(G) \to A$, that assigns to each subgroup $H \subseteq G$, the intersection of all the Sylow p-subgroups containing H. Then r is a continuous map, and $ir \geq 1_{S_p(G)}$. By Proposition 8.2.1, A is an equivariant strong deformation retract of $S_p(G)$. □

Proposition 8.3.4 motivates the following definition.

Definition 8.3.5. Let X be a reduced lattice. Define the subspace $\mathfrak{i}(X) \subseteq X$ by $\mathfrak{i}(X) = \{\bigwedge A \mid A$ is a lower bounded subset of maximal elements of $X\}$. Analogously, define $\mathfrak{s}(X) = \{\bigvee A \mid A$ is an upper bounded subset of minimal elements of $X\}$. Here, $\bigwedge A$ denotes the infimum of A and $\bigvee A$ its supremum.

Following Stong's proof of Proposition 8.3.4, one can prove that the retraction $r : X \to \mathfrak{i}(X)$, which sends x to the infimum of the maximal elements of X that are greater than x, is continuous and that $\mathfrak{i}(X)$ is a strong deformation retract of X. Similarly, $\mathfrak{s}(X) \subseteq X$ is a strong deformation retract.

Example 8.3.6. Let $n \geq 2$ and let P_n be the poset of proper positive divisors of n with the order given by: $a \leq b$ if a divides b. If n is square free, P_n is homeomorphic to $\mathcal{X}(\dot{\sigma})$ where σ is a $(k-1)$-simplex, k being the number of primes dividing n. In fact, if $p_1, p_2, \ldots, p_k$ are the prime divisors of n, and $\sigma = \{p_1, p_2, \ldots, p_k\}$ is a simplex, the map $f : P_n \to \mathcal{X}(\dot{\sigma})$ defined by $f(d) = \{p_i \mid p_i$ divides $d\}$, is a homeomorphism. In particular, $|\mathcal{K}(P_n)| = |(\dot{\sigma})'|$ is homeomorphic to the $(k-2)$-dimensional sphere.

If n is not square free, we show that P_n is contractible. Note that P_n is a reduced lattice with the infimum induced by the greatest common divisor. Since n is not square free, the product of the prime divisors of n is a proper divisor of n and it is the maximum of $\mathfrak{s}(P_n)$. Thus, $\mathfrak{s}(P_n)$ is contractible and then, so is P_n.

Proposition 8.3.7. *Let X be a reduced lattice. The following are equivalent*

1. *X is a minimal finite space.*

2. *$\mathfrak{i}(X) = \mathfrak{s}(X) = X$.*

Proof. If X is minimal, the unique strong deformation retract of X is X itself. Therefore $\mathfrak{i}(X) = \mathfrak{s}(X) = X$. Conversely, suppose this equality holds and that $x \in X$ is a down beat point with $y \prec x$. Since $x \in X = \mathfrak{s}(X)$, x is the supremum of a set M of minimal elements of X. Since x is not minimal, every element of M is strictly smaller than x, and therefore y is an upper bound of M. This contradicts the fact that $x = \bigvee M$. Then X does not have down beat points and analogously it has no up beat point, so it is minimal. □

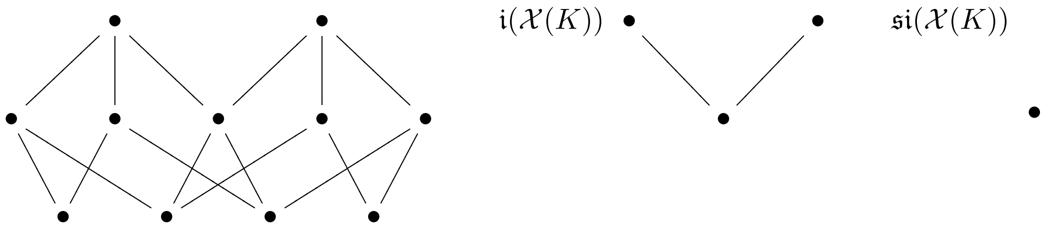
If X is a reduced lattice, $\mathfrak{i}(X)$ is a strong deformation retract of X, which is a reduced lattice by Proposition 8.3.3. Analgously $\mathfrak{s}(\mathfrak{i}(X))$ is a strong deformation retract of X and it is a reduced lattice. The sequence

$$X \supseteq \mathfrak{i}(X) \supseteq \mathfrak{si}(X) \supseteq \mathfrak{isi}(X) \supseteq \dots$$

is well defined and it stabilizes in a space Y which is a strong deformation retract of X and a minimal finite space by Proposition 8.3.7. Therefore, in order to obtain the core of a reduced lattice, one can carry out alternatively the constructions $\mathfrak{i}$ and $\mathfrak{s}$, starting from anyone.

Example 8.3.8. Let K be the simplicial complex which consists of two 2-simplices with a common 1-face. Since K is strong collapsible, so is $\mathcal{X}(K)$. Another way to see this is the following: $\mathcal{X}(K)$ is a reduced lattice with two maximal elements, $\mathfrak{i}(\mathcal{X}(K))$ has just three points, and $\mathfrak{si}(\mathcal{X}(K))$ is the singleton.

$\mathcal{X}(K)$ $\quad$ $\mathfrak{i}(\mathcal{X}(K))$ $\quad$ $\mathfrak{si}(\mathcal{X}(K))$



Although there are many reduced lattices which are minimal finite spaces, a reduced lattice X is a minimal finite model if and only if it is discrete. For if X is not discrete, there is a point $x \in X$ which is not minimal and we can apply Osaki's open reduction (Theorem 6.1.1) to obtain a smaller model X/U_x.

Let X be a finite T_0-space and Y a reduced lattice. If $f, g : X \to Y$ are two maps which coincide in the set $Max(X)$ of maximal elements of X, then $f \simeq g$. Define $h : X \to Y$ by $h(x) = \bigwedge\{f(x') \mid x' \in Max(X) \text{ and } x' \geq x\}$. Clearly h is continuous and $h \geq f$. Analogously $h \geq g$ and then $f \simeq g$.

If X and Y are two finite T_0-spaces and $f : X \to Y$ is a continuous map, there exists $g : X \to Y$ homotopic to f and such that $g(Max(X)) \subseteq Max(Y)$. Consider $g \in Max(F_f) \subseteq Y^X$. Suppose there exists $x \in Max(X)$ such that $g(x) \notin Max(Y)$. Then, there exists $y > g(x)$, and the map $\widetilde{g} : X \to Y$ which coincides with g in $X \smallsetminus \{x\}$ and such that $\widetilde{g}(x) = y$ is continuous and $\widetilde{g} > g$, which is a contradiction. Therefore $g(Max(X)) \subseteq Max(Y)$.

We deduce that if X is a finite T_0-space and Y is a reduced lattice, then $\#[X, Y] \leq (\#Max(Y))^{\#Max(X)}$. Here, $[X, Y]$ denotes the set of homotopy classes of maps $X \to Y$.

8.4 Fixed points, Lefschetz number and the $f^\infty(X)$

In the previous Sections of this Chapter we studied fixed point sets of actions over finite spaces. Now we turn our attention to fixed point sets of continuous maps between finite spaces and their relationship with the fixed point sets of the associated simplicial maps.

We will recall first some basic facts about Lefschetz Theorems and the fixed point theory for finite posets. Some references for this are [4, 29]. Then we will prove a stronger version of the Lefschetz Theorem for simplicial automorphisms.

We also introduce the construction $f^{\infty}(X)$ of a map $f : X \to X$ which has applications to the study of weak homotopy equivalences between minimal finite models.

If X is a topological space and $f : X \to X$ is a continuous map, we denote by $X^f = \{x \in X \mid f(x) = x\}$ the set of fixed points of f. For a simplicial map $\varphi : K \to K$, K^{φ} denotes the full subcomplex spanned by the vertices fixed by φ.

Let M be a finitely generated $\mathbb{Z}$-module, and $T(M)$ its torsion submodule. An endomorphism $\varphi : M \to M$ induces a morphism $\overline{\varphi} : M/T(M) \to M/T(M)$ between finite-rank free $\mathbb{Z}$-modules. The *trace* $tr(\varphi)$ of φ is the trace of $\overline{\varphi}$. If K is a compact polyhedron, $H_n(K)$ is finitely generated for every $n \geq 0$. If $f : K \to K$ is a continuous map, the *Lefschetz number* of f is defined by

$$\lambda(f) = \sum_{n\geq 0}(-1)^n tr(f_n), \tag{8.2}$$

where $f_n : H_n(K) \to H_n(K)$ are the induced morphisms in homology.

Notice that the Lefschetz number of the identity $1_K : K \to K$ coincides with the Euler characteristic of K.

The Lefschetz Theorem states the following

Theorem 8.4.1. *Let K be a compact polyhedron and let $f : K \to K$ be a continuous map. Then, if $\lambda(f) \neq 0$, f has a fixed point.*

In particular, if K is contractible, $\lambda(f) = 1$ for every map $f : K \to K$ and then f has a fixed point. This generalizes the well-known Theorem of Brouwer for discs.

If X is a finite T_0-space, its homology is finitely generated as well and therefore we can define the Lefschetz number $\lambda(f)$ of a map $f : X \to X$ as in 8.2. Note that $\lambda(f) = \lambda(|\mathcal{K}(f)|)$ by Remark 1.4.7.

The Lefschetz Theorem version for finite spaces is the following

Theorem 8.4.2. *Let X be a finite T_0-space and $f : X \to X$ a continuous map. Then $\lambda(f) = \chi(X^f)$. In particular, if $\lambda(f) \neq 0$, $X^f \neq \emptyset$.*

For details we refer the reader to [4, 29].

Proposition 8.4.3. *Let $\varphi : K \to K$ be a simplicial automorphism. Then $|K|^{|\varphi|} = |(K')^{\varphi'}|$.*

Proof. Let $x \in |K'| = |K|$, $x = \sum \alpha_i b(S_i)$ is a convex combination of the barycenters of the simplices $S_0 \subsetneq S_1 \subsetneq \ldots \subsetneq S_k$ of K ($\alpha_i > 0$ for every i). Suppose $x \in |(K')^{\varphi'}|$. Then $b(S_i)$ is fixed by φ' for every i, or equivalently $\varphi(S_i) = S_i$. If we see $x \in |K|$, $x = \sum \alpha_i \sum\limits_{v\in S_i} \frac{v}{\#S_i}$ and $|\varphi|(x) = \sum \alpha_i \sum\limits_{v\in S_i} \frac{\varphi(v)}{\#S_i}$. Since $\varphi(S_i) = S_i$, $\sum\limits_{v\in S_i} \frac{\varphi(v)}{\#S_i} = \sum\limits_{v\in S_i} \frac{v}{\#S_i}$, and then $|\varphi|(x) = x$. This proves one inclusion.

Conversely, suppose $x \in |K|^{|\varphi|}$. Then $\sum \alpha_i \sum\limits_{v\in S_i} \frac{v}{\#S_i} = x = |\varphi|(x) = \sum \alpha_i \sum\limits_{v\in S_i} \frac{\varphi(v)}{\#S_i}$. Let $v \in S_i \smallsetminus S_{i-1}$. Then, the coordinate of v in x is $\alpha_i + \alpha_{i+1} + \ldots + \alpha_k$. Since φ is an isomorphism, the coordinate of $\varphi(v)$ in $|\varphi|(x)$ is also $\alpha_i+\alpha_{i+1}+\ldots+\alpha_k$. If $\varphi(v) \in S_j \smallsetminus S_{j-1}$, the coordinate of $\varphi(v)$ in x is $\alpha_j+\alpha_{j+1}+\ldots+\alpha_k$. Since $|\varphi(x)| = x$, $\alpha_i+\alpha_{i+1}+\ldots+\alpha_k = \alpha_j+\alpha_{j+1}+\ldots+\alpha_k$ and then $i = j$. In particular $\varphi(S_i) \subseteq S_i$ and then $\varphi(S_i) = S_i$ for every i. Therefore $x \in |(K')^{\varphi'}|$, which proves the other inclusion.

□

Since $X^f \subseteq X$, $\mathcal{K}(X^f)$ is the full subcomplex of $\mathcal{K}(X)$ spanned by the vertices fixed by f. By definition, this subcomplex is $\mathcal{K}(X)^{\mathcal{K}(f)}$. Therefore we have,

Remark 8.4.4. Let X be a finite T_0-space and let $f : X \to X$ be a continuous map. Then $\mathcal{K}(X^f) = \mathcal{K}(X)^{\mathcal{K}(f)}$.

Corollary 8.4.5. *Let K be a finite simplicial complex and $\varphi : K \to K$ a simplicial automorphism. Then $\mathcal{X}(K)^{\mathcal{X}(\varphi)}$ is a finite model of $|K|^{|\varphi|}$.*

Proof. By 8.4.3, $|K|^{|\varphi|} = |(K')^{\varphi'}| = |\mathcal{K}(\mathcal{X}(K))^{\mathcal{K}(\mathcal{X}(\varphi))}|$ and by 8.4.4, this coincides with $|\mathcal{K}(\mathcal{X}(K)^{\mathcal{X}(\varphi)})|$ which is weak homotopy equivalent to $\mathcal{X}(K)^{\mathcal{X}(\varphi)}$. □

The following is a stronger version of Lefschetz Theorem 8.4.1 for simplicial automorphisms.

Corollary 8.4.6. *Let K be a finite simplicial complex and let $\varphi : K \to K$ be a simplicial automorphism. Then $\chi(|K|^{|\varphi|}) = \lambda(|\varphi|)$.*

Proof. The diagram

$$\begin{array}{ccc} |K| & \xrightarrow{|\varphi|} & |K| \\ \downarrow{\scriptstyle \mu_K} & & \downarrow{\scriptstyle \mu_K} \\ \mathcal{X}(K) & \xrightarrow{\mathcal{X}(\varphi)} & \mathcal{X}(K) \end{array}$$

commutes up to homotopy and $(\mu_{K*})_n : H_n(|K|) \to H_n(\mathcal{X}(K))$ is an isomorphism for every $n \geq 0$. Then $|\varphi|_* = (\mu_{K*})^{-1}\mathcal{X}(\varphi)_*\mu_{K*} : H_n(|K|) \to H_n(|K|)$ and $tr((|\varphi|_*)_n) = tr((\mathcal{X}(\varphi)_*)_n)$. Therefore $\lambda(|\varphi|) = \lambda(\mathcal{X}(\varphi))$. By 8.4.5 and the finite space version of the Lefschetz Theorem, $\chi(|K|^{|\varphi|}) = \chi(\mathcal{X}(K)^{\mathcal{X}(\varphi)}) = \lambda(\mathcal{X}(\varphi)) = \lambda(|\varphi|)$. □

In [30], R. Oliver proves the following result using "standard theorems from the homological theory of $\mathbb{Z}_p$ actions". Here we exhibit a completely different proof using the results of above.

Proposition 8.4.7. *(Oliver) Assume that $\mathbb{Z}_n$ acts on a $\mathbb{Q}$-acyclic finite simplicial complex K. Then $\chi(|K|^{\mathbb{Z}_n}) = 1$.*

Proof. Let g be a generator of $\mathbb{Z}_n$ and $\varphi : K \to K$ the multiplication by g. Then $\chi(|K|^{\mathbb{Z}_n}) = \chi(|K|^{|\varphi|}) = \lambda(|\varphi|) = 1$, since K is $\mathbb{Q}$-acyclic. □

Suppose that X is a finite model of the circle and that $f : X \to X$ is a map. Then $f_* : H_1(X) \to H_1(X)$ is a map $\mathbb{Z} \to \mathbb{Z}$. However, the only possible morphisms that can appear in this way are $0, 1_{\mathbb{Z}}$ and $-1_{\mathbb{Z}}$. We prove this and a more general fact in the following result.

Proposition 8.4.8. *Let $f : X \to X$ be an endomorphism of a finite T_0-space X and let $n \geq 0$. Let $f_* : H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q})$ be the induced map in homology. If $dim_{\mathbb{Q}} H_n(X;\mathbb{Q}) = r$, f_* is a complex matrix of order r well defined up to similarity. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of f_*. Then $\lambda = 0$ or λ is a root of unity.*

Proof. Since X is finite, there exist $s \neq t \in \mathbb{N}$ such that $f^s = f^t$. Then $f_*^s = f_*^t$ and $\lambda^s = \lambda^t$. □

Corollary 8.4.9. *In the hipothesis of the previous proposition, $-r \leq tr(f_*) \leq r$. In particular, since f_* has integer entries, $tr(f_*) \in \{-r, -r+1, \dots, r-1, r\}$.*

We say that a topological space X has the *fixed point property* if any map $f : X \to X$ has a fixed point.

For instance, compact polyhedra or finite spaces with trivial reduced homology have the fixed point property by the Lefschetz Theorems, but there are spaces with the fixed point property without trivial homology (see Example 8.4.12).

The following is a well-known result:

Proposition 8.4.10. *Let X be a finite T_0-space, and let $f, g : X \to X$ be two homotopic maps. Then, f has a fixed point if and only if g has a fixed point.*

Proof. Without loss of generality, we can assume that $g \leq f$. If $f(x) = x$, $g(x) \leq f(x) = x$. Then $g^{i+1}(x) \leq g^i(x)$ for every $i \geq 0$ and then there exists i such that $g^{i+1}(x) = g^i(x)$. Therefore, $g^i(x)$ is a fixed point of g. □

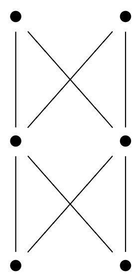
Corollary 8.4.11. *The fixed point property is a homotopy type invariant of finite T_0-spaces.*

Proof. Let X, Y be homotopy equivalent finite T_0-spaces, X with the fixed point property. Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse g. Let $h : Y \to Y$ be a continuous map. We show that h has a fixed point. The map $ghf : X \to X$ fixes some point $x \in X$. Then $f(x) \in Y$ is a fixed point of $fgh : Y \to Y$. Since $h \simeq fgh$, h has a fixed point. □

A different proof of this result appears for example in [40], Corollary 3.16.

Example 8.4.12. (Baclawski and Björner) The fixed point property is not a weak homotopy invariant, nor a simple homotopy invariant. In [4] Example 2.4, Baclawski and Björner exhibit the regular CW-complex K which is the border of a piramid with square base. Therefore $X = \mathcal{X}(K)$ is a finite model of S^2. Let $f : X \to X$ be a continuous map. If f is onto, it is an automorphism and then the vertex of the top of the piramid is fixed by f since it is the unique point covered by 4 points. If f is not onto, $\mathcal{K}(f) : S^2 \to S^2$ is not onto and then $\mathcal{K}(f)$ is nullhomotopic. Therefore $\lambda(f) = \lambda(|\mathcal{K}(f)|) = 1$ and then $X^f \neq \emptyset$.

On the other hand, the minimal finite model of S^2



is simple homotopy equivalent to X and does not have the fixed point property since the simetry is fixed point free.

Now we introduce the construction $f^\infty(X)$.

Definition 8.4.13. Let X be a finite T_0-space and $f : X \to X$ a continuous map. We define $f^\infty(X) = \bigcap\limits_{i\geq 1} f^i(X) \subseteq X$.

Remark 8.4.14. Given $f : X \to X$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $f^n(X) = f^\infty(X)$. Let $k \in \mathbb{N}$ be the order of $f|_{f^\infty(X)}$ in the finite group $Aut(f^\infty(X))$. If $n \geq n_0$ and k divides n, $f^n(X) = f^\infty(X)$ and $f^n|_{f^\infty(X)} = 1_{f^\infty(X)}$. In this case we will say that $n \in \mathbb{N}$ is *suitable* for f.

Remark 8.4.15. $f^\infty(X) = \{x \in X \mid \exists\ n \in \mathbb{N} \text{ such that } f^n(x) = x\}$.

Proposition 8.4.16. *Let X be a finite T_0-space and let $f, g : X \to X$ be two homotopic maps. Then $f^\infty(X)$ is homotopy equivalent to $g^\infty(X)$.*

Proof. We can assume that $g \leq f$. By Remark 8.4.14, there exists $n \in \mathbb{N}$ which is suitable for f and g simultaneously. Then one can consider $f^n|_{g^\infty(X)} : g^\infty(X) \to f^\infty(X)$ and $g^n|_{f^\infty(X)} : f^\infty(X) \to g^\infty(X)$. Since

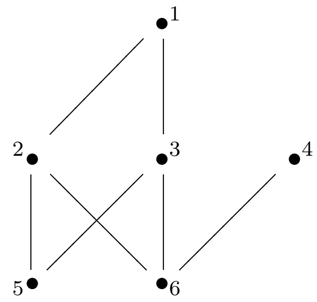
$$f^n|_{g^\infty(X)} g^n|_{f^\infty(X)} \leq f^{2n}|_{f^\infty(X)} = 1_{f^\infty(X)},$$

$f^n|_{g^\infty(X)} g^n|_{f^\infty(X)} \simeq 1_{f^\infty(X)}$. Analogously, $g^n|_{f^\infty(X)} f^n|_{g^\infty(X)} \simeq 1_{g^\infty(X)}$. □

Proposition 8.4.17. *Let X be a finite T_0-space and let $Y \subseteq X$ be a subspace. Then there exists a continuous map $f : X \to X$ such that $f^\infty(X) = Y$ if and only if Y is a retract of X.*

Proof. If $Y = f^\infty(X)$ for some f, choose $n \in \mathbb{N}$ suitable for f. Then $f^n : X \to Y$ is a retraction. Conversely, if $r : X \to Y$ is a retraction, $r^\infty(X) = Y$. □

Example 8.4.18. Let X be the following finite T_0-space



Define $f : X \to X$ such that 5 and 6 are fixed, $f(1) = f(2) = f(3) = 2$, $f(4) = 3$. Since X is contractible and $f(X)$ is a finite model of S^1, $f(X)$ is not a retract of X. However, $f^{\infty}(X) = \{2, 5, 6\}$ is a retract of X.

Remark 8.4.19. X has the fixed point property if and only if all its retracts have the fixed point property with respect to automorphisms. The first implication holds in general: if X is a topological space with the fixed point property, every retract of X also has that property. Conversely, if $f : X \to X$ is a continuous map, $f^{\infty}(X)$ is a retract of X and $f|_{f^{\infty}(X)} : f^{\infty}(X) \to f^{\infty}(X)$ is an automorphism. Then $f|_{f^{\infty}(X)}$ has a fixed point and therefore f.

Stong proves that a homotopy equivalence between minimal finite spaces is a homeomorphism (Corollary 1.3.7). We prove an analogue for weak homotopy equivalences and minimal finite models.

Proposition 8.4.20. *Let X be a finite T_0-space and let $f : X \to X$ be a weak homotopy equivalence. Then the inclusion $i : f^{\infty}(X) \hookrightarrow X$ is a weak homotopy equivalence. In particular, if X is a minimal finite model, f is a homeomorphism.*

Proof. Let $n \in \mathbb{N}$ be suitable for f. Then $f^n : X \to f^{\infty}(X)$, and the compositions $f^n i = 1_{f^{\infty}(X)}$, $if^n = f^n : X \to X$ are weak homotopy equivalences. Therefore i is a homotopy equivalence.

If X is a minimal finite model, $f^{\infty}(X) \subseteq X$ cannot have less points than X, then $f^{\infty}(X) = X$ and $f : X \to X$ is onto. Therefore, it is a homeomorphism. □

Observe that with the same proof of the last proposition, one can prove that if $f : X \to X$ is a homotopy equivalence, then $i : f^{\infty}(X) \hookrightarrow X$ is a homotopy equivalence. In particular, if X is a representative of minimum cardinality of its homotopy type (ie, a minimal finite space), f is a homeomorphism. This was already proved by Stong using beat points, but this is a different proof which does not use this concept.

Corollary 8.4.21. *Let X and Y be minimal finite models. Suppose there exist weak homotopy equivalences $f : X \to Y$ and $g : Y \to X$. Then f and g are homeomorphisms.*

Proof. The composition $gf : X \to X$ is a weak homotopy equivalence and then a homeomorphism by Proposition 8.4.20. Analogously gf is a homeomorphism. Then the result follows. □

Remark 8.4.22. In 1.4.16 we proved that there is no weak homotopy equivalence $\mathbb{S}(D_3) \to \mathbb{S}(D_3)^{op}$. We give here an alternative approach using the previous result and our description of the minimal finite models of graphs.

Suppose there exists a weak homotopy equivalence $f : \mathbb{S}(D_3) \to \mathbb{S}(D_3)^{op}$. Then f^{op} is also a weak homotopy equivalence. Since $\mathbb{S}(D_3)$ is a minimal finite model (see Section 3.2), so is $\mathbb{S}(D_3)^{op}$. By Corollary 8.4.21, $\mathbb{S}(D_3)$ is homeomorphic to its opposite, which is clearly absurd.

Proposition 8.4.23. *Let X be a finite T_0-space and $f, g : X \to X$ two maps. Then $(gf)^\infty(X)$ and $(fg)^\infty(X)$ are homeomorphic.*

Proof. Let $x \in (gf)^\infty(X)$, then there exists $n \in \mathbb{N}$ such that $(gf)^n(x) = x$. Therefore $(fg)^n(f(x)) = f(x)$, and $f(x) \in (fg)^\infty(X)$. Then $f|_{(gf)^\infty(X)} : (gf)^\infty(X) \to (fg)^\infty(X)$. Analogously $g|_{(fg)^\infty(X)} : (fg)^\infty(X) \to (gf)^\infty(X)$. The compositions of these two maps are the identities, and therefore, they are homeomorphisms. □

Remark 8.4.24. Let X be a finite T_0-space, and $f : X \to X$ a map. Then $(f')^\infty(X') = f^\infty(X)'$. A chain $x_1 < x_2 < \ldots < x_k$ is in $(f')^\infty(X')$ if and only if there exists n such that $(f')^n(\{x_1, x_2, \ldots, x_k\}) = \{x_1, x_2, \ldots, x_k\}$. This is equivalent to say that there exists n such that $f^n(x_i) = x_i$ for every $1 \leq i \leq k$ or in other words, that $\{x_1, x_2, \ldots, x_k\} \subseteq f^\infty(X)$.

To finish this Chapter, we introduce a nice generalization of the construction of $f^\infty(X)$ for the case of composable maps not necessarily equal nor with the same domain or codomain.

Suppose $X_0 \overset{f_0}{\to} X_1 \overset{f_1}{\to} \ldots$ is a sequence of maps between finite spaces. Define $Y_n = f_{n-1}f_{n-2}\ldots f_0(X_0) \subseteq X_n$ the image of the composition of the first n maps of the sequence.

Proposition 8.4.25. *There exist $n_0 \in \mathbb{N}$ such that Y_n is homeomorphic to Y_{n_0} for every $n \geq n_0$.*

Proof. Since $(\#Y_n)_{n\in\mathbb{N}}$ is a decreasing sequence, there exists $n_1 \in \mathbb{N}$ such that $\#Y_n$ is constant for $n \geq n_1$. Therefore $f_n : Y_n \to Y_{n+1}$ is a bijection for $n \geq n_1$.

Let $C_n = \{(x, x') \in Y_n \times Y_n \mid x \leq x'\}$. The map $f_n : Y_n \to Y_{n+1}$ induces a one-to-one function $F_n : C_n \to C_{n+1}$, $F_n(x, x') = (f_n(x), f_n(x'))$ for $n \geq n_1$. Therefore $(\#C_n)_{n\geq n_1}$ is increasing and bounded by $(\#Y_{n_1})^2$. Hence, there exists $n_0 \geq n_1$ such that F_n is a bijection and then $f_n : Y_n \to Y_{n+1}$ a homeomorphism for $n \geq n_0$. □

The space Y_{n_0} constructed above is well defined up to homeomorphism and it is denoted by $(f_n)_{n\in\mathbb{N}}^\infty(X_0)$. We show that in the case that all the spaces X_n are equal, i.e. $X_n = X$ for every $n \geq 0$, $(f_n)_{n\in\mathbb{N}}^\infty(X)$ is a retract of X, as in the original case. Since X is finite, there exists a subspace $Y \subseteq X$ and an increasing sequence $(n_i)_{i\in\mathbb{N}}$ of positive integers such that $Y_{n_i} = Y$ for every $i \in \mathbb{N}$. Let $g_i = f_{n_i-1}f_{n_i-2}\ldots f_{n_1}|_{Y_{n_1}} : Y_{n_1} \to Y_{n_i}$. These maps are permutations of the finite set Y, therefore there are two equal, say $g_i = g_j$ with $i < j$. Then $f_{n_j-1}f_{n_j-2}\ldots f_{n_i}|_{Y_{n_i}} = 1_Y$, so Y is a retract of X.

Chapter 9

The Andrews-Curtis Conjecture

One of the most important mathematical problems of all times is the Poincaré conjecture which states that every compact simply connected 3-manifold without boundary is homeomorphic to S^3. Versions of the conjecture for greater dimensions were proved by Smale, Stallings, Zeeman and Freedman. The problem was open for a century until G. Perelman finally proved it some years ago. Perelman proof uses hard differential geometry results. An alternative combinatorial proof of the Poincaré conjecture would be a great achievement.

In [45], E. Zeeman shows that the Dunce hat D is not collapsible but $D \times I$ is a collapsible polyhedron. Zeeman conjectures that the same holds for any contractible 2-complex:

Conjecture 9.0.1 (Zeeman). If K is contractible polyhedron, then $K \times I$ is collapsible.

Zeeman proves in [45] that his conjecture implies the Poincaré conjecture. Conjecture 9.0.1 is still not proved nor disproved.

Let $n \geq 1$. We say that a complex K n-deforms to another complex L if we can obtain L from K by a sequence of collapses and expansions in such a way that all the complexes involved in the deformation have dimension less than or equal to n.

The geometric Andrews-Curtis conjecture is weaker than Zeeman's, but is also open.

Conjecture 9.0.2 (Andrews-Curtis). Any contractible 2-polyhedron 3-deforms to a point.

The analogous version for greater dimensions is known to be true. More specifically,

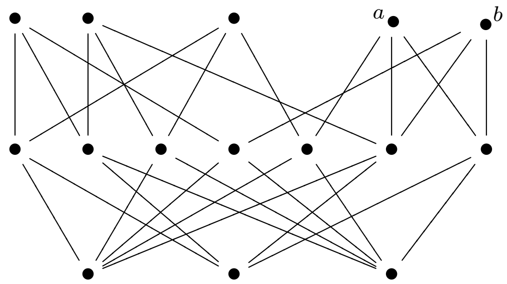
Theorem 9.0.3 (Whitehead-Wall)**.** *Let $n \geq 3$. If K and L are polyhedra of dimension less than or equal to n, then K $(n+1)$-deforms to L.*

The Geometric Andrews-Curtis conjecture is equivalent to the so called Andrews-Curtis conjecture which states that any *balanced presentation* of the trivial group can be transformed into the trivial presentation by a sequence of *Nielsen transformations* (see [2, 34] for further infomation).

In this Chapter we will define a large class of simplicial complexes called *quasi constructible complexes* which are built recursively by attaching smaller quasi constructible complexes. Using techniques of finite spaces we will prove that contractible quasi constructible complexes satisfy the Andrews-Curtis conjecture. Quasi constructible complexes generalize the notion of *constructible complexes* which was deeply studied by M. Hachimori in [19].

9.1 Quasi-constructible complexes

The content of this Section is partially motivated by the following example studied in Chapter 7



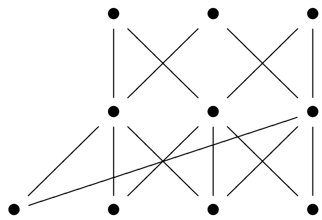
This space is the face poset of an h-regular structure of the Dunce hat and it has no weak points. However, there are two maximal points a, b such that $U_a \cup U_b$ is contractible, and therefore $X \overset{e}{\nearrow} Y = X \cup \{c\}$ where $a < c > b$. Now, $Y \searrow Y \smallsetminus \{a, b\}$. Thus $\mathcal{K}(X)$ 3-deforms to $\mathcal{K}(Y \smallsetminus \{a, b\})$ which has one point less that X.

For which complexes K it is possible to choose two maximal elements a, b such that $U_a \cup U_b$ is contractible? and when is it possible to perform repeatedly those moves to obtain a space with maximum and therefore collapsible?

Let X be a finite T_0-space of height at most 2 and let a, b be two maximal elements of X such that $U_a \cup U_b$ is contractible. Then we say that there is a *qc-reduction* from X to $Y \smallsetminus \{a, b\}$ where $Y = X \cup \{c\}$ with $a < c > b$. We say that X is *qc-reducible* if we can obtain a space with a maximum by performing qc-reductions starting from X.

Note that a, b and c are all weak points of Y. Since spaces with maximum are collapsible, qc-reducible finite spaces are simple homotopy equivalent to a point. Furthermore, if X is qc-reducible, all the spaces involved in the transfromation $X \nearrow\searrow *$ are of height less than or equal to 3. Therefore if X is qc-reducible, $\mathcal{K}(X)$ 3-deforms to a point.

Example 9.1.1. The following space is collapsible but not qc-reducible. In fact we can not perform any qc-reduction starting from X.



Proposition 9.1.2. *Let X be a finite T_0-space of height at most 2 and such that $H_2(X) = 0$. Let a, b be two maximal elements of X. Then the following are equivalent:*

1. $U_a \cup U_b$ *is contractible.*

2. $U_a \cap U_b$ *is nonempty and connected.*

3. $U_a \cap U_b$ *is contractible.*

Proof. The non-Hausdorff suspension $\mathbb{S}(U_a \cap U_b) = (U_a \cap U_b) \cup \{a, b\}$ is a strong deformation retract of $U_a \cup U_b$. A retraction is given by $r : U_a \cup U_b \to \mathbb{S}(U_a \cap U_b)$ with $r(x) = a$ for every $x \in U_a \smallsetminus U_b$ and $r(x) = b$ for $x \in U_b \smallsetminus U_a$. Therefore, by 2.7.3, $U_a \cup U_b$ is contractible if and only if $U_a \cap U_b$ is contractible.

Since $\mathcal{K}(X)$ has dimension at most 2, $H_3(\mathcal{K}(X), \mathcal{K}(\mathbb{S}(U_a \cap U_b))) = 0$. By the long exact sequence of homology, $H_2(\mathcal{K}(\mathbb{S}(U_a \cap U_b))) = 0$ and then $H_1(U_a \cap U_b) = 0$. Thus, if $U_a \cap U_b$ is nonempty and connected, it is contractible since $ht(U_a \cap U_b) \leq 1$. $\square$

Remark 9.1.3. If X is a contractible finite T_0-space of height at most 2, it can be proved by induction in $\#X$ that there exist two maximal elements a, b such that $U_a \cup U_b$ is contractible. However when a qc-reduction is performed, the resulting space can be not contractible.

Definition 9.1.4. A finite simplicial complex K of dimension at most 2 is said to be *quasi constructible* if K has just one maximal simplex or, recursively, if it can be written as $K = K_1 \cup K_2$ in such a way that

- K_1 and K_2 are quasi constructible,
- $K_1 \cap K_2$ is nonempty and connected, and
- no maximal simplex of K_1 is in K_2 and no maximal simplex of K_2 is in K_1.

The name of this complexes is suggested by the particular case of *constructible* complexes studied in [19].

Definition 9.1.5. An homogeneous finite n-simplicial complex K is *n-constructible* if $n = 0$, if K has just one maximal simplex or if $K = K_1 \cup K_2$ where K_1 and K_2 are n-constructible and $K_1 \cap K_2$ is $(n-1)$-constructible.

A homogeneous 1-complex is 1-constructible if and only if it is connected. Therefore, 2-constructible complexes are quasi constructible. A wedge of two 2-simplices is quasi constructible but not 2-constructible. This example also shows that collapsible 2-complexes need not be 2-constructible. However we prove that collapsible 2-complexes are quasi constructible.

Lemma 9.1.6. *Let K be a finite simplicial complex and let K_1, K_2 be two subcomplexes such that $K_1 \cap K_2$ is a vertex v (i.e. $K = K_1 \bigvee_v K_2$). Then K is collapsible if and only if K_1 and K_2 are collapsible.*

Proof. Suppose $K_1 \neq v \neq K_2$. If K is collapsible and $\sigma \subseteq \sigma'$ is a collapsible pair of K such that $K \smallsetminus \{\sigma, \sigma'\}$ is collapsible, then $\sigma \subsetneq \sigma'$ is a collapsible pair of K_1 or K_2. Without loss of generality assume the first. Then $(K_1 \searrow \{\sigma,\sigma'\}) \bigvee_v K_2 = K \searrow \{\sigma,\sigma'\}$ is collapsible. By induction $K_1 \searrow \{\sigma,\sigma'\}$ and K_2 are collapsible.

If K_1 and K_2 are collapsible, they collapse to any of their vertices. In particular $K_1 \searrow v$ and $K_2 \searrow v$. The collapses of K_1 and K_2 together show that $K \searrow v$. □

Theorem 9.1.7. *Let K be a finite simplicial complex of dimension less than or equal to 2. If K is collapsible, then it is quasi constructible.*

Proof. If K is collapsible and not a point, there exist a collapsible pair $\sigma \subsetneq a\sigma$ such that $L = K \searrow \{\sigma, a\sigma\}$ is collapsible. By induction L is quasi constructible. $K = L \cup a\sigma$ and $L \cap a\sigma = a\dot{\sigma}$ is connected. If no maximal simplex of L is a face of $a\sigma$, K is quasi constructible as we want to prove. However this might not be the case.

If $a\sigma$ is a 1-simplex and a is a maximal simplex of L, $L = a$ and then K is a 1-simplex which is quasi constructible.

Assume $a\sigma$ is a 2-simplex and let b, c be the vertices of σ.

Consider this first case: ab is a maximal simplex of L but ac is not. We claim that $L \searrow \{ab\}$ has two connected components. Certainly, since L is contractible, from the Mayer-Vietoris sequence,

$$\widetilde{H}_1(L) \to \widetilde{H}_0(a \cup b) \to \widetilde{H}_0(ab) \oplus \widetilde{H}_0(L \searrow \{ab\}) \to \widetilde{H}_0(L)$$

we deduce that $\widetilde{H}_0(L \searrow \{ab\}) = \mathbb{Z}$. Therefore, there exist subcomplexes $L_1 \ni a$ and $L_2 \ni b$ of L such that $L = L_1 \bigvee_a ab \bigvee_b L_2$.

By 9.1.6, L_1 and L_2 are collapsible and therefore quasi constructible.

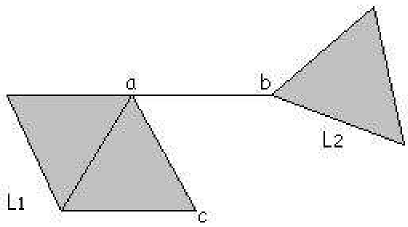


Figure 9.1: L

Now, L_1 and $a\sigma$ are quasi constructible, $L_1 \cap a\sigma = ac$ is connected and $\{ac\}$ is not maximal in L_1 nor in $a\sigma$. Thus $L_1 \cup a\sigma$ is quasi constructible. If L_2 is just the point b, $K = L_1 \cup a\sigma$ is quasi constructible. If L_2 is not a point, $\{b\}$ is not a maximal simplex of L_2 and then $K = (L_1 \cup a\sigma) \cup L_2$ is quasi constructible since $(L_1 \cup a\sigma) \cap L_2 = b$ is connected.

The second case: ac is maximal in L but ab is not is analogous to the first.

The third case is: ab and ac are maximal simplices of L. As above $L \searrow \{ab\}$ and $L \searrow \{ac\}$ have two connected components. Therefore, there exist subcomplexes L_1, L_2 and L_3 of L such that $a \in L_1$, $b \in L_2$, $c \in L_3$ and $L = L_2 \bigvee_b ab \bigvee_a L_1 \bigvee_a ac \bigvee_c L_3$. Since L is collapsible, by 9.1.6, L_i are also collapsible and by induction, quasi constructible. If $L_1 \neq a$, $L_2 \neq b$ and $L_2 \neq c$, we prove that K is quasi constructible as follows: $a\sigma \cup L_1$ is quasi constructible since $a\sigma \cap L_1 = a$ is connected and $\{a\}$ is not maximal in $a\sigma$ nor in L_1. Then $(a\sigma \cup L_1) \cup L_2$ is quasi constructible since $(a\sigma \cup L_1) \cap L_2 = b$ is connected and $\{b\}$ is maximal in none of them. Similarly, $K = (a\sigma \cup L_1 \cup L_2) \cup L_3$ is quasi constructible. If some of the complexes L_i are just single points, this simplifies the proof since we can remove those from the writing of $K = a\sigma \cup L_1 \cup L_2 \cup L_3$. □

On the other hand, contractible 2-constructible complexes need not be collapsible as the next example shows.

Example 9.1.8. The following example of a contractible 2-constructible and non-collapsible complex is a slight modification of one defined by Hachimori (see [19], Section 5.4). Let K be the 2-homogeneous simplicial complex of Figure 9.2.

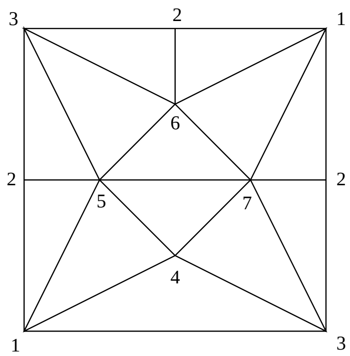


Figure 9.2: K.

This complex is 2-constructible (in fact it is shellable). For instance, one can construct it adjoining 2-simplices in the following order: $567, 457, 347, 237, 127, 167, 126, 236, 356, 235, 125, 145, 134$. In each adjuntion both the complex and the 2-simplex are 2-constructible and their intersection is 1-constructible. Moreover, K is collapsible.

Now take two copies K_1 and K_2 of K and identify the 1-simplex 13 of both copies. The resulting complex L is contractible since K_1 and K_2 are contractible. Moreover, K_1 and K_2 are 2-constructible and their intersection is 1-constructible, therefore L is 2-constructible. On the other hand, L is not collapsible since it does not have free faces.

We will see in 9.1.10 that quasi constructible complexes 3-deform to a point. In particular this is true for this complex.

The notion of constructibility is in turn a generalization of the concept of *shellability* [11]. Shellable complexes are collapsible.

Theorem 9.1.9. *Let K be a finite simplicial complex of dimension less than or equal to 2. Then the following are equivalent:*

1. *K is quasi constructible and $H_2(|K|) = 0$,*

2. *$\mathcal{X}(K)$ is qc-reducible,*

3. K is quasi constructible and contractible.

Proof. Let K be quasi constructible and suppose $H_2(|K|)$=0. If K has just one maximal simplex, $\mathcal{X}(K)$ has maximum and it is qc-reducible. Otherwise, $K = K_1 \cup K_2$ where K_1 and K_2 are quasi constructible and $K_1 \cap K_2$ is connected and nonempty. Moreover the maximal simplices of K_1 are not in K_2 and viceversa. Since $H_3(|K|, |K_i|) = 0$, $H_2(|K_i|) = 0$ and by an inductive argument, $\mathcal{X}(K_i)$ is qc-reducible for $i = 1, 2$. Carring out the same qc-reductions in $\mathcal{X}(K)$ we obtain a space Y with two maximal elements a_1 and a_2 such that $U_{a_1} \cap U_{a_2} = \mathcal{X}(K_1 \cap K_2)$ which is connected and nonempty. Moreover, $H_2(Y) = H_2(\mathcal{X}(K)) = 0$ and therefore, by 9.1.2, a last qc-reduction transforms Y in a space with maximum.

Now suppose that K is such that $\mathcal{X}(K)$ is qc-reducible. Then we can make qc-reductions to obtain a space with maximum. If $\mathcal{X}(K)$ does not have maximum, in the last step, before the last qc-reduction, one has a contractible space Y with two maximal elements a_1 and a_2. Consider the simplicial complex K_1 generated by all the maximal simplices of K that were eventually replaced by a_1 when performing the qc-reductions. Define K_2 similarly. Then, $\mathcal{X}(K_1)$ and $\mathcal{X}(K_2)$ are qc-reducible and by induction K_1 and K_2 are quasi constructible. Moreover $\mathcal{X}(K_1 \cap K_2) = U_{a_1} \cap U_{a_2}$ is connected and nonempty by 9.1.2 and then so is $K_1 \cap K_2$. Hence K is quasi constructible. On the other hand, since $\mathcal{X}(K)$ is qc-reducible, it is homotopically trivial and therefore $|K|$ is contractible. □

In fact, the equivalence between 1 and 3 can be proved easily without going through 2 (see 9.1.11).

Recall that if K is a 2-complex, K 3-deforms to K'. Therefore we have the following

Corollary 9.1.10. *If K is quasi constructible and contractible, it 3-deforms to a point, i.e. contractible quasi constructible complexes satisfy the geometric Andrews-Curtis conjecture 9.0.2.*

Remark 9.1.11. By the Theorem of Van-Kampen, quasi constructible complexes are simply connected. In particular, their reduced Euler characteristic is non-negative.

In the next we adapt an argument of Hachimori to show that there are many contractible 2-complexes which are not quasi constructible. The results and their proof are essentially the same as [19]. A vertex v of a finite complex K is a *bridge* if $K \smallsetminus v$ has more connected components than K. Following Hachimori we say that a vertex v of a finite 2-simplicial complex K is *splittable* if the graph $lk(v)$ has a bridge.

Remark 9.1.12. Suppose $K = K_1 \cup K_2$ is a 2-complex such that no maximal simplex of K_1 is in K_2 and viceversa. In this case $K_1 \cap K_2$ is a graph. Assume that there exists a vertex v which is a leaf of $K_1 \cap K_2$, i.e. $lk_{K_1 \cap K_2}(v) = v'$ is a point. We prove that v is splittable in K. Since $vv' \in K_1 \cap K_2$, vv' is not maximal in either of the subcomplexes K_1 and K_2. Let $v_i \in K_i$ such that $vv'v_i \in K_i$ for $i = 1, 2$. The vertices v_1 and v_2 are connected in $lk_K(v)$ via v'. Suppose that they are also connected in $lk_K(v) \smallsetminus v'$. Then, there exists $w \in lk_K(v) \smallsetminus v'$ such that vw is a simplex of K_1 and K_2 simultaneously. This contradicts the fact that $lk_{K_1 \cap K_2}(v) = v'$. Therefore v' is a bridge of $lk_K(v)$.

Proposition 9.1.13. *Let K be a contractible finite 2-simplicial complex with no bridges and with at most one splittable point. If K is not a 2-simplex, then it is not quasi constructible.*

Proof. Suppose that K is quasi constructible. Then there exists quasi constructible subcomplexes K_1 and K_2 as in Definition 9.1.4. $K_1 \cap K_2$ is a connected graph with more than one vertex, provided that K has no bridges. By the previous remark, it has at most one leaf and therefore it is not a tree. In particular $\widetilde{\chi}(K_1 \cap K_2) < 0$. Since K is contractible, by 9.1.11 we have that

$$0 = \widetilde{\chi}(K) = \widetilde{\chi}(K_1) + \widetilde{\chi}(K_2) - \widetilde{\chi}(K_1 \cap K_2) > 0,$$

which is a contradiction. □

In particular we deduce that any triangulation of the Dunce hat is not quasi constructible, since it has just one splittable point.

Remark 9.1.14. Gillman and Rolfsen proved that the Poincaré conjecture is equivalent to the Zeeman's conjecture restricted to some complexes called *standard spines* (see [34]). In particular, with the proof of Poincaré conjecture, the Geometric Andrews-Curtis conjecture is known to be true for such complexes. It is easy to see that standard spines have no bridges nor splittable points and therefore they are not quasi constructible. Therefore our result enlarges the class of 2-complexes for which the conjecture is known to be valid.

Any triangulation of the Dunce hat is not quasi constructible and it is easy to see that it is not a standard spine either since it has a splittable point.

It seems very natural to consider the dual notion of qc-reducibility in order to obtain a larger class of complexes satisfying the Andrews-Curtis conjecture. However we will see that if K is such that $\mathcal{X}(K)^{op}$ is qc-reducible, then K is collapsible. If X is a finite T_0-space of height at most 2 and a, b are two minimal elements such that $F_a \cup F_b$ is contractible. Then we say that there is a *qc^{op}-reduction* from X to $Y \smallsetminus \{a, b\}$ where $Y = X \cup \{c\}$ with $a > c < b$. We say that X is *qc^{op}-reducible* if we can obtain a space with a minimum by carrying out qc^{op}-reductions begining from X, or, in other words, if X^{op} is qc-reducible.

If K is a finite simplicial complex and V is a subset of vertices of K, we will denote by $st(V) \subseteq |K|$ the union of the open stars of the vertices in V, i.e.

$$st(V) = (\bigcup_{v \in V} \overset{\circ}{st}(v)),$$

where $\overset{\circ}{st}(v) = |K| \smallsetminus |K \smallsetminus v| = \bigcup_{\sigma \ni v} \overset{\circ}{\sigma} \subseteq |K|$.

We introduce the dual notion of quasi constructibility which is the following.

Definition 9.1.15. Let $K = (V_K, S_K)$ be a finite simplicial complex of dimension at most 2. We say that a subset $V \subseteq V_K$ of vertices is *quasi^{op} constructible in* K if $\#V = 1$ or if, recursivelly, $V = V_1 \cup V_2$ with V_i quasi^{op} constructible in K for $i = 1, 2$, $V_1 \cap V_2 = \emptyset$ and $st(V_1) \cap st(V_2)$ is a connected nonempty subspace of the geometric realization $|K|$.

The complex K is said to be *quasi^{op} constructible* if V_K is quasi^{op} constructible in K.

In order to understand the topology of $st(V_1) \cap st(V_2)$, we will generalize the result that says that $\mathcal{X}(K)$ is a finite model of K, giving an alternative proof of this fact.

Proposition 9.1.16. *Let K be a finite simplicial complex and let $Y \subseteq S_K$ be a subset of simplices of K. Let $X = \bigcup_{\sigma \in Y} \overset{\circ}{\sigma} \subseteq |K|$ and let $f : X \to Y \subseteq \mathcal{X}(K)^{op}$ be the map defined by $f(x) = \sigma$ if $x \in \overset{\circ}{\sigma}$. Then, f is a weak homotopy equivalence.*

Proof. We first note that f is continuous. If $\sigma \in Y$,

$$f^{-1}(U_\sigma) = \bigcup_{\sigma \subseteq \tau \in Y} \overset{\circ}{\tau} = (\bigcup_{\sigma \subseteq \tau \in S_K} \overset{\circ}{\tau}) \cap X = X \smallsetminus |\sigma^c|$$

is open in X since σ^c is a subcomplex of K. To prove that f is a weak homotopy equivalence we use the Theorem of McCord 1.4.2. We only have to show that $f^{-1}(U_\sigma)$ is contractible. In fact, $\overset{\circ}{\sigma}$ is a strong deformation retract of $f^{-1}(U_\sigma)$. Let $x \in \overset{\circ}{\tau}$ with $\sigma \subseteq \tau \in Y$, $x = t\alpha + (1-t)\beta$ for some $0 < t \leq 1$, $\alpha \in \overset{\circ}{\sigma}$ and $\beta \in (\tau \smallsetminus \sigma)^\circ$. Define $r : f^{-1}(U_\sigma) \to \overset{\circ}{\sigma}$ by $r(x) = \alpha$. Then r is a retraction and $H : f^{-1}(U_\sigma) \times I \to f^{-1}(U_\sigma)$, $H(x,s) = (1-s)x + s\alpha$ defines a homotopy between $1_{f^{-1}(U_\sigma)}$ and ir. □

Proposition 9.1.17. *Let K be a finite T_0-space of height at most 2. Then K is quasi^{op} constructible and contractible if and only if $\mathcal{X}(K)$ is qc^{op}-reducible.*

Proof. Suppose $|K|$ is contractible. We prove that if $V \subseteq V_K$ is quasi^{op} constructible in K, then $\bigcup_{v \in V} F_{\{v\}} \subseteq \mathcal{X}(K)$ is qc^{op}-reducible. If $\#V = 1$, $\bigcup_{v \in V} F_{\{v\}}$ has minimum and there is nothing to do. Assume that $V = V_1 \cup V_2$ where V_1 and V_2 are disjoint and quasi^{op} constructible in K, and $st(V_1) \cap st(V_2)$ is connected and nonempty. By induction $\bigcup_{v \in V_1} F_{\{v\}}$ and $\bigcup_{v \in V_2} F_{\{v\}}$ are qc^{op}-reducible. Then $\bigcup_{v \in V} F_{\{v\}}$ qc^{op}-reduces to a space X with two minimal elements a_1 and a_2. Moreover, $F_{a_1} \cap F_{a_2} = \{\sigma \in S_K \mid$ there exist $v_1 \in V_1$ and $v_2 \in V_2$ with $v_1, v_2 \in \sigma\}$ is weak homotopy equivalent to $st(V_1) \cap st(V_2)$ by Proposition 9.1.16. In particular, $F_{a_1} \cap F_{a_2}$ is connected and nonempty, and since $\mathcal{X}(K)$ is homotopically trivial, by Proposition 9.1.2, X is contractible. Therefore a last qc^{op}-reductions transforms X into a space with minimum, so $\bigcup_{v \in V} F_{\{v\}}$ is qc^{op}-reducible. Now, if in addition K is quasi^{op}-constructible, V_K is quasi^{op} constructible in K and then $\mathcal{X}(K) = \bigcup_{v \in V_K} F_{\{v\}}$ is qc^{op}-reducible.

Conversely, let $V \in V_K$ be a subset of vertices of K. We will prove that if $\bigcup_{v \in V} F_{\{v\}} \subseteq \mathcal{X}(K)$ is qc^{op}-reducible, then V is quasi^{op} constructible in K. If $\#V = 1$ there is nothing to prove. In other case, before the last step we will have reduced $\bigcup_{v \in V} F_{\{v\}}$ into a contractible space X with two minimal points a_1 and a_2. Let V_i be the subset of V of vertices that were eventually replaced by a_i for $i = 1, 2$. Then $\bigcup_{v \in V_i} F_{\{v\}}$ is qc^{op}-reducible and by induction V_i is quasi^{op} constructible for $i = 1, 2$. By Proposition 9.1.16, $st(V_1) \cap st(V_2)$ is weak homotopy equivalent to $F_{a_1} \cap F_{a_2}$ which is connected and nonempty by Proposition 9.1.2. Then V is quasi^{op} constructible in K.

Finally, applying this result to $V = V_K$ we deduce that if $\mathcal{X}(K)$ is qc^{op}-reducible, then K is quasi^{op} constructible. In this case $\mathcal{X}(K)$ is homotopically trivial and then $|K|$ is contractible. □

In particular, we deduce that if K is quasi^{op} constructible and contractible, it 3-deforms to a point. Unfortunately, this does not enlarge the class of complexes satisfying the Andrews-Curtis conjecture, since quasi^{op} constructible complexes are collapsible as we will see.

Lemma 9.1.18. *Let K be a finite simplicial complex of dimension less than or equal to 2. If $V \subseteq V_K$ is quasi^{op} constructible in K, then $\widetilde{\chi}(st(V)) \geq 0$.*

Proof. If $\#V = 1$, $st(V)$ is contractible and then $\widetilde{\chi}(st(V)) = 0$. Suppose that $V = V_1 \cup V_2$ where V_1 and V_2 are disjoint, quasi^{op} constructible in K and such that $st(V_1) \cap st(V_2)$ is connected and nonempty. By induction,

$$\widetilde{\chi}(st(V)) = \widetilde{\chi}(st(V_1)) + \widetilde{\chi}(st(V_2)) - \widetilde{\chi}(st(V_1) \cap st(V_2)) \geq -\widetilde{\chi}(st(V_1) \cap st(V_2)).$$

By Proposition 9.1.16, $st(V_1) \cap st(V_2)$ is weak homotopy equivalent to $\overline{V}_1 \cap \overline{V}_2 \subseteq \mathcal{X}(K)$ which is a finite T_0-space of height at most 1. Since it is connected and nonempty, $\widetilde{\chi}(st(V_1) \cap st(V_2)) = \widetilde{\chi}(\overline{V}_1 \cap \overline{V}_2) \leq 0$ and then $\widetilde{\chi}(st(V)) \geq 0$. □

Theorem 9.1.19. *Let K be a contractible quasi^{op} constructible simplicial complex. Then K is collapsible.*

Proof. If $K = *$, there is nothing to do. Suppose $V_K = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, V_1 and V_2 quasi^{op} constructible in K and $st(V_1) \cap st(V_2)$ nonempty and connected. Since $|K|$ is contractible,

$$0 = \widetilde{\chi}(|K|) = \widetilde{\chi}(st(V_1)) + \widetilde{\chi}(st(V_2)) - \widetilde{\chi}(st(V_1) \cap st(V_2)).$$

By Lemma 9.1.18, $\widetilde{\chi}(st(V_i)) \geq 0$ for $i = 1, 2$ and then $\widetilde{\chi}(\overline{V}_1 \cap \overline{V}_2) = \widetilde{\chi}(st(V_1) \cap st(V_2)) \geq 0$. Moreover, $\overline{V}_1 \cap \overline{V}_2 \subseteq \mathcal{X}(K)$ is nonempty, connected and its height is less than or equal to 1. Therefore, it is contractible. In particular, there exists a simplex $\sigma \in K$ which is a leaf (maybe the unique vertex) of the graph $\mathcal{K}(\overline{V}_1 \cap \overline{V}_2)$. We claim that σ is not a 2-simplex, because if that was the case, it would have two of its vertices a, b in V_i and the third c in V_j for $i \neq j$. Then $\{a, c\}$ and $\{b, c\}$ would be covered by σ in $\overline{V}_1 \cap \overline{V}_2$ contradicting the fact that σ is a leaf of $\mathcal{K}(\overline{V}_1 \cap \overline{V}_2)$. Thus σ is a 1-simplex.

Let $a \in V_1$ and $b \in V_2$ be the vertices of σ. Since σ is a leaf of $\mathcal{K}(\overline{V}_1 \cap \overline{V}_2)$, we consider two different cases:

(1) $\overline{V}_1 \cap \overline{V}_2 = \{\sigma\}$ or

(2) $\sigma \in K$ is a free face of a simplex $\sigma' = \{a, b, c\} \in K$.

We study first the case (1). For $i = 1, 2$, let K_i be the full subcomplex of K spanned by the vertices of V_i. Then $K = K_1 \cup K_2 \cup \{\sigma\} = K_1 \bigvee_a \sigma \bigvee_b K_2$. Since K is contractible, then K_1 and K_2 are contractible as well. Moreover, since V_i is quasi^{op} constructible in

K, it is also quasi^{op} constructible in K_i. Note that if V and V' are subsets of V_i, then $st_{K_i}(V) \cap st_{K_i}(V') = st_K(V) \cap st_K(V')$. Thus, K_1 and K_2 are contractible and quasi^{op} constructible. By induction, they are collapsible. Therefore $K = K_1 \bigvee_a \sigma \bigvee_b K_2$ is also collapsible.

Now we consider the second case (2). Let $L = K \smallsetminus \{\sigma, \sigma'\}$. By hypothesis $K \searrow^e L$. We claim that L is quasi^{op} constructible. To prove that, we will show first that V_1 and V_2 are quasi^{op} constructible in L. We prove by induction that if $V \subseteq V_1$ is quasi^{op} constructible in K, then it also is in L. If $\#V = 1$ this is trivial. Suppose $V = V' \cup V''$ with V' and V'' disjoint, quasi^{op} constructible in K and such that $st_K(V') \cap st_K(V'') \overset{we}{\approx} \overline{V'}^{\mathcal{X}(K)} \cap \overline{V''}^{\mathcal{X}(K)}$ is nonempty and connected. By induction V' and V'' are quasi^{op} constructible in L. We have to show that $\overline{V'}^{\mathcal{X}(L)} \cap \overline{V''}^{\mathcal{X}(L)} = (\overline{V'}^{\mathcal{X}(K)} \cap \overline{V''}^{\mathcal{X}(K)}) \smallsetminus \{\sigma, \sigma'\}$ is nonempty and connected.

Since σ has only one vertex in V_1, it cannot have a vertex in V' and other in V''. Therefore, $\sigma \notin \overline{V'}^{\mathcal{X}(K)} \cap \overline{V''}^{\mathcal{X}(K)}$. If $\sigma' \notin \overline{V'}^{\mathcal{X}(K)} \cap \overline{V''}^{\mathcal{X}(K)}$, then $\overline{V'}^{\mathcal{X}(L)} \cap \overline{V''}^{\mathcal{X}(L)} = (\overline{V'}^{\mathcal{X}(K)} \cap \overline{V''}^{\mathcal{X}(K)})$ is nonempty and connected. If $\sigma' \in \overline{V'}^{\mathcal{X}(K)} \cap \overline{V''}^{\mathcal{X}(K)}$, then $c \in V_1$ and σ' covers just one element of $\overline{V'}^{\mathcal{X}(K)} \cap \overline{V''}^{\mathcal{X}(K)}$, which is $\{a, c\}$. Hence, σ' is a down beat point of $\overline{V'}^{\mathcal{X}(K)} \cap \overline{V''}^{\mathcal{X}(K)}$ and in particular $\overline{V'}^{\mathcal{X}(L)} \cap \overline{V''}^{\mathcal{X}(L)}$ is homotopy equivalent to $\overline{V'}^{\mathcal{X}(K)} \cap \overline{V''}^{\mathcal{X}(K)}$. Then, it is nonempty and connected and therefore V is quasi^{op} constructible in L.

Since V_1 is quasi^{op} constructible in K it follows that it is quasi^{op} constructible in L. Analogously, V_2 is quasi^{op} constructible in L.

Now, by assumption $st_K(V_1) \cap st_K(V_2) \overset{we}{\approx} \overline{V_1}^{\mathcal{X}(K)} \cap \overline{V_2}^{\mathcal{X}(K)}$ is nonempty and connected. Since σ is a free face of K, it is an up beat point of $\overline{V_1}^{\mathcal{X}(K)} \cap \overline{V_2}^{\mathcal{X}(K)}$. On the other hand, σ' is a down beat point of $\overline{V_1}^{\mathcal{X}(K)} \cap \overline{V_2}^{\mathcal{X}(K)} \smallsetminus \{\sigma\}$ since there is a 1-face of σ' with both vertices in V_1 or in V_2. Hence, $\overline{V_1}^{\mathcal{X}(L)} \cap \overline{V_2}^{\mathcal{X}(L)} = \overline{V_1}^{\mathcal{X}(K)} \cap \overline{V_2}^{\mathcal{X}(K)} \smallsetminus \{\sigma, \sigma'\}$ is a strong deformation retract of $\overline{V_1}^{\mathcal{X}(K)} \cap \overline{V_2}^{\mathcal{X}(K)}$, and then it is connected and nonempty. Thus, $V_L = V_1 \cup V_2$ is quasi^{op} constructible in L, or in other words, L is quasi^{op} constructible.

Since $K \searrow^e L$, L is contractible and quasi^{op} constructible. By induction L is collapsible and therefore, so is K. □

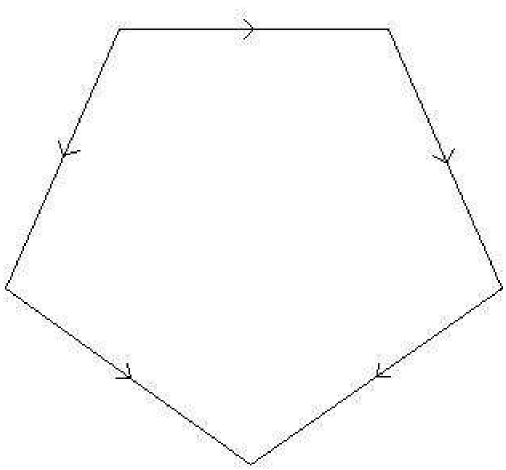
The converse of this result is false as we prove in the next example.

Example 9.1.20. The complex K studied in Example 9.1.8 is a collapsible homogeneous 2-complex with a unique free face. We prove that a complex satisfying these hypothesis cannot be quasi^{op} constructible.

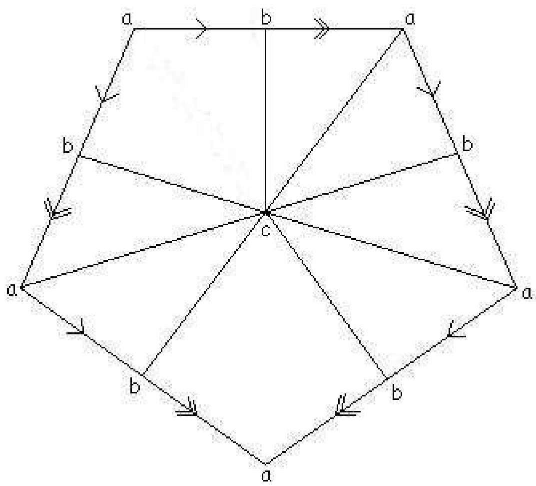
Suppose that K is quasi^{op} constructible. Since K has more than one vertex, V_K can be written as a disjoint union of quasi^{op} constructible subsets V_1 and V_2 in K such that $\overline{V}_1 \cup \overline{V}_2$ is contractible. The case (1) of the proof of Theorem 9.1.19 cannot ocurr since K is homogeneous. Therefore, $\mathcal{K}(\overline{V}_1 \cap \overline{V}_2)$ has dimension exactly 1 and it is a tree. Then, it has at least two leaves, which must be 1-simplices and free faces of K. However this is absurd since K has only one free face.

In this Thesis we have studied many methods of reduction and techniques that allow to recognize homotopically trivial finite spaces. The methods that we introduced allowed us, among other things, to characterize the simple homotopy theory of polyhedra in terms of finite spaces and to analize some known conjectures from a totally new viewpoint. We have also shown that these procedures are not completely effective to describe weak homotopy types of finite spaces. We will next exhibit a homotopically trivial finite space in which these methods fail altogether.

Consider the following pentagon whose edges are identified as indicated by the arrows.



This is a contractible CW-complex since the attaching map of the 2-cell is a homotopy equivalence $S^1 \to S^1$. We give to this space an h-regular structure K as follows



Since K is contractible, $\mathcal{X}(K)$ is homotopically trivial finite space of 21 points by Theorem 7.1.7. It is easy to check that $\mathcal{X}(K)$ has no weak points (nor γ-points). In fact no h-regular CW-complex has down weak points and it is not hard to see that 1-cells are not up weak points in this example. We only have to show that $\hat{F}_a$, $\hat{F}_b$ and $\hat{F}_c$ are not contractible, but this is clear since their associated graphs contain a cycle.

It is not possible to make a qc-reduction on $\mathcal{X}(K)$, since for any 2-cells e, e' of K, $\overline{e} \cap \overline{e}' \subseteq K$ is not connected. It can be also proved that no qc^{op}-reduction can be made in $\mathcal{X}(K)$ since the subspaces $F_a \cap F_b, F_a \cap F_c, F_b \cap F_c \subseteq \mathcal{X}(K)$ are nonconnected.

Osaki's reduction methods 6.1.1, 6.1.2 are not applicable either.

Therefore, the methods studied in this work can not be used directly to reduce $\mathcal{X}(K)$, however $\mathcal{X}(K)$ is homotopically trivial and then, it has the same simple homotopy type of a point.

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