Complexity of high dimensional Gaussian random fields with isotropic increments

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Outline

Model and problem

- The model is

$$H_N(x) = X_N(x) + \frac{\mu}{2}\|x\|^2, \quad x \in \mathbb{R}^N.$$

Here $\mu \in \mathbb{R}$, $\|x\|$ is the Euclidean norm of x, X_N is a Gaussian random field with isotropic increments (GRFIC)

$$\mathbb{E}[(X_N(u) - X_N(v))^2] = ND\Big(\frac{1}{N}\|u - v\|_2^2\Big), \quad u, v \in \mathbb{R}^N.$$

$X_N(x)$ is also known as a locally isotropic Gaussian random field. The function D is known as the structure function of X_N.

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- Problem:
  - Given smooth H_N, how many critical points does H_N have?
  - Where are most of the critical points?
  - How about local minima or saddles with given index?

Classification of LIGRF for all N: Yaglom 1957

1. Isotropic fields (short-range correlation, or SRC): $\exists\, B : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\mathbb{E}[X_N(u)X_N(v)] = NB\Big(\frac{1}{N}\|u-v\|_2^2\Big)$$

   where B has the representation

$$B(r) = c_0 + \int_{(0,\infty)} e^{-rt^2}\nu(\mathrm{d}t), \quad c_0 \in \mathbb{R}_+,$$

   ν is a finite measure on $(0,\infty)$. In this case, $D(r) = 2(B(0) - B(r))$.

2. Non-isotropic field with isotropic increments (long-range correlation, or LRC): D has representation

$$D(r) = \int_{(0,\infty)} (1 - e^{-rt^2})\nu(\mathrm{d}t) + Ar, \quad A \in \mathbb{R}_+,$$

   ν is a σ-finite measure with

$$\int_{(0,\infty)} \frac{t^2}{1+t^2}\nu(\mathrm{d}t) < \infty.$$

An example of structure function

Assume $X_N(0)=0$ for LRC.
$\mathbb{E}(X_N(u)X_N(v)) = \frac{N}{2}(D(\frac{1}{N}\|u\|_2^2) + D(\frac{1}{N}\|v\|_2^2) - D(\frac{1}{N}\|u-v\|_2^2))$.

Example

We assume $c_0 = 0$ and $A = 0$. For fixed $\varepsilon > 0$ and $\gamma > 0$, let

$$\nu(\mathrm{d}x) = 2e^{-\varepsilon x^2} x^{2\gamma-3}\mathrm{d}x.$$

$\gamma > 1$ corresponds to SRC and $0 < \gamma \le 1$ LRC. If $\gamma > 1$,

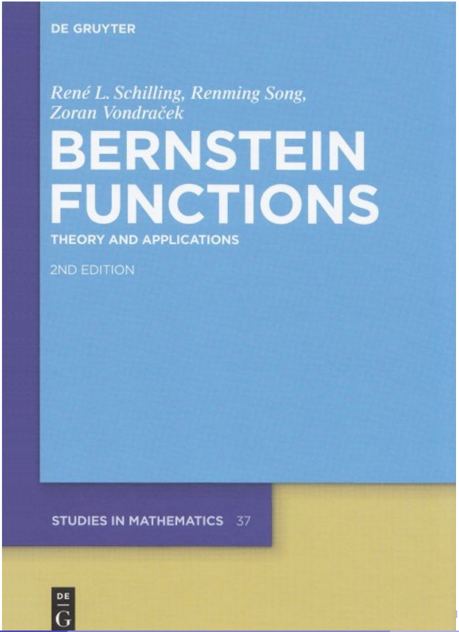
$$B(r) = \int_0^\infty 2e^{-rt^2} e^{-\varepsilon t^2} t^{2\gamma-3}\mathrm{d}t = \frac{\Gamma(\gamma-1)}{(r+\varepsilon)^{\gamma-1}}.$$

If $0 < \gamma < 1$,

$$D(r) = \frac{\Gamma(\gamma)}{1-\gamma}[(r+\varepsilon)^{1-\gamma} - \varepsilon^{1-\gamma}].$$

If $\gamma = 1$, $D(r) = \log(1 + \frac{r}{\varepsilon})$.

More examples of structure functions



More examples of structure functions

16.3 Exponential functions

No	Function $f(\lambda)$	Comment
18	$\sqrt{\lambda}\,(1-e^{-2a\sqrt{\lambda}}), \quad a>0$	14.2(43) of [107], 2.25 and 7.78 in [283], Theorem 8.2 (v)
19	$\sqrt{\lambda}\,(1+e^{-2a\sqrt{\lambda}}), \quad a>0$	14.60(3) of [107], 2.25 and 7.78 in[283], Theorem 8.2 (v)
20	$\dfrac{\lambda\,(1-e^{-2\sqrt{\lambda+a}})}{\sqrt{\lambda+a}}, \quad a>0$	Appendix 1.17 of [68], Theorem 8.2 (v). See §16.12.2
21	$\lambda\,(1+\lambda)^{1/\lambda}-\lambda-\dfrac{\lambda}{\lambda+1}$	[5], p. 457, Theorem 8.2 (v)
22	$e\lambda-\lambda\left(1+\dfrac{1}{\lambda}\right)^{\lambda}-\dfrac{\lambda}{\lambda+1}$	Theorem 3 of [5], Theorem 8.2 (v)

Background and history

- The definition of locally isotropic fields was formulated by Kolmogorov for the application in statistical theory of turbulence in 1941.

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- Many works thereafter, both in mathematics and physics, by Bray–Dean (2007), Fyodorov–Williams (2007), Fyodorov–Bouchaud (2008), Fyodorov–Nadal (2012), Klimovsky (2012), Fyodorov (2015), Cheng–Schwartzman (2018), Yamada–Vilenkin (2018), etc.

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- Complexity for spherical (mixed) p-spin model and variants: Auffinger–Ben Arous–Cerny (2011), Auffinger–Ben Arous (2013), Subag (2017), Ben Arous–Mei–Montanari–Nica (2019), etc.

Outline

Assumptions and Notation

We consider non-isotropic Gaussian random fields with isotropic increments X_N.

- Assumption I (smoothness): $0 < |D^{(4)}(0)| < \infty$.
- Assumption II (pinning): $X_N(0) = 0$.
- For Borel set $E \subset \mathbb{R}$ and $B_N \subset \mathbb{R}^N$, let

$$\mathrm{Crt}_{N,k}(E, B_N) = \#\{x \in B_N : \nabla H_N(x) = 0, \frac{1}{N} H_N(x) \in E, \\ i(\nabla^2 H_N(x)) = k\},$$

$$\mathrm{Crt}_N(E, B_N) = \#\{x \in B_N : \nabla H_N(x) = 0, \frac{1}{N} H_N(x) \in E\}.$$

Here

$$i(\nabla^2 H_N(x)) = \#\text{ negative eigenvalues of } \nabla^2 H_N(x).$$

In this talk, we will focus on $\mathbb{E}\mathrm{Crt}_{N,k}(E, B_N)$ for $k \in \mathbb{Z}_+$ fixed.

Some preparations

Assume domain growth condition: Let $Z_N \sim N(0, I_N)$. There exist Ξ or Θ such that the sequence of sets B_N satisfy

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}(Z_N \in |\mu| B_N/\sqrt{D'(0)}) = -\Xi \le 0, \qquad \mu \neq 0,$$
$$\lim_{N\to\infty} \frac{1}{N} \log |B_N| = \Theta, \qquad \mu = 0.$$

Define

$$J_1(x) = \begin{cases} \int_x^{-\sqrt{2}} \sqrt{z^2 - 2}\mathrm{d}z, & x \le -\sqrt{2}, \\ \infty, & \text{otherwise}, \end{cases}$$

and

$$\phi(x) = -\frac{1}{2}x^2 - \frac{\mu x}{\sqrt{-D''(0)}}.$$

Warm-up: no restriction on critical values

Theorem (Total number of critical points of index k)

Let $k \in \mathbb{Z}_+$. Then we have

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_{N,k}(\mathbb{R}, B_N) = \begin{cases} \frac{\mu^2}{4D''(0)} - \log \frac{|\mu|}{\sqrt{-2D''(0)}} - \frac{1}{2} - \Xi + I_k, & \mu \neq 0, \\ \log \sqrt{-2D''(0)} - \frac{1}{2}\log[D'(0)] - \frac{3}{2} - \frac{1}{2}\log(2\pi) + \Theta, & \mu = 0. \end{cases}$$

where the constant I_k (decreasing in k) is given as

$$I_k = \begin{cases} \phi(-\sqrt{2}) = -1 + \frac{\sqrt{2}\mu}{\sqrt{-D''(0)}}, & \mu \leq \sqrt{-2D''(0)}, \\ -\frac{1}{2}{x_1}^2 - \frac{\mu x_1}{\sqrt{-D''(0)}} - J_1(x_1), & \mu > \sqrt{-2D''(0)}, k = 0, \\ -\frac{1}{2}x_2^2 - \frac{\mu x_2}{\sqrt{-D''(0)}} - (k+1)J_1(x_2), & \mu > \sqrt{-2D''(0)}, k \geq 1, \end{cases}$$

and x_1, x_2 are explicit constants depending only on $\mu, D''(0)$ and k.

Proof: Based on the Kac–Rice formula

Roughly, $\int f(\nabla H_N(x))\mathrm{d}(\nabla H_N(x)) = \int f(\nabla H_N(x))|\det \nabla^2 H_N(x)|\mathrm{d}x$,

$$\mathrm{Crt}_{N,k}(E, B_N) = \int_{B_N} \delta_0(\nabla H_N(x))|\det \nabla^2 H_N(x)| \mathbf{1}\Big\{\frac{1}{N}H_N(x) \in E, i(\nabla^2 H_N(x)) = k\Big\}\mathrm{d}x.$$

Writing $p_{\nabla H_N(x)}(t)$ for the p.d.f. of $\nabla H_N(x)$ at t,

$$\begin{aligned}\mathbb{E}\mathrm{Crt}_{N,k}(E, B_N) &= \int_{B_N} \mathbb{E}[|\det \nabla^2 H_N(x)| \\ &\mathbf{1}\Big\{\frac{1}{N}H_N(x) \in E, i(\nabla^2 H_N(x)) = k\Big\}|\nabla H_N(x) = 0]p_{\nabla H_N(x)}(0)\mathrm{d}x \\ &= \int_{B_N}\int_E \mathbb{E}[|\det \nabla^2 H_N(x)|\mathbf{1}\{i(\nabla^2 H_N(x)) = k\} \\ &|\nabla H_N(x) = 0, H_N(x) = Nt]p_{\nabla H_N(x)}(0)\mathbb{P}(\frac{1}{N}H_N(x) \in \mathrm{d}t)\mathrm{d}x.\end{aligned}$$

Covariances

For $x \in \mathbb{R}^N$,

$$\begin{aligned}
\mathrm{Cov}[H_N(x), \partial_i H_N(x)] &= D'(\frac{\|x\|^2}{N})x_i, \\
\mathrm{Cov}[\partial_i H_N(x), \partial_j H_N(x)] &= D'(0)\delta_{ij}, \\
\mathrm{Cov}[H_N(x), \partial_{ij} H_N(x)] &= 2D''(\frac{\|x\|^2}{N})\frac{x_i x_j}{N} + [D'(\frac{\|x\|^2}{N}) - D'(0)]\delta_{ij} \\
\mathrm{Cov}[\partial_k H_N(x), \partial_{ij} H_N(x)] &= 0, \\
\mathrm{Cov}[\partial_{lk} H_N(x), \partial_{ij} H_N(x)] &= -2D''(0)[\delta_{jl}\delta_{ik} + \delta_{il}\delta_{kj} + \delta_{kl}\delta_{ij}]/N,
\end{aligned}$$

where δ_{ij} are the Kronecker delta function.

The case $E = \mathbb{R}$

By independence,

$$\mathbb{E}\mathrm{Crt}_{N,k}(\mathbb{R}, B_N) = \int_{B_N} \mathbb{E}[|\det \nabla^2 H_N(x)| \mathbf{1}\{i(\nabla^2 H_N(x)) = k\}] \\ p_{\nabla H_N(x)}(0)\mathrm{d}x,$$

$$p_{\nabla H_N(x)}(0) = \frac{1}{(2\pi)^{N/2} D'(0)^{N/2}} \exp\Big(- \frac{\mu^2 \|x\|^2}{2D'(0)}\Big),$$

$$\begin{aligned} \mathrm{Var}(\partial_{ii} H_N(x)^2) &= -6D''(0)/N, \\ \mathrm{Var}(\partial_{ij} H_N(x)^2) &= -2D''(0)/N, i \neq j, \\ \mathrm{Cov}(\partial_{ii} H_N(x), \partial_{kk} H_N(x)) &= -2D''(0)/N, i \neq k, \\ \mathrm{Cov}(\partial_{ij} H_N(x), \partial_{kl} H_N(x)) &= 0, \text{otherwise.} \end{aligned}$$

The case $E = \mathbb{R}$: Relation to GOE

Let $M = M^N$ be an $N \times N$ matrix taken from the Gaussian Orthogonal Ensemble (GOE) with

$$\mathbb{E}(M_{ij}) = 0, \quad \mathbb{E}(M_{ij}^2) = \frac{1+\delta_{ij}}{2N}$$

and z an independent $N(0,1)$ r.v. Then

$$\begin{aligned}\nabla^2 H_N(x) &\overset{d}{=} \sqrt{-4D''(0)}M - \Big(\sqrt{\frac{-2D''(0)}{N}}z - \mu\Big) I_N \\ &= \sqrt{-4D''(0)}\Big[M - \Big(\frac{1}{\sqrt{2N}}z - \frac{\mu}{\sqrt{-4D''(0)}}\Big) I_N\Big].\end{aligned}$$

Let $z' = \frac{1}{\sqrt{2N}}z - \frac{\mu}{\sqrt{-4D''(0)}}$, $m = -\mu/\sqrt{-4D''(0)}$ and

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$$

be eigenvalues of $\mathrm{GOE}(N)$.

A key identity

$$\mathbb{E}[|\det \nabla^2 H_N(x)| \mathbf{1}\{i(\nabla^2 H_N(x)) = k\}]$$

$$= [-4D''(0)]^{N/2} \mathbb{E} \prod_{i=1}^{N} |\lambda_i - z'| \mathbf{1}\{\lambda_k \le z' \le \lambda_{k+1}\}$$

$$= [-4D''(0)]^{N/2} \int_{\lambda_1 \le \cdots \le \lambda_N} \mathbb{E}_{z'} \prod_{i=1}^{N} |\lambda_i - z'| \mathbf{1}\{\lambda_k \le z' \le \lambda_{k+1}\}$$

$$\frac{1}{Z_N} \prod_{1 \le i < j \le N} |\lambda_i - z'| e^{-\frac{N}{2} \sum_i \lambda_i^2} \prod_i \mathrm{d}\lambda_i$$

$$= [-4D''(0)]^{N/2} \sqrt{\frac{N}{\pi}} \Big(\frac{N+1}{N}\Big)^{\frac{(N+2)(N+1)}{4}} \frac{e^{-Nm^2} Z_{N+1}}{Z_N}$$

$$\int_{y_1 \le \cdots \le y_{N+1}} e^{-\frac{1}{2}(N+1)y_{k+1}^2 + 2\sqrt{N(N+1)} m y_{k+1}} \frac{1}{Z_{N+1}}$$

$$\prod_{i=1}^{N+1} e^{-\frac{(N+1)y_i^2}{2}} \prod_{1 \le i < j \le N+1} |y_i - y_j| \mathrm{d}y_1 \cdots \mathrm{d}y_{N+1}$$

$$= \frac{\sqrt{2}[-4D''(0)]^{N/2} \Gamma(\frac{N+1}{2})}{\sqrt{\pi} N^{N/2} e^{Nm^2}} \mathbb{E}_{\mathrm{GOE}(N+1)} e^{-\frac{1}{2}(N+1)\lambda_{k+1}^2 + 2\sqrt{N(N+1)} m \lambda_{k+1}}.$$

A further reduction

We get

$$\mathbb{E}\mathrm{Crt}_{N,k}(\mathbb{R}, B_N) = C_N \mathbb{E}_{\mathrm{GOE}(N+1)} e^{-\frac{1}{2}(N+1)\lambda_{k+1}^2 - \frac{\sqrt{N(N+1)}\mu\lambda_{k+1}}{\sqrt{-D''(0)}}},$$

where

$$\lim_{N\to\infty} \frac{1}{N} \log C_N$$
$$= \begin{cases} \log \frac{\sqrt{-2D''(0)}}{|\mu|} + \frac{\mu^2}{4D''(0)} - \frac{1}{2} - \Xi, & \mu \neq 0, \\ \log \sqrt{-2D''(0)} - \frac{1}{2}\log[D'(0)] - \frac{1}{2} - \frac{1}{2}\log(2\pi) + \Theta, & \mu = 0. \end{cases}$$

It suffices to consider

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}_{\mathrm{GOE}(N+1)} e^{(N+1)\phi(\lambda_{k+1})} = ?$$

where $\phi(x) = -\frac{1}{2}x^2 - \frac{\mu x}{\sqrt{-D''(0)}}$.

To finish: LDP

Theorem (Ben Arous–Dembo–Guionnet 2001, Auffinger–Ben Arous–Cerny 2013)

The kth smallest eigenvalue of an $N \times N$ GOE matrix satisfies a LDP with speed N and a good rate function

$$J_k(x) = \begin{cases} k \int_x^{-\sqrt{2}} \sqrt{z^2 - 2} \mathrm{d}z, & x \le -\sqrt{2}, \\ \infty, & \textit{otherwise}. \end{cases}$$

By Varadhan's Lemma,

$$\sup_{x \in \mathbb{R}} \phi(x) - J_{k+1}(x) \le \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\mathrm{GOE}(N+1)} e^{(N+1)\phi(\lambda_{k+1})}$$
$$\le \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\mathrm{GOE}(N+1)} e^{(N+1)\phi(\lambda_{k+1})} \le \sup_{x \in \mathbb{R}} \phi(x) - J_{k+1}(x).$$

Let $I_k = \sup_{x \in \mathbb{R}}[\phi(x) - J_{k+1}(x)]$.

Outline

Main result: Local minima

We always assume Assumptions I, II & III,
$B_N = \{x \in \mathbb{R}^N : R_1 \le \|x\|/\sqrt{N} < R_2\}$ and write

$$\mathrm{Crt}_{N,k}(E, [R_1, R_2)) = \mathrm{Crt}_{N,k}(E, B_N).$$

Theorem

Let $0 \le R_1 < R_2 \le \infty$ *and* E *be an open set of* $\mathbb{R}$. *Suppose* $|\mu| + \frac{1}{R_2} > 0$. *Then*

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_{N,0}(E, (R_1, R_2)) = \frac{1}{2}\log[-4D''(0)] - \frac{1}{2}\log D'(0) + \frac{1}{2} + \sup_{(\rho,t,y)\in F} [\psi(\sigma_{\mathrm{sc}}, \rho, t, y) - \mathcal{I}^-(\rho, t, y)],$$

where $F = \{(\rho, t, y) : \rho \in (R_1, R_2), t \in \bar{E}, y \le -\sqrt{2}\}$ *and the functions* $\psi, \mathcal{I}^-$ *only depend on* D *and* μ.

Main result: Index $k \geq 1$

Theorem

Let $0 \leq R_1 < R_2 \leq \infty$ *and* E *be an open set of* $\mathbb{R}$. *Suppose* $|\mu| + \frac{1}{R_2} > 0$. *Then for any fixed* $k \in \mathbb{N}$,

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_{N,k}(E,(R_1,R_2)) = \frac{1}{2}\log[-4D''(0)] - \frac{1}{2}\log D'(0) + \frac{1}{2}$$
$$+ \max\Big\{ \sup_{(\rho,t,y)\in F} [\psi(\sigma_{\mathrm{sc}},\rho,t,y) - kJ_1(y)],$$
$$\sup_{(\rho,t,y)\in F} [\psi(\sigma_{\mathrm{sc}},\rho,t,y) - \mathcal{I}^+(\rho,t,y) - (k-1)J_1(y)]\Big\},$$

where $F = \{(\rho,t,y) : \rho \in (R_1,R_2), t \in \bar{E}, y \leq -\sqrt{2}\}$ *and the functions* ψ, $\mathcal{I}^+$ *depend only on* μ, D.

It is necessary to have both terms in 'max'.

Proof: E open set

- By Kac–Rice formula

$$\mathbb{E}\mathrm{Crt}_{N,k}(E, B_N) = \int_{B_N} \mathbb{E}[|\det \nabla^2 H_N(x)|$$

$$\mathbf{1}\{\frac{1}{N} H_N(x) \in E, i(\nabla^2 H_N(x)) = k\} | \nabla H_N(x) = 0] p_{\nabla H_N(x)}(0) \mathrm{d}x$$

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- First new phenomenon: H_N and $\nabla H_N(x)$ not independent.
- No worry: Note that $\mathrm{Cov}(H_N(x),\nabla H_N(x))=D'(\frac{\|x\|^2}{N})x^\mathsf{T}$, $\mathrm{Cov}(\nabla H_N(x))=D'(0)I_N$. Consider

$$Y(x)=\frac{H_N(x)}{N}-\frac{D'(\frac{\|x\|^2}{N})\sum_{i=1}^N x_i\partial_i H_N(x)}{ND'(0)}.$$

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$$Y(x)=\frac{H_N(x)}{N}-\frac{D'(\frac{\|x\|^2}{N})\sum_{i=1}^N x_i\partial_i H_N(x)}{ND'(0)}.$$

- $Y(x)\perp\!\!\!\perp\nabla H_N(x)$. So

$$\mathbb{E}\mathrm{Crt}_{N,k}(E,B_N)=\int_{B_N}\mathbb{E}[|\det\nabla^2 H_N(x)|$$

$$\mathbf{1}\{Y(x)\in E, i(\nabla^2 H_N(x))=k\}]p_{\nabla H_N(x)}(0)\mathrm{d}x.$$

General case: Difficulty

$$\mathbb{E}\mathrm{Crt}_{N,k}(E,B_N)=\int_{B_N}\int_E \mathbb{E}[|\det\nabla^2 H_N(x)|\mathbf{1}\{i(\nabla^2 H_N(x))=k\}|Y=t]$$
$$p_{\nabla H_N(x)}(0)\mathbb{P}(Y\in \mathrm{d}t)\mathrm{d}x.$$

But for $i\neq j, k\neq l, \{i,j\}\neq\{k,l\}$,

$$\mathrm{Cov}[(\partial_{ij}H_N(x),\partial_{kl}H_N(x))|Y=t]=-\frac{1}{N}\frac{\alpha x_i x_j}{N}\frac{\alpha x_k x_l}{N},$$

where we let

$$\alpha=\alpha(\|x\|^2/N)=\frac{2D''(\|x\|^2/N)}{\sqrt{D(\frac{\|x\|^2}{N})-\frac{D'(\|x\|^2/N)^2}{D'(0)}\frac{\|x\|^2}{N}}},$$
$$\beta=\beta(\|x\|^2/N)=\frac{D'(\|x\|^2/N)-D'(0)}{\sqrt{D(\frac{\|x\|^2}{N})-\frac{D'(\|x\|^2/N)^2}{D'(0)}\frac{\|x\|^2}{N}}}.$$

Remedy: Rotation

Define $\color{red}{A} = U\nabla^2 H_N(x)U^\mathsf{T}$ where U is an $N \times N$ orthogonal matrix s.t.

$$U(\frac{\alpha x x^\mathsf{T}}{N} + \beta I_N)U^\mathsf{T} = \begin{pmatrix} \frac{\alpha\|x\|^2}{N} + \beta & 0 & \cdots & 0 \\ 0 & \beta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta \end{pmatrix}.$$

Then

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Then

$$\mathrm{Cov}[(A_{ij}, A_{i'j'})|Y = t]$$
$$= \begin{cases} \frac{-6D''(0)}{N} - \frac{1}{N}(\frac{\alpha\|x\|^2}{N} + \beta)^2, & i = j = i' = j' = 1, \\ \frac{-2D''(0)}{N} - \frac{1}{N}(\frac{\alpha\|x\|^2}{N} + \beta)\beta, & i = j = 1 \neq i' = j', \text{ or } i' = j' = 1 \neq i = j \\ \frac{-6D''(0)}{N} - \frac{\beta^2}{N}, & i = j = i' = j' \neq 1, \\ \frac{-2D''(0)}{N} - \frac{\beta^2}{N}, & 1 \neq i = j \neq i' = j' \neq 1, \\ \frac{-2D''(0)}{N}, & i = i' \neq j = j', \text{ or } i = j' \neq j = i', \\ 0, & \text{otherwise.} \end{cases}$$

Assumption III and Relation to GOE

Assumption III: For $x \in \mathbb{R}^N \setminus \{0\}$,

$$-2D''(0) > (\frac{\alpha\|x\|^2}{N} + \beta)\beta, \quad -4D''(0) > (\frac{\alpha\|x\|^2}{N} + \beta)\frac{\alpha\|x\|^2}{N}.$$

Under Assumption III,

$$(A|Y = t) \stackrel{d}{=} \begin{pmatrix} z_1' & \xi^\mathsf{T} \\ \xi & G_{**} \end{pmatrix} =: G,$$

where $z_1' = \sigma_1 z_1 - \sigma_2 z_2 + m_1$,

$$G_{**} = \sqrt{-4D''(0)}(\mathrm{GOE}_{N-1} - z_3' I_{N-1}),$$
$$z_3' = \left(\sigma_2 z_2 + \frac{\|x\|\sqrt{\alpha\beta}}{N} z_3 - m_2\right)/\sqrt{-4D''(0)},$$

$\xi \sim N(0, \frac{-2D''(0)}{N} I_{N-1})$, $z_1, z_2, z_3 \sim N(0,1)$ i.i.d.

Some notation

$$m_1 = \mu + \frac{(t - \frac{\mu\|x\|^2}{2N} + \frac{\mu D'(\frac{\|x\|^2}{N})\|x\|^2}{D'(0)N})(\frac{2D''(\frac{\|x\|^2}{N})\|x\|^2}{N} + D'(\frac{\|x\|^2}{N}) - D'(0))}{D(\frac{\|x\|^2}{N}) - \frac{D'(\frac{\|x\|^2}{N})^2\|x\|^2}{D'(0)N}},$$

$$m_2 = \mu + \frac{(t - \frac{\mu\|x\|^2}{2N} + \frac{\mu D'(\frac{\|x\|^2}{N})\|x\|^2}{D'(0)N})(D'(\frac{\|x\|^2}{N}) - D'(0))}{D(\frac{\|x\|^2}{N}) - \frac{D'(\frac{\|x\|^2}{N})^2\|x\|^2}{D'(0)N}}$$

$$\sigma_1 = \sqrt{\frac{-4D''(0) - (\alpha\|x\|^2/N + \beta)\alpha\|x\|^2/N}{N}},$$

$$\sigma_2 = \sqrt{\frac{-2D''(0) - (\alpha\|x\|^2/N + \beta)\beta}{N}}.$$

$$\mathbb{E}\mathrm{Crt}_{N,k}(E, B_N) = \int_{B_N}\int_E \mathbb{E}|\det G|\mathbf{1}\{i(G) = k\} p_{\nabla H_N(x)}(0) \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(t-m_Y)^2}{2}} \mathrm{d}t\mathrm{d}x.$$

Side remark about Assumption III

Following [SSV12], a function $f:(0,\infty)\to(0,\infty)$ is a Thorin–Bernstein function if and only if $\lim_{x\to0+}f(x)$ exists and its derivative has the representation

$$f'(x)=\frac{a}{x}+b+\int_{(0,\infty)}\frac{1}{x+t}\sigma(\mathrm{d}t),$$

where $a,b\geq 0$ and σ is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)}\frac{1}{1+t}\sigma(\mathrm{d}t)<\infty$.

Lemma

If D is a Thorin–Bernstein function with $a=0$, then Assumption III holds.

In fact,

$$\text{Thorin–Bernstein}\Rightarrow\beta^2<-\frac{2}{3}D''(0)\Rightarrow\text{Assumption III}.$$

To proceed: Two cases for $i(G) = k$

Lemma (Lazutkin 1988)

Let S be a symmetric block matrix, and write its inverse S^{-1} in block form with the same block structure:

$$S = \begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} A' & B' \\ (B')^{\mathsf{T}} & C' \end{pmatrix}.$$

Then, $\operatorname{sgn}(S) = \operatorname{sgn}(A) + \operatorname{sgn}(C')$*, with* $\operatorname{sgn}(M)$ *denoting the signature of the matrix* M*.*

By the interlacement property, for all $j \in \{1, \ldots, N-1\}$,

$$\lambda_j(G) \le \lambda_j(G_{**}) \le \lambda_{j+1}(G).$$

Writing $\zeta(z_1', z_3') = z_1' - [-4D''(0)]^{-1/2} \langle \xi, (\mathrm{GOE}_{N-1} - z_3' I_{N-1})^{-1} \xi \rangle$, by Schur complement formula, $(G^{-1})_{11} = 1/\zeta$. For $k \ge 1$,

$$\{i(G) = k\} = \{i(G_{**}) = k, \zeta > 0\} \cup \{i(G_{**}) = k-1, \zeta < 0\}.$$

General formula

Using spherical coordinates, $\rho = \frac{\|x\|}{\sqrt{N}}$,

$$\mathbb{E}\mathrm{Crt}_{N,k}(E; R_1, R_2) = S_{N-1} N^{(N-1)/2} \int_{R_1}^{R_2} \int_E \mathbb{E}[|\det G|(\mathbf{1}\{i(G_{**}) = k, \zeta > 0\} + \mathbf{1}\{i(G_{**}) = k-1, \zeta < 0\})] \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(t-m_Y)^2}{2\sigma_Y^2}} \frac{1}{(2\pi)^{N/2} D'(0)^{N/2}} e^{-\frac{N\mu^2\rho^2}{2D'(0)}} \rho^{N-1} \mathrm{d}t\mathrm{d}\rho,$$

where $\lim_{N\to\infty} \frac{1}{N} \log(S_{N-1} N^{\frac{N-1}{2}}) = \frac{1}{2}\log(2\pi) + \frac{1}{2}$, and

$$m_Y = \frac{\mu\rho^2}{2} - \frac{\mu D'(\rho^2)\rho^2}{D'(0)}, \quad \sigma_Y^2 = \frac{1}{N}\Big(D(\rho^2) - \frac{D'(\rho^2)^2\rho^2}{D'(0)}\Big).$$

Simplification

Define $Q = Q(z_3') = [-4D''(0)]^{-1/2}\xi^{\mathsf{T}}(\mathrm{GOE}_{N-1} - z_3' I_{N-1})^{-1}\xi$. By rotational invariance of Gaussians, with Z_i's being i.i.d. $N(0,1)$,

$$Q(z_3') = \frac{\sqrt{-D''(0)}}{N}\sum_{i=1}^{N-1}\frac{Z_i^2}{\lambda_i - z_3'}.$$

By conditioning on $z_3' = y$,

$$\begin{aligned}
&\mathbb{E}[|\det G|(\mathbf{1}\{i(G_{**}) = k, \zeta > 0\}] \\
&= [-4D''(0)]^{\frac{N-1}{2}}\mathbb{E}[|\det(\mathrm{GOE}_{N-1} - z_3' I_{N-1})|\mathbf{1}\{\lambda_k < z_3' < \lambda_{k+1}\} \\
&\quad \mathbb{E}(|z_1' - Q(z_3')|\mathbf{1}\{z_1' - Q(z_3') > 0\}|\lambda_1^{N-1}, z_3', Z_1^{N-1})] \\
&= [-4D''(0)]^{\frac{N-1}{2}}\int_{\mathbb{R}}\mathbb{E}[|\det(\mathrm{GOE}_{N-1} - yI_{N-1})|\mathbf{1}\{\lambda_k < y < \lambda_{k+1}\} \\
&\quad (\mathsf{a}_N\Phi(\frac{\sqrt{N}\mathsf{a}_N}{\mathsf{b}}) + \frac{\mathsf{b}}{\sqrt{2\pi N}}e^{-\frac{N\mathsf{a}_N^2}{2\mathsf{b}^2}})] \\
&\quad \frac{\sqrt{-4ND''(0)}\exp\{-\frac{N(\sqrt{-4D''(0)}y + m_2)^2}{2(-2D''(0) - \beta^2)}\}}{\sqrt{2\pi(-2D''(0) - \beta^2)}}\mathrm{d}y,
\end{aligned}$$

where we used $\mathbb{E}[(a+bz)\mathbf{1}\{a+bz>0\}] = a\Phi(\frac{a}{b}) + \frac{b}{\sqrt{2\pi}}e^{-\frac{a^2}{2b^2}}$, and

$$\mathsf{a}_N = \mathsf{a}_N(\rho,t,y) = W_N(\rho,t,y) - Q(y),$$
$$\mathsf{b}^2 = \mathsf{b}^2(\rho) = \frac{-2D''(0)(-4D''(0)-2\beta^2-\alpha^2\rho^4)}{-2D''(0)-\beta^2}$$

are conditional mean and variance of z_1' given $z_3' = y$, respectively. For any $x \in \mathbb{R}$ and $b > 0$, note that

$$\lim_{N\to\infty} \frac{1}{N} \log\left(x\Phi(\frac{\sqrt{N}x}{b}) + \frac{b}{\sqrt{2\pi N}}e^{-\frac{Nx^2}{2b^2}}\right) = -\frac{(x_-)^2}{2b^2},$$

where $x_- = x \wedge 0$. We want to replace $Q(y)$ with a deterministic quantity in the large N limit.

Reduction: $k = 0$

- Since LDP of empirical measures of GOE has speed N^2, it suffices to consider $y < -\sqrt{2}$.
- For $k = 0$, on $\{y < \lambda_1\}$, we may find Lipschitz function $f_{y,\delta}$ to approximate $\frac{1}{\lambda_i - y}$ and

$$Q(y) = \frac{\sqrt{-D''(0)}}{N} \sum_{i=1}^{N-1} \frac{Z_i^2}{\lambda_i - y} \geq \frac{\sqrt{-D''(0)}}{N} \sum_{i=1}^{N-1} f_{y,\delta}(\lambda_i) Z_i^2 =: \tilde{Q}(y)$$
$$\to \sqrt{-D''(0)}\mathrm{m}(y)$$

  where $\mathrm{m}(y)$ is the Stieltjes transform of semicirle law.
- Then use modified Gärtner–Ellis Theorem for large deviations for left tail of $Q(y)$.
- Use many times the concentration/deviation inequality of empirical measure of λ_i on the scale of N^2 from σ_{sc} due to Maida and Maurel-Segala 2014.

Concluding remark for $k = 0$

Write

$$\mathsf{a}_N \to W(\rho, t, y) - \sqrt{-D''(0)}\mathsf{m}(y) =: \mathsf{a}.$$

Recall

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_{N,0}(E, (R_1, R_2)) = \frac{1}{2}\log[-4D''(0)] - \frac{1}{2}\log D'(0) + \frac{1}{2} + \sup_{(\rho,t,y)\in F} [\psi(\sigma_{\mathrm{sc}}, \rho, t, y) - \mathcal{I}^-(\rho, t, y)].$$

Here

$$\mathcal{I}^-(\rho, t, y) = \inf_{x\in[0,\mathsf{m}(y)]} \Lambda^*(x) + \frac{1}{2\mathsf{b}^2}[(\mathsf{a} + \sqrt{-D''(0)}x)_-]^2,$$

$\Lambda^*(x)$ is the Fenchel–Legendre transform of Λ, the limiting log moment generating function of $\tilde{Q}(y)$.

Reduction: $k \geq 1$

- For $k \geq 1$, the more difficulty term is $\mathbb{E}[|\det G|(\mathbf{1}\{i(G_{**}) = k-1, \zeta < 0\}]$.
- On $\{\lambda_{k-1} < y < \lambda_k\}$, we may find Lipschitz function f_y to approximate $\frac{1}{\lambda_i - y}$ and

$$Q(y) = \frac{\sqrt{-D''(0)}}{N} \sum_{i=1}^{N-1} \frac{Z_i^2}{\lambda_i - y} \leq \frac{\sqrt{-D''(0)}}{N} \sum_{i=k}^{N-1} f_y(\lambda_i) Z_i^2 =: \hat{Q}(y)$$
$$\to \sqrt{-D''(0)}\mathrm{m}(y)$$

  where $\mathrm{m}(y)$ is the Stieltjes transform of semicirle law.
- Then use modified Gärtner–Ellis Theorem for large deviations for right tail of $Q(y)$.
- Extra work to handle boundary case, as Λ is not steep.

Concluding remark for $k \geq 1$

Recall

$$\{i(G) = k\} = \{i(G_{**}) = k, \zeta > 0\} \cup \{i(G_{**}) = k-1, \zeta < 0\}.$$

and

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_{N,k}(E,(R_1,R_2)) = \frac{1}{2}\log[-4D''(0)] - \frac{1}{2}\log D'(0) + \frac{1}{2}$$
$$+ \max\Big\{ \sup_{(\rho,t,y)\in F} [\psi(\sigma_{\mathrm{sc}},\rho,t,y) - kJ_1(y)],$$
$$\sup_{(\rho,t,y)\in F} [\psi(\sigma_{\mathrm{sc}},\rho,t,y) - \mathcal{I}^+(\rho,t,y) - (k-1)J_1(y)]\Big\}.$$

Here

$$\mathcal{I}^+(\rho,t,y) = \inf_{x<0} \Lambda^*(x) + \frac{1}{2\mathsf{b}^2}[(\mathsf{a} + \sqrt{-D''(0)}x)_+]^2.$$

Some general strategy

- Prove exponential tightness first:

$$\limsup_{T\to\infty} \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N([-T,T]^c, \mathbb{R}_+) = -\infty,$$

$$\limsup_{R\to\infty} \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N(\mathbb{R}, [0,R]) = -\infty.$$

  This allows considering just $\bar{E}$ compact and $R_2 < \infty$.
- Use LDP for empirical measure $L(\lambda_{i=1}^N)$ of GOE with speed N^2 to consider just $L(\lambda_{i=1}^N) \in B(\sigma_{\mathrm{sc}}, \delta)$.
- Use $\mathbb{P}(\max|\lambda_i| > K) \le e^{-NK^2/9}$ to consider only $\max|\lambda_i| \le K$ for K large enough.

A simple idea of all arguments

$$\lim_{N\to\infty} \frac{1}{N} \log(e^{-10N} \pm e^{-100N} \pm e^{-N^2}) = -10.$$

A simple idea of all arguments

$$\lim_{N\to\infty} \frac{1}{N} \log(e^{-10N} \pm e^{-100N} \pm e^{-N^2}) = -10.$$

Thanks for your attention!

$$\psi(\sigma_{\rm sc},\rho,t,x):=\Psi_*(x)-\frac{(t-\frac{\mu\rho^2}{2}+\frac{\mu D'(\rho^2)\rho^2}{D'(0)})^2}{2(D(\rho^2)-\frac{D'(\rho^2)^2\rho^2}{D'(0)})}-\frac{\mu^2\rho^2}{2D'(0)}+\log\rho$$

$$-\frac{-2D''(0)}{-2D''(0)-\frac{[D'(\rho^2)-D'(0)]^2}{D(\rho^2)-\frac{D'(\rho^2)^2\rho^2}{D'(0)}}}$$

$$\times\Big(x+\frac{1}{\sqrt{-4D''(0)}}\Big[\mu+\frac{(t-\frac{\mu\rho^2}{2}+\frac{\mu D'(\rho^2)\rho^2}{D'(0)})(D'(\rho^2)-D'(0))}{D(\rho^2)-\frac{D'(\rho^2)^2\rho^2}{D'(0)}}\Big]\Big)^2,$$

$$\Psi_*(x):=\int\log|x-y|\mathrm{d}\sigma_{\rm sc}(y)$$

$$=\begin{cases}\frac12x^2-\frac12-\frac12\log 2, & |x|\le\sqrt2,\\ \frac12x^2-\frac12-\log 2-\frac12|x|\sqrt{x^2-2}+\log(|x|+\sqrt{x^2-2}), & |x|>\sqrt2.\end{cases}$$