Superposition of functions on monotonic functions.

By

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In this note the domain and range of functions is the interval $I=\langle 0,1\rangle$. A function f is said to have the Baire property (Baire property in the restricted sense) if $f^{-1}(F)$ has the Baire property (Baire property in the restricted sense) for each closed set F [1]). Let B_r denote the class of sets having the Baire property in the restricted sense.

Our principal result is contained in the following

Theorem. *Let $g(x)$ be a general monotonic function. Then if $f(y)$ has the Baire property in the restricted sense, $f[g(x)]$ has the Baire property in the restricted sense.*

A monotonic function $g(x)$ is continuous except at a denumerable set of points $A=\{a_i\}$. Let Y denote the set $g(I)$. Except over a denumerable set of closed intervals $\{I_n\}$ on the x-axis, the function $g(x)$ has a single-valued inverse.

Let $K=A+I_1+I_2+\ldots$; $H=I-K$; $h=g/H$ (i.e. the partial function defined over H); $K^*=g(K)$; $H^*=g(H)=h(H)$.

The sets H and H^* are G_δ sets and h is a homeomorphism between H and H^*. The set K^* is denumerable.

Suppose that $f(y)$ has the Baire property in the restricted sense and let $\varphi(x)=f[g(x)]$. Then $\varphi^{-1}(F)=g^{-1}[f^{-1}(F)]=g^{-1}(B)$ where $B\epsilon B_r$, and $g^{-1}(B)=g^{-1}(BH^*+BK^*)=g^{-1}(BH^*)+g^{-1}(BK^*)=$ $=h^{-1}(BH^*)+g^{-1}(BK^*)$.

[1]) C. Kuratowski, *Topologie I*, Monogr. Matem., Warszawa-Lwów 1933, pp. 191, 194.

Since h is a homeomorphism and $BH^*\epsilon B_r$, hence $h^{-1}(BH^*)\epsilon B_r$ [1]), since BK^* is denumerable and g monotonic then $g^{-1}(BK^*)$ is an F_σ. Thus $\varphi^{-1}(F)=g^{-1}(B)\epsilon B_r$.

Remarks. Let $g(x)$ be a general monotonic function. Consider now two cases: 1) $f(y)$ fails to have the Baire property, 2) $f(y)$ has the Baire property only in the large sense. In both cases $f[g(x)]$ can: a) have the Baire property in the restricted sense, or b) have the Baire property only in the large sense, or c) fail to have the Baire property.

1^0 For instance, let $f(y)$ be a function which has the Baire property in the restricted sense on the interval $0<y<1/3$, which has the Baire property on $1/3<y<2/3$, and which does not have the Baire property on $2/3<y<1$. Then if $g(x)=x$, $f[g(x)]$ does not have the Baire property. If $g(x)=(1/3)(1+x)$, $f[g(x)]$ has the Baire property. If $g(x)=(1/3)x$, $f[g(x)]$ has the Baire property in the restricted sense.

2^0 Consider an $f(y)$ which has the Baire property. For instance, let $f(y)$ be a function which has the Baire property only in the large sense on the interval $0<y<1/2$ and the Baire property in the restricted sense on the interval $1/2<y<1$. Then if $g(x)=x$, $f[g(x)]$ has the Baire property only in the large sense. If $g(x)=(1/2)(1+x)$, $f[g(x)]$ has the Baire property in the restricted sense.

Consider a monotonic function which transforms the set $(0,1)$ into some subset of Cantor's non-dense set. Consider any set A on the x-axis not having the Baire property. This set is transformed into a set A' on the y-axis, which is a non-dense set since it is a subset of Cantor's non-dense set. For $f(y)$ take the characteristic function of A', then $f(y)$ has the Baire property and $f[g(x)]$ has not the Baire property.

[1]) C. Kuratowski, l.c., top of pg. 56.