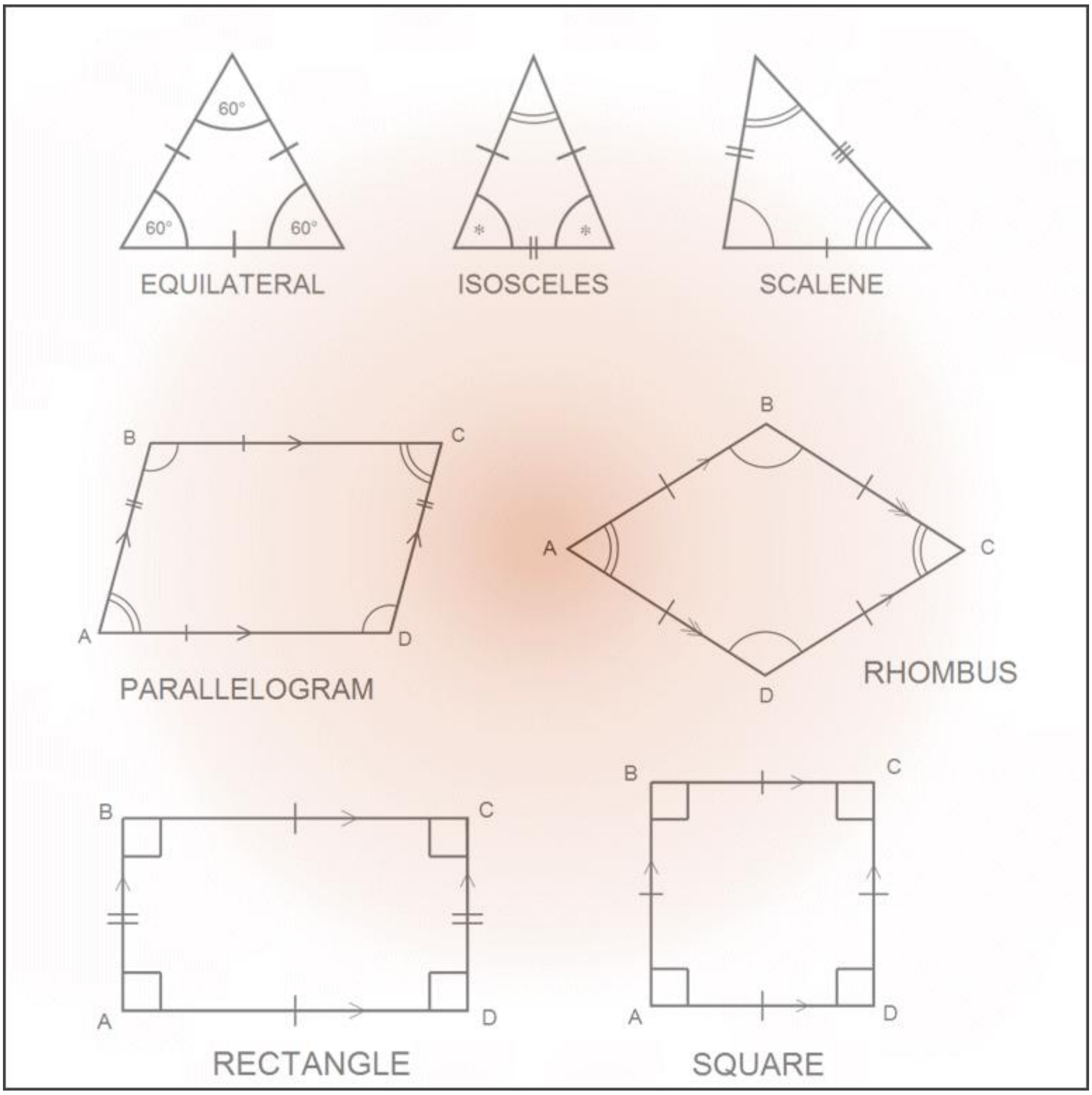
M.K. HOME TUITION

Mathematics Revision Guides
Level: GCSE Higher Tier

PROPERTIES OF TRIANGLES AND QUADRILATERALS

(plus polygons in general)

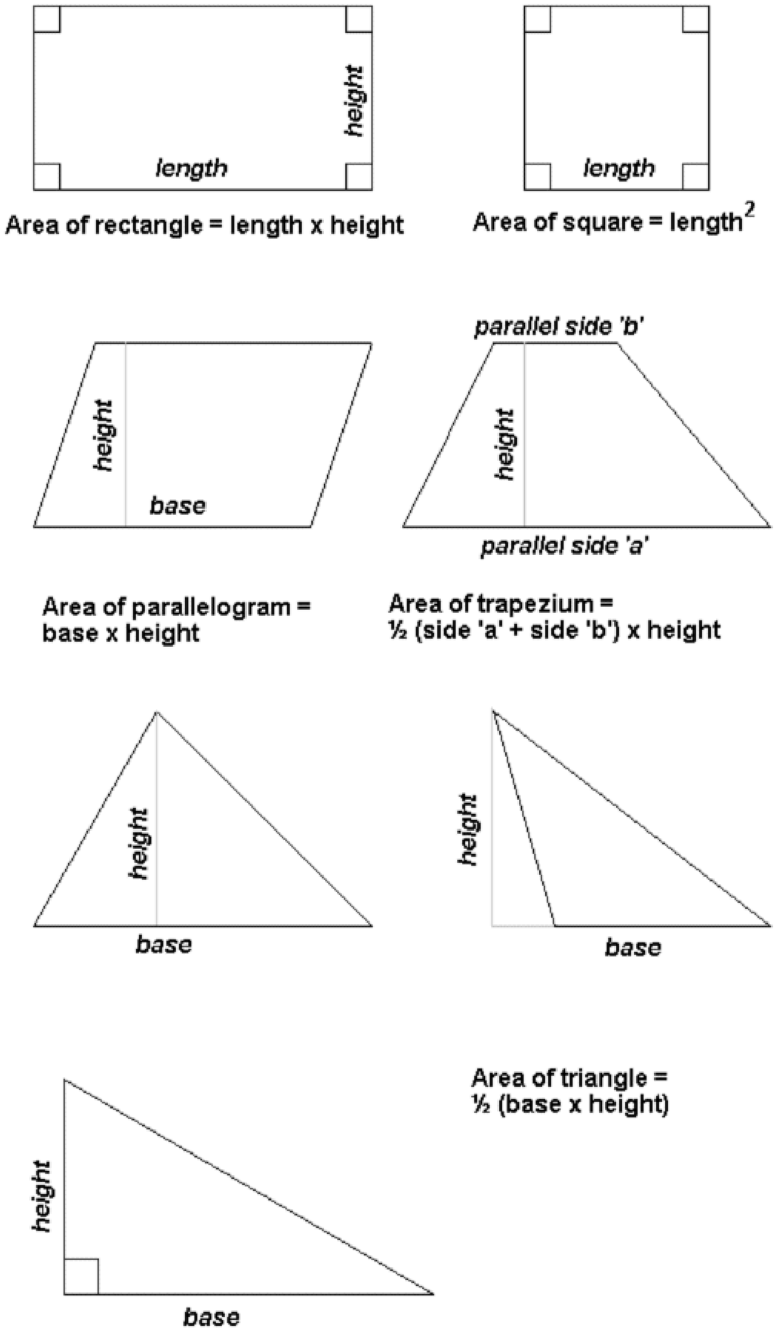


Version: 2.32 Date: 22-10-2016

PROPERTIES OF TRIANGLES AND QUADRILATERALS

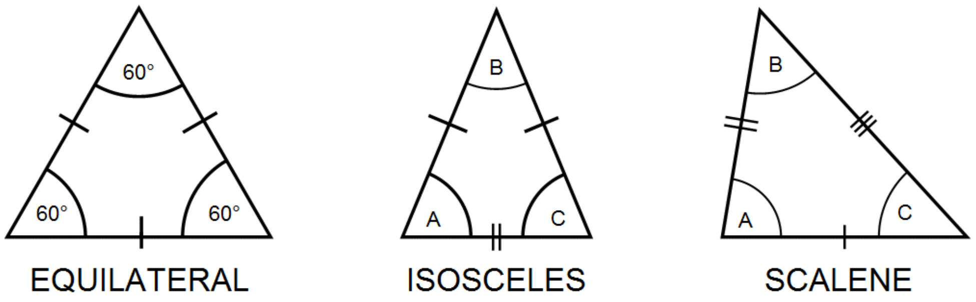
Revision.

Here is a recap of the methods used for calculating areas of quadrilaterals and triangles. There are more examples in the Foundation Tier document “Measuring Shapes”.



Types of triangles.

Triangles can be classified in various ways, based either on their symmetry or their angle properties.



An **equilateral** triangle has all three sides equal in length, and all three angles equal to 60°.

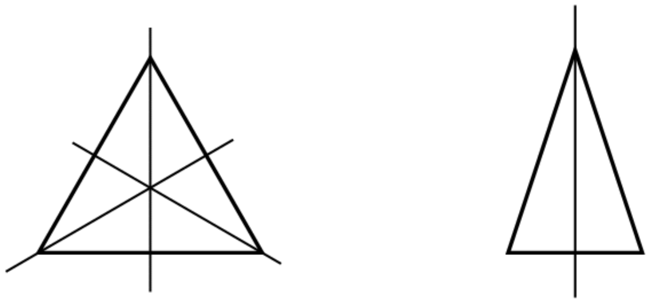
An **isosceles** triangle has two sides equal, and the two angles opposite the equal sides also the same. Thus in the diagram, angle A = angle C.

A **scalene** triangle has all three sides and angles different.

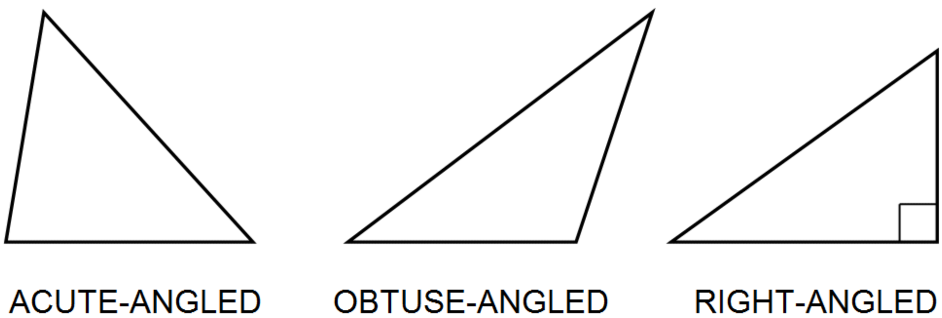
All triangles have an interior angle sum of 180°.

A scalene triangle has no symmetry at all, but an isosceles triangle has one line of symmetry.

An equilateral triangle has three lines of symmetry, and rotational symmetry of order 3 in addition.



Another way of classifying triangles is by the types of angle they contain.

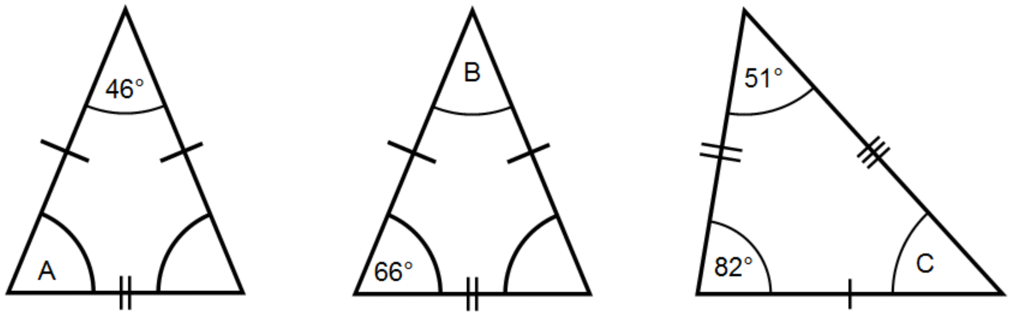


An acute-angled triangle has all its angles less than 90°. A right-angled triangle has one 90° angle, and an obtuse-angled triangle has one obtuse angle.

Because the sum of the internal angles of any triangle is 180°, it follows that no triangle can have more than one right angle or obtuse angle.

Example (1):

i) Find the angles A, B and C in the triangles below.



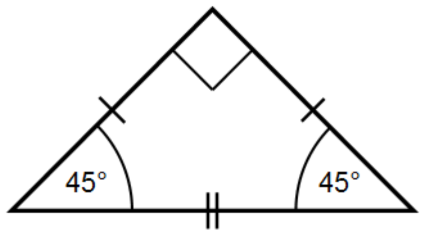
ii) Is it possible for a right-angled triangle to be isosceles? Explain with a sketch.

i) In the first triangle, the given angle is the 'different' one, so therefore the two equal angles must add to (180-46)°, or 134°. Angle A = half of 134°, or 67°.

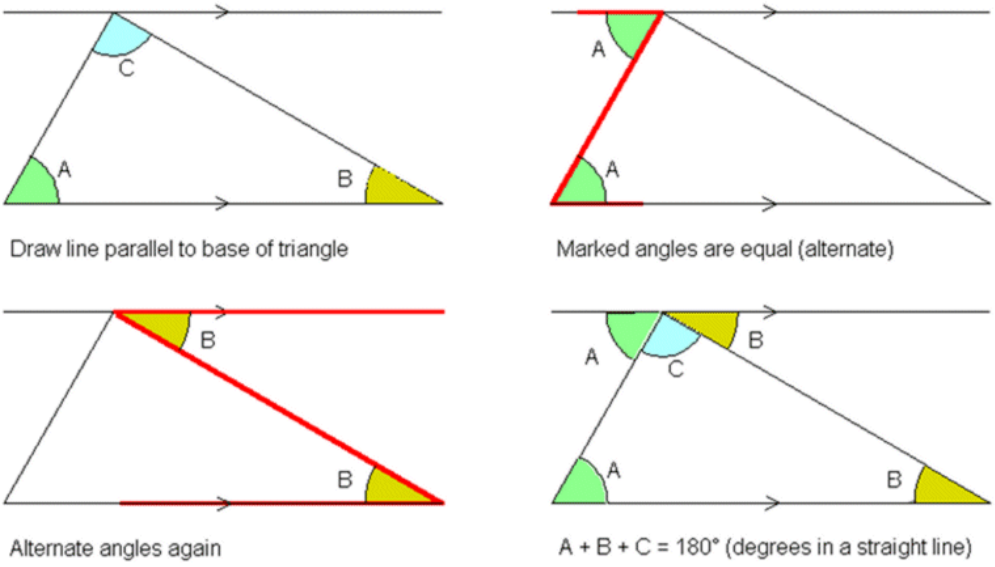
In the second triangle, the 66° angle is one of the two equal ones.
Hence angle B = 180° - (66 + 66)°, or 48°.

The third triangle is scalene, so angle C = 180° - (51 + 82)° = 47°.

ii) Although no triangle can have two right angles, it is perfectly possible to have an isosceles right-angled triangle. Two such triangles can be joined at their longest sides to form a square, and this shape of triangle occurs in a set square of a standard geometry set.



Here is the proof that the interior angles of any triangle add up to 180°.



The exterior angle sum of a triangle.

Since an exterior angle is obtained by extending one of the sides, it follows that the sum of an interior angle and the resulting exterior angle is 180°. See the diagram.

The interior angle-sum, A + B + C, = 180°.

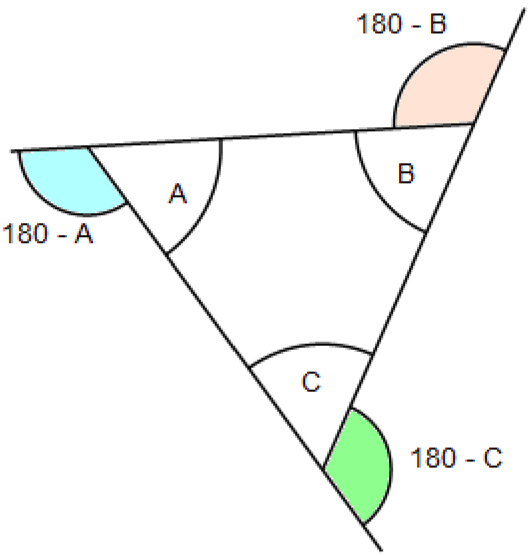
The exterior angle sum is

((180 - A) + (180 – B) + (180 – C))°

or (540 – (A + B + C)) °

or 540° - 180° = 360°.

∴ The exterior angle sum of a triangle is 360°.



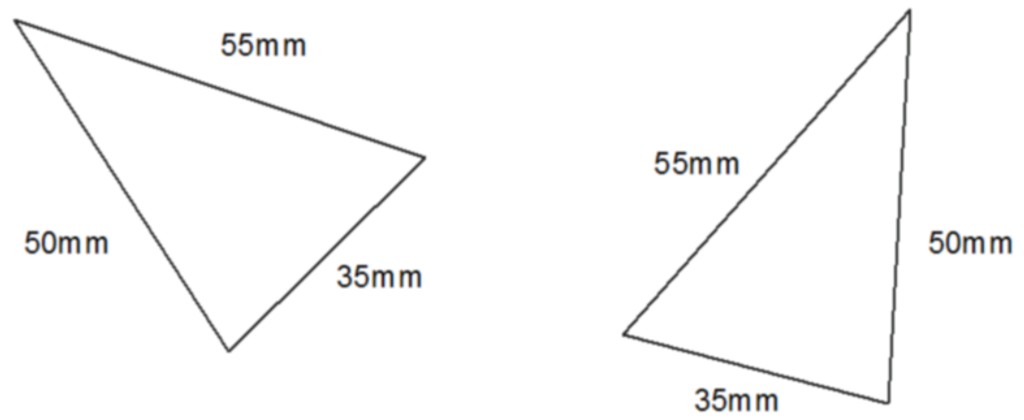
Congruent Triangles.

Two triangles are said to be congruent if they are equal in size and shape.

In other words, if triangle ***B*** can fit directly onto triangle ***A*** by any combination of the three standard transformations (translation, rotation, reflection), then triangles ***A*** and ***B*** are congruent.

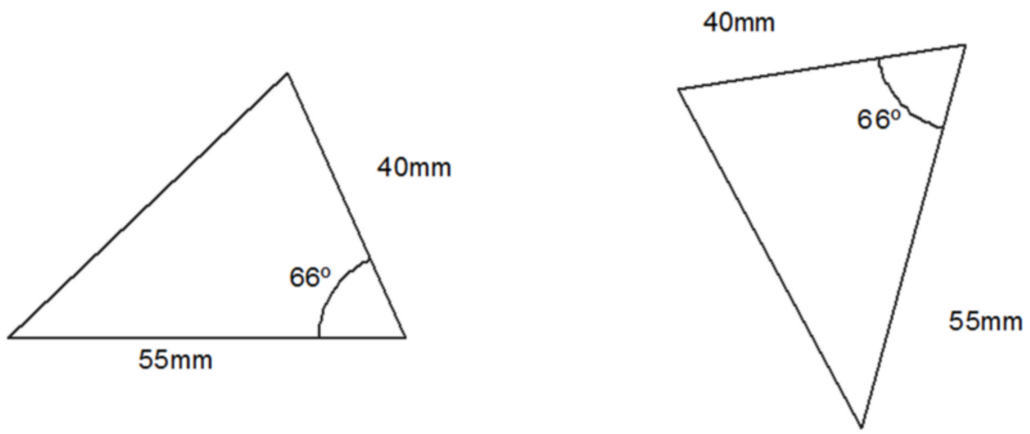
Conditions for congruency.

3 sides (Side – Side – Side or SSS)



If three sides of one triangle are equal to the three sides of the other, then the triangles are congruent. (Note that the triangles are mirror images of each other, but they are still congruent.)

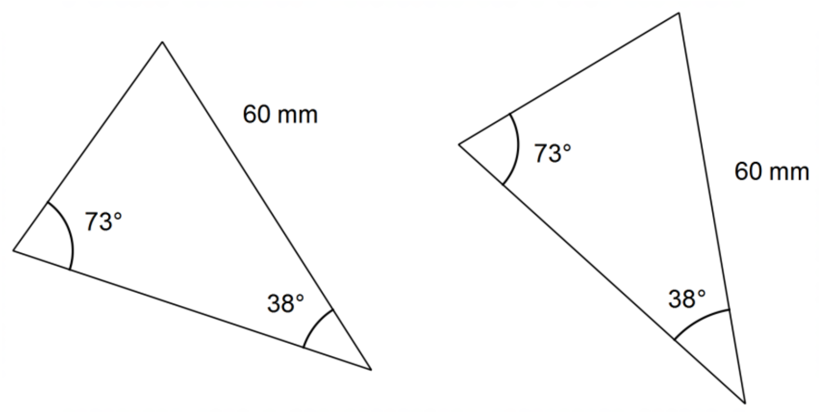
2 sides and the included angle (Side – Angle – Side or SAS)



If two sides and the ***included*** angle of one triangle are equal to two sides and the ***included*** angle of the other, then the triangles are congruent.

(The included angle is simply the angle between the two given sides).

2 angles and a corresponding side (Angle – Angle – Side or AAS)



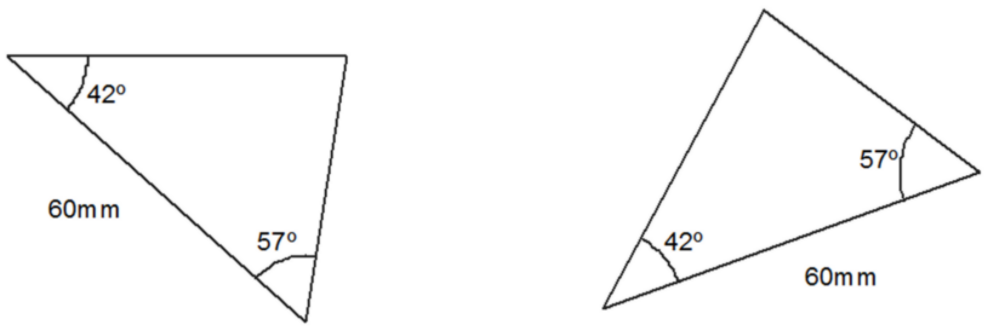
If two angles and a ***corresponding*** side of one triangle are equal to two angles and a ***corresponding*** side of the other, then the triangles are congruent.

Here, the 60 mm sides correspond, as each one is opposite the 73° angle.

The variation below is the "**ASA**" form where two angles and the ***included*** side of one triangle are equal to two angles and the ***included*** side of the other.

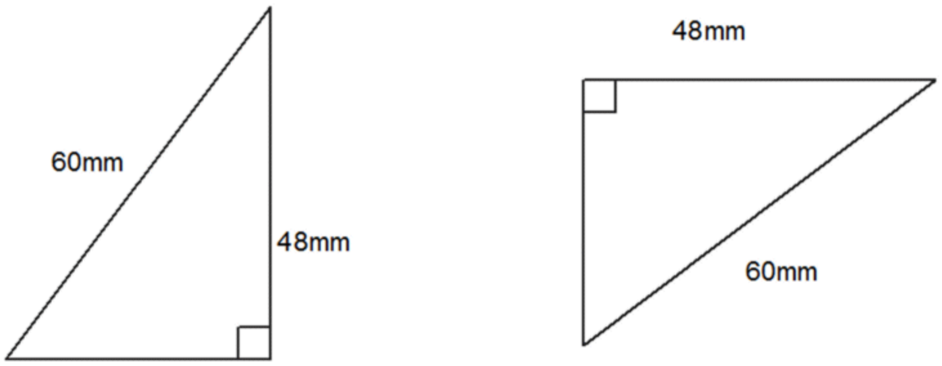
We could also say that the sides correspond, as each one is opposite the unmarked angle (here 81° by subtraction).

Of course, if *two* angles of each triangle are equal, then the third angle must also be equal.



Hypotenuse and one other side (Right Angle – Hypotenuse – Side or RHS).

(Applies only to right-angled triangles)

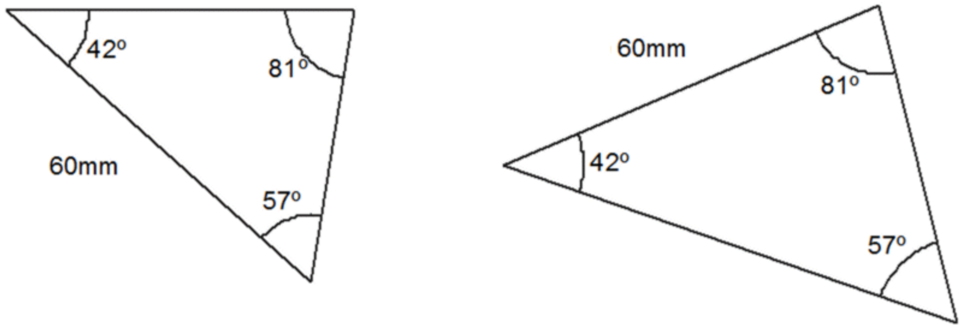


If the hypotenuse and one side of one right-angled triangle are equal to the hypotenuse and one side of the other, then the triangles are congruent.

Conditions insufficient for congruency.

The four conditions above guarantee congruency of triangles, but the following do not:

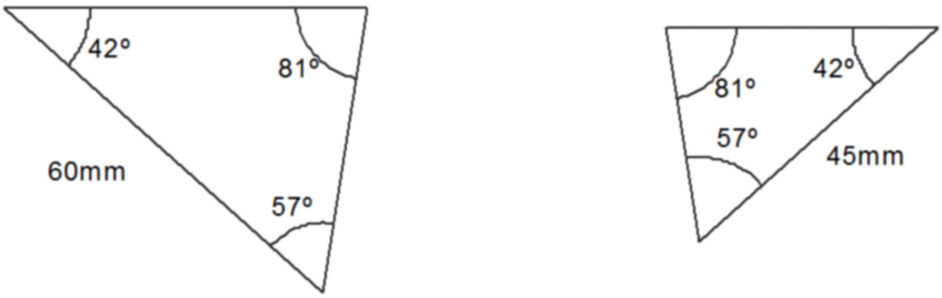
2 angles and a non-corresponding side



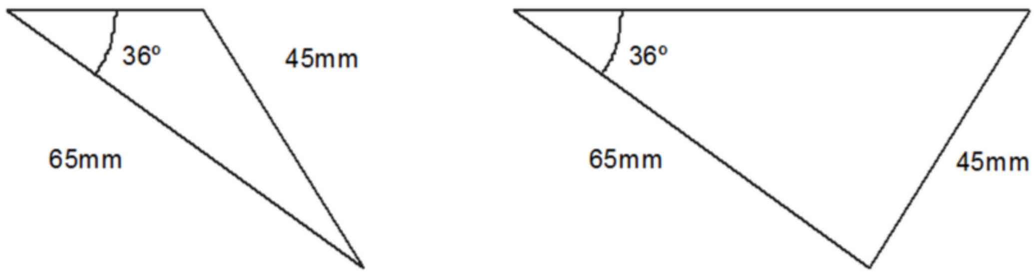
The two triangles above are **not congruent** because the 60mm sides do not correspond. In the left-hand triangle, the 60mm side is opposite the 81° angle, but in the right-hand triangle, the 60mm side is opposite the 57° angle. The triangles do have the same shape, and are therefore **similar.**

3 angles

This is a generalisation of the above case - the sides opposite the 81° angle are not equal . The triangles have the same shape, therefore they are **similar**, but not congruent.



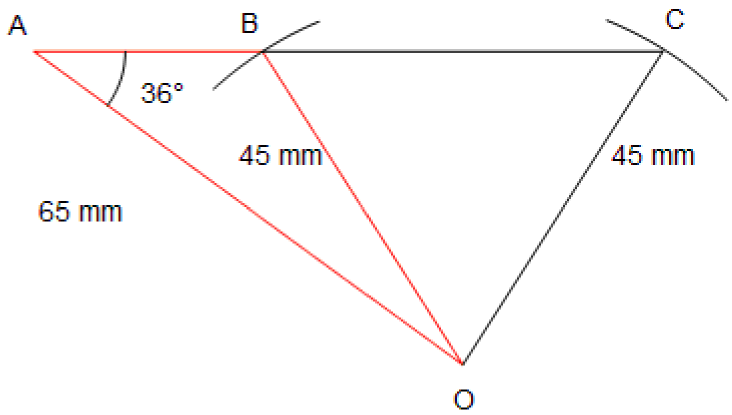
An angle *not* included and 2 sides



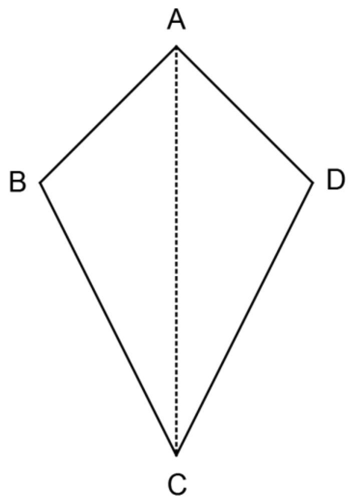
The triangles above have two sides equal at 45mm and 65mm, but the angle of 36° is crucially **not the included** angle. They are emphatically not congruent - in fact they are not even similar.

Angle-**S**ide-**S**ide does not guarantee congruency – so don't be an **ASS** !

The reason is as follows: when attempting to construct the 45mm side, the compass arc centred on O cuts the third side either at B or C, giving two possible constructs – namely triangles OAB and OAC.



Example (2a): ABCD is a kite. (See the section on quadrilaterals later in the document.)



Prove that the diagonal AC divides the kite into two congruent triangles ABC and ADC.

A kite has adjacent pairs of sides equal,
so AB = AD and BC = CD.

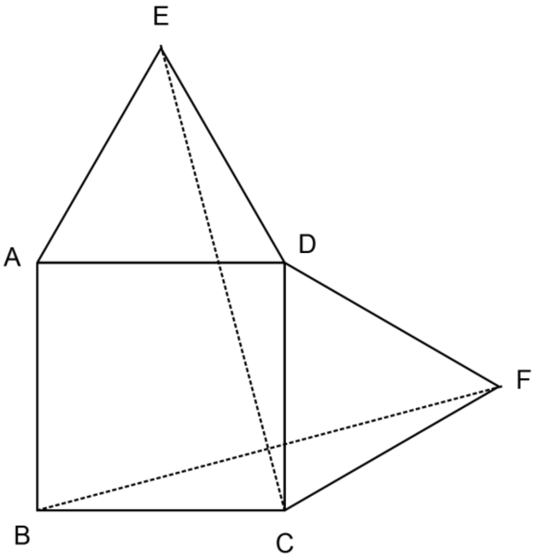
Also, the diagonal of this kite, AC, is common to both triangles.

Hence three sides of triangle ABC = three sides of triangle ADC.

$\therefore$ **triangles ABC and ADC are congruent (SSS) .**

Example (2b): The illustrated figure is made up of a square and two equilateral triangles, each of which has a side length of one unit.

Prove that triangles CDE and BCF are congruent.



The interior angle of a square is 90° and the interior angle of an equilateral triangle is 60°.

$\therefore$ $\angle$ EDC = $\angle$ BCF = 90° + 60° = 150°.

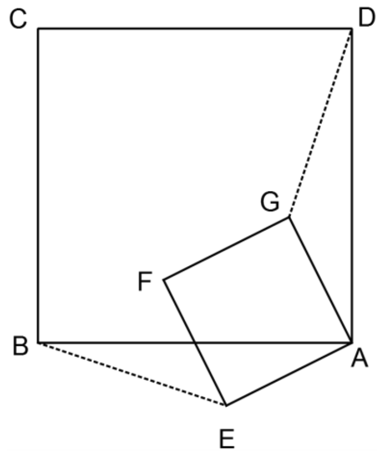
Also, BC = CF = 1 unit and CD = DE = 1 unit.

Hence two sides and the included angle of triangle CDE are equal to two sides and the included angle of triangle BCF.

$\therefore$ **triangles CDE and BCF are congruent (SAS) .**

Example (2c) : Figures ABCD and AEFG are squares.

Prove, using congruent triangles, that DG = EB.



The triangles in question are AEB and AGD.
Now, AG = AE and AD = AB, so two sides of one triangle are equal to two sides of the other.

Let the included angle BAE of triangle AEB = x°.

Because $\angle$ BAE + $\angle$ BAG = $\angle$ EAG = 90°,
$\angle$ BAG = (90-x)°.

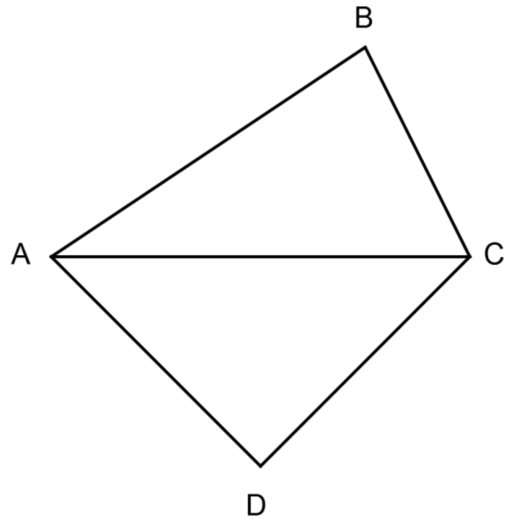
Also, $\angle$BAD = 90°, so $\angle$ DAG = 90° - $\angle$ BAG,
or 90° - (90 – x)° = x°.

$\therefore$ the included angles BAE and DAG are both equal to x°.

Two sides and the included angle of triangle AEB are equal to two sides and the included angle of triangle AEG.

$\therefore$ **triangles AEB and AEG are congruent (SAS) .**

Example (2d) : ABCD is a quadrilateral divided into two triangles by the diagonal AC. Given that the areas of the triangles ABC and ADC are equal, prove that diagonal AC bisects diagonal BD.

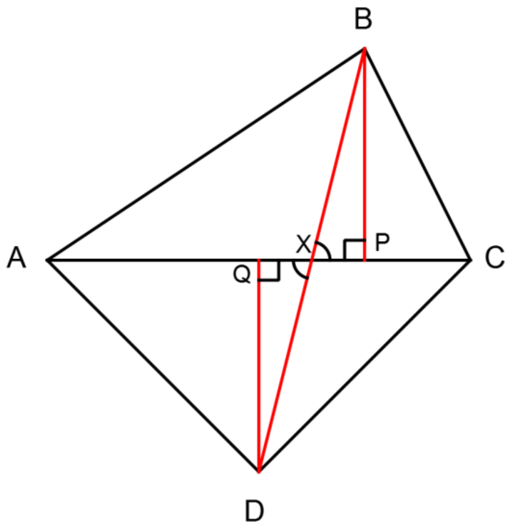


The two triangles ABC and ADC share the same base AC, and because their areas are equal, their perpendicular heights must also be equal.

Recall the triangle area formula:

Area = ½ (base) × (height) rearranged to

Height = (2 × area) ÷ (base)



We draw a perpendicular from B to the base AC, meeting at P, and another from D to the base, meeting at Q.
The perpendiculars are equal in length, i.e. BP = DQ, and ∠ BPX = ∠ DQX = 90°.

Next we draw the diagonal BD, forming two right-angled triangles BPX and DQX.

Because ∠ BXP and ∠ DXQ are vertically opposite, they are equal. Each angle is also opposite the corresponding perpendicular.

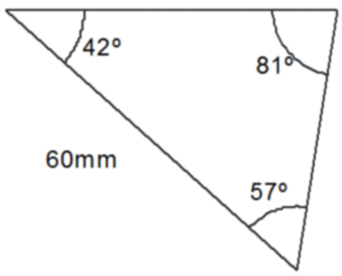
Triangles BPX and DQX are therefore congruent (AAS = AAS) and so BX = XD.

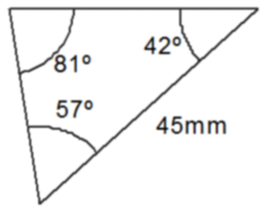
∴ **Diagonal AC bisects diagonal BD.**

Similar triangles.

Two figures are said to be **similar** if they have the same shape, regardless of actual size, so the conditions are less strict than those for congruency.

The two triangles below are similar because the angles are equal, despite the difference in side lengths.

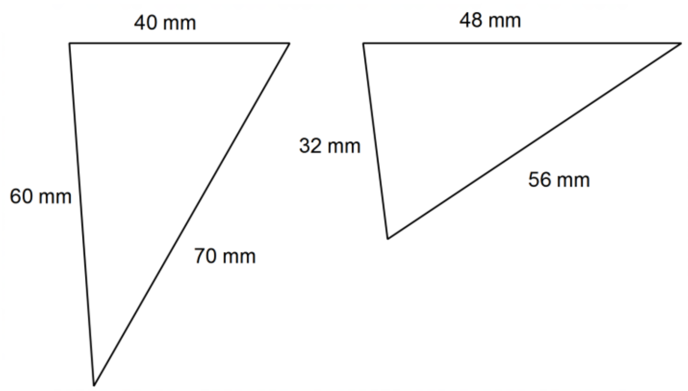




If three angles of one triangle equal three angles of the other, then the triangles are similar.

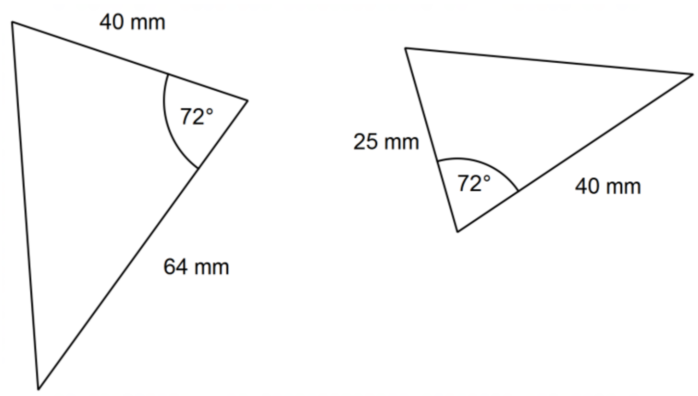
The two triangles below have their side lengths in the same ratio.

The ratio 40 : 60 : 70 can be simplified to 4 : 6: 7 if we divide the side lengths in millimetres by 10, and so can the ratio 32 : 48 : 56 if we divide them by 8.



If three sides of one triangle are in the same ratio as three sides of the other, then the triangles are similar.

The triangles below have two sides and their included angle stated, and the two stated sides are in the same ratio because the ratios 40 : 64 and 25 : 40 can both be simplified to 5 : 8 by division. The triangles also have the same included angle.



If two sides of one triangle are in the same ratio as two sides of the other, and the included angles between the two pairs of sides are the same, then the triangles are similar.

Congruency and similarity can also be defined in terms of transformations.

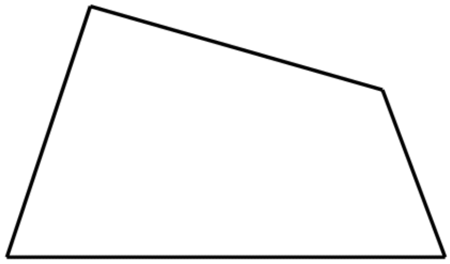
If shape *A* can be transformed to shape *B* by translation, rotation or reflection (either singly or combined) , then figures *A* and *B* are **congruent.**

If shape *A* can be transformed to shape *B* by translation, rotation, reflection or enlargement (either singly or combined) , then figures *A* and *B* are **similar.**

These definitions hold true for all shapes, not just triangles.

Introduction to quadrilaterals.

Quadrilaterals are plane figures bounded by four straight sides, and they too can be classified into various types, based mainly on symmetry and the properties of their sides and diagonals.



QUADRILATERAL

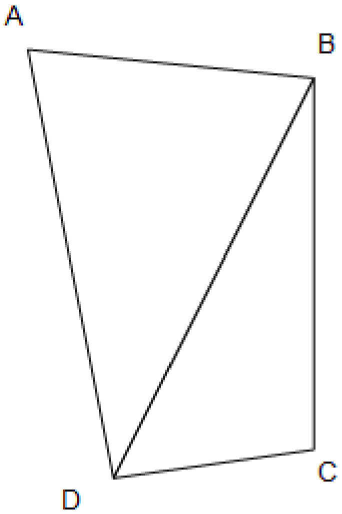
A general quadrilateral need not have any symmetry, nor any parallel sides. Its angles and sides could also all be different.

Angle sum of a quadrilateral.

The square and rectangle have all their interior angles equal to 90°, and therefore their interior angle sum is 360°.

This holds true for all quadrilaterals, because *any* quadrilateral can be divided into two triangles by one of its diagonals, e.g. quadrilateral ABCD can be divided into the two triangles ABD and BCD by adding diagonal BD.

The interior angle sum of a triangle is 180°,and therefore the interior angle sum of a quadrilateral is 360°.



To find the exterior angle sum of a *convex* quadrilateral, we use the same reasoning as that for the triangle.

The interior angle-sum, A + B + C + D, = 360°.

The exterior angle sum is

((180 - A) + (180 – B) + (180 – C) + (180 – D))°

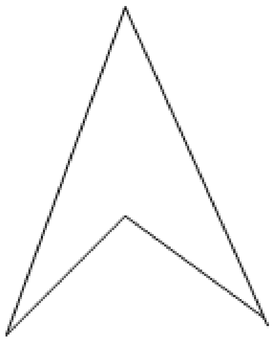
or (720 – (A + B + C + D)) °

or 720° - 360° = 360°.

∴ **The exterior angle sum of a convex quadrilateral is 360°.**

Note that the idea of an exterior angle is meaningless when we are dealing with a reflex angle.

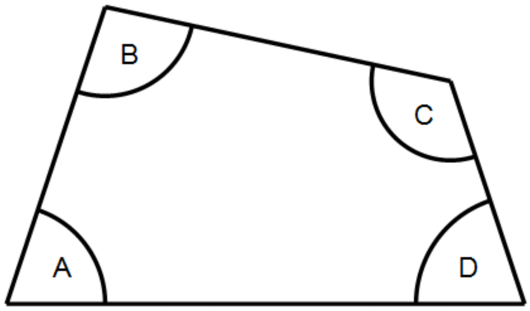
A concave quadrilateral, such as the example shown right, has a reflex angle, and if it also has a line of symmetry, it is known as an arrowhead or delta.



CONCAVE QUADRILATERAL

This same argument can be used to show that the exterior angle-sum of *any* convex polygon is 360°.

Example (3): Find angle C in this quadrilateral, given that angle A = 70°, angle B = 96° and angle D = angle A.

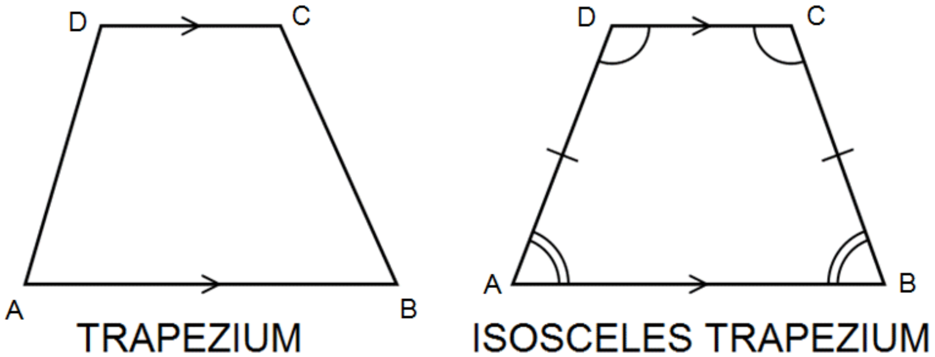


C = 360 ° – (70 + 96 + 70)°
= (360 – 236)° = 124°.

We shall now start to examine the properties of some more special quadrilaterals.

The trapezium.

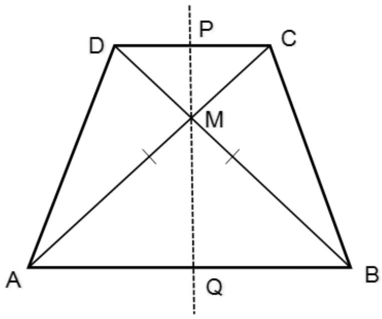
If one pair of sides is parallel, the quadrilateral is a **trapezium**.



A trapezium whose non-parallel sides are equal in length is an **isosceles trapezium.**
In the diagram, sides BC and AD are equal in length.
An isosceles trapezium also has adjacent pairs of angles equal.
Thus, angles A and B are equal, as are angles C and D.

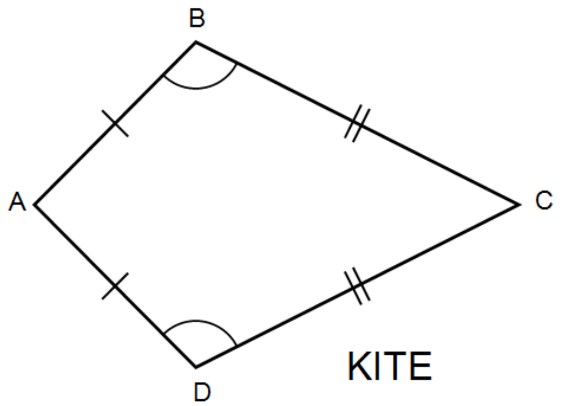
Furthermore, the diagonals of an isosceles trapezium are also equal in length, thus AC = BD. There is also a line of symmetry passing through the midpoints P and Q of the parallel sides as well as point M, the intersection of the diagonals.

(Although, in this example, the diagonals appear to meet at right angles, this is not generally true.)

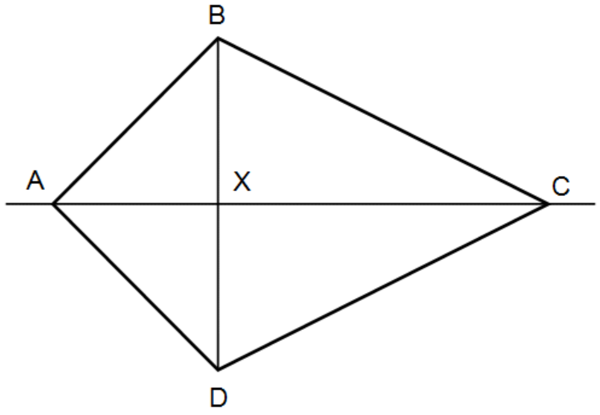


The kite.

A kite has one pair of opposite angles and both pairs of adjacent sides equal. In the diagram, sides AB and AD are equal in length, as are CB and CD. Also, angles B and D are equal.



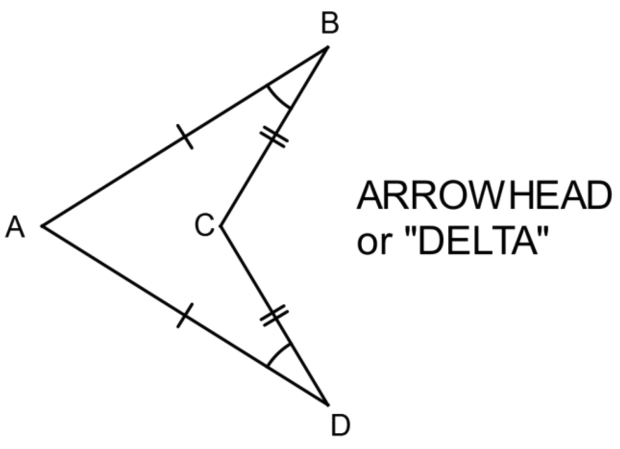
A kite is symmetrical about one diagonal, which bisects it into two mirror-image congruent triangles. In this case, the diagonal AC is the line of symmetry, and triangles ABC and ADC are thus congruent. Hence AC also bisects the angles BAD and BCD, BX = XD, and the diagonals AC and BD intersect at right angles.



The arrowhead or "delta".

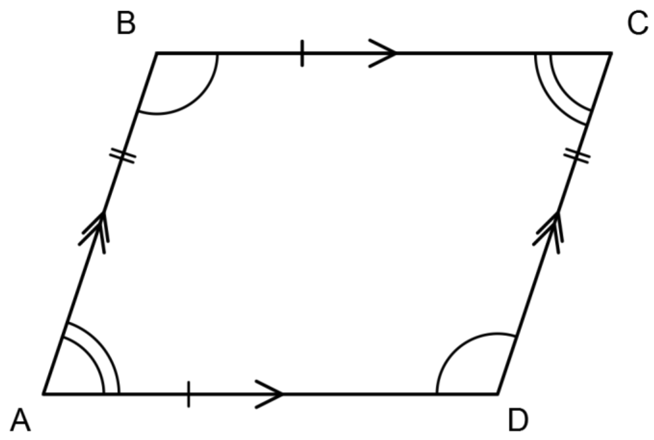
This quadrilateral has one reflex angle (here BCD), but shares all the other properties of the kite, although one of the "diagonals" (here BD) lies outside the figure.

This is an example of a concave quadrilateral.



The parallelogram.

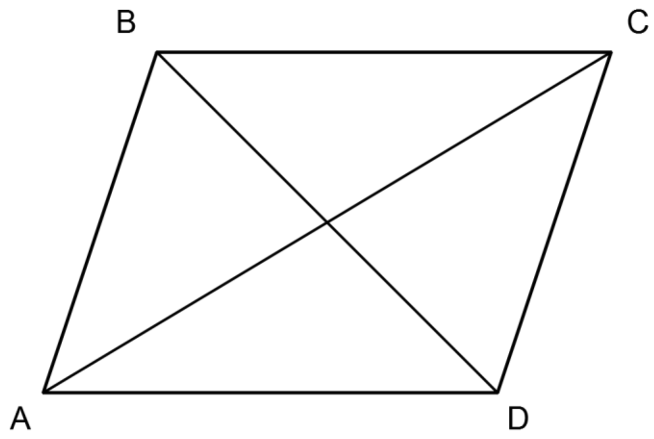
Whereas a trapezium has at one pair of sides parallel, a **parallelogram** has both pairs of opposite sides equal and parallel, as well as having opposite pairs of angles equal. Thus, in the diagram, side AD = side BC, and side AB = side CD. Additionally, angle A = angle C, and angle B = angle D.



PARALLELOGRAM

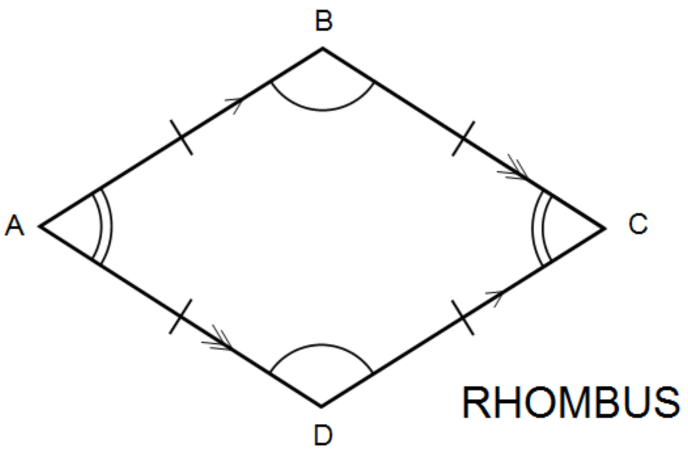
The diagonals of a parallelogram bisect each other, so in the diagram below, AM = MC and BM = MD. We also have rotational symmetry of order 2 about the point M, the midpoint of each diagonal.

Note however that a parallelogram does not generally have a line of symmetry.



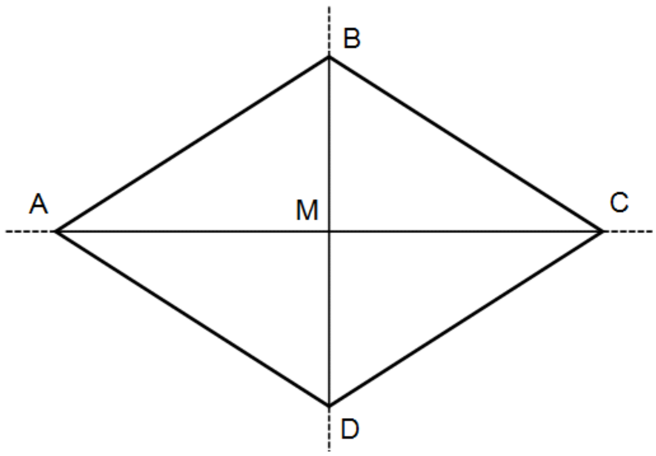
The rhombus.

A **rhombus** has all the properties of a parallelogram, but it also has all its sides equal in length.
Hence here, AB = BC = CD = DA.



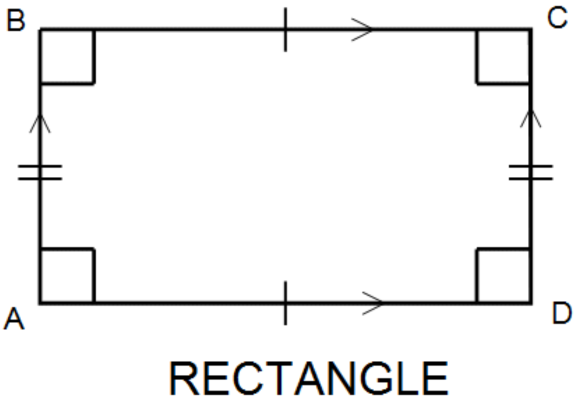
Although a parallelogram does not as a rule have any lines of symmetry, a rhombus has two lines of symmetry coinciding with the diagonals, as well as order-2 rotational symmetry.

Thus, triangles ABC and ACD are congruent, as are triangles BAD and BCD.
The diagonals bisect each other at right angles, so AM = MC and BM = MD.
The diagonals also bisect the angles of the rhombus.



The rectangle.

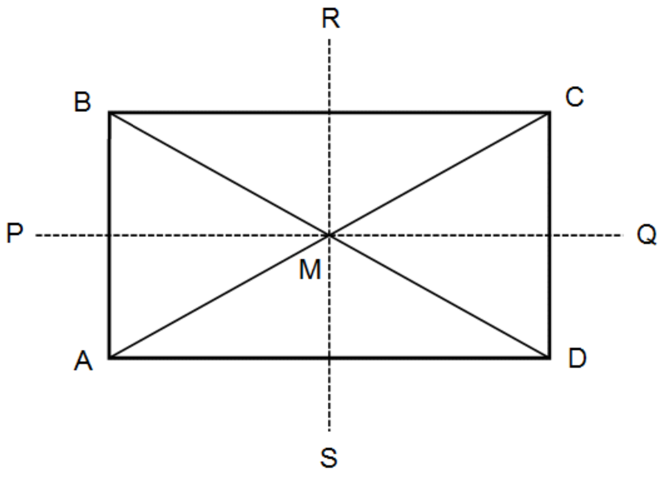
A **rectangle** also has all the properties of a parallelogram, but it additionally has all its four angles equal to 90°. Hence angles A, B, C and D are all right angles.



As stated earlier, a parallelogram does not generally have any lines of symmetry, but a rectangle has two lines of symmetry passing through lines joining the midpoints of opposite pairs of sides, i.e. PQ and RS in the diagram. A rectangle also has order-2 rotational symmetry.

The diagonals of a rectangle are equal in length and bisect each other, therefore AC = BD, and AM = MC = BM = MD. They do not generally bisect at right angles, though.

Thus, triangles AMB and CMD are congruent, as are triangles BMC and AMD.



It can be seen that both the rectangle and the rhombus are not quite ‘perfect’ when it comes to symmetry, although they complement each other.

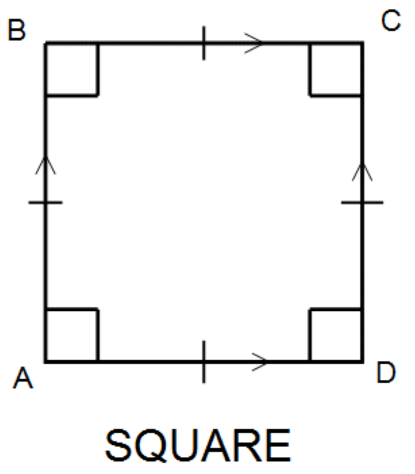
The rhombus has all its sides equal, but not all its angles; the rectangle has all its angles equal to 90°, but not all its sides. The diagonals of a rhombus meet at right angles, but are not equal in length, whereas those of a rectangle are equal in length but do not meet at right angles.

That leaves us with one type of quadrilateral combining the ‘best’ of both the rectangle and the rhombus.

The square.

A **square** is the most ‘perfect’ of quadrilaterals, combining the symmetries and properties of the rectangle and the rhombus.

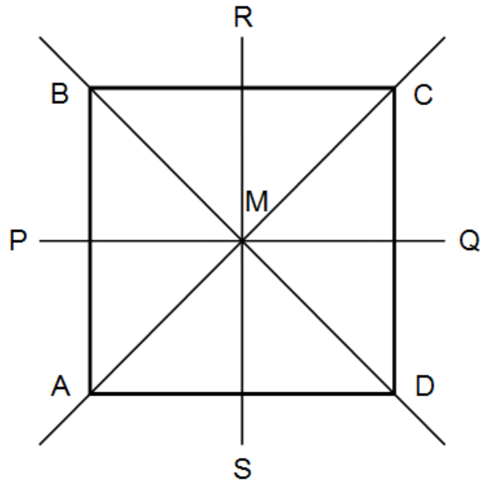
All four sides are equal. Thus AB = BC = CD = DA.
All four angles A, B, C, D are right angles.
Both pairs of sides are parallel.



A square also has greater symmetry than a rectangle or a rhombus.

There are now four lines of symmetry; two passing through the diagonals AC and BD, and the other two passing through the midpoints of opposite sides at PQ and RS. A square also has rotational symmetry of order 4.

The diagonals bisect each other and are equal in length. Hence AC = BD, and AM = MB = CM = MD. They also bisect their respective angles - AC bisects angles A and C, and BD bisects angles B and D.

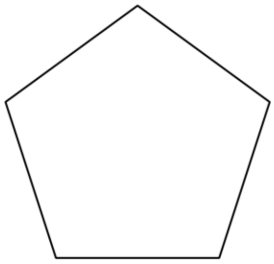


Polygons.

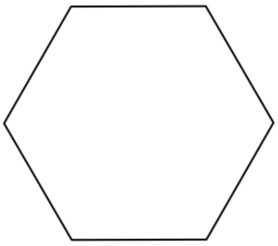
A polygon is any plane figure with three or more straight sides.

A regular polygon has all of its sides and angles equal.

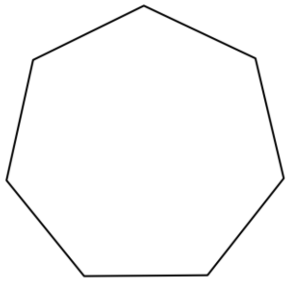
A few examples are shown as follows:



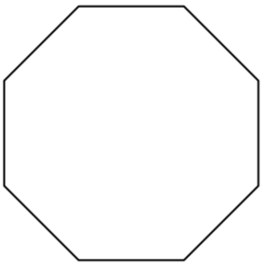
Regular pentagon
5 equal sides and angles



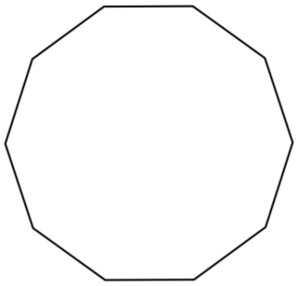
Regular hexagon
6 equal sides and angles



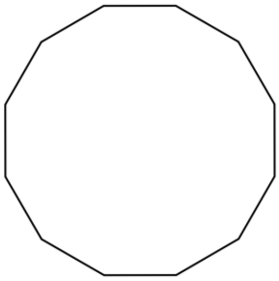
Regular heptagon
7 equal sides and angles



Regular octagon
8 equal sides and angles



Regular decagon
10 equal sides and angles



Regular dodecagon
12 equal sides and angles

Since the exterior angles of any convex polygon sum to 360°, it follows that the exterior angle of a regular polygon (in degrees) is the number of sides divided into 360.

Thus, for example, the exterior angle of a regular pentagon is $\frac{360}{5}$ ° or 72°, and its interior angle is (180 – 72)° or 108°.

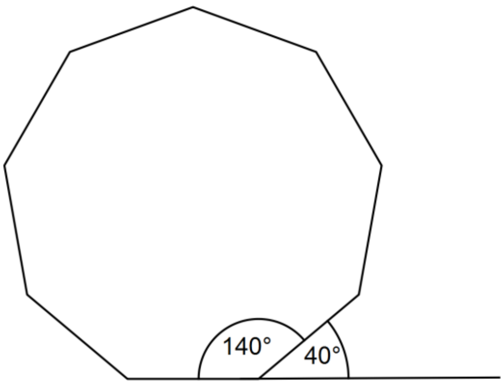
Similarly, a regular hexagon has an exterior angle of $\frac{360}{6}$ ° or 60°, and its interior angle is 120°.

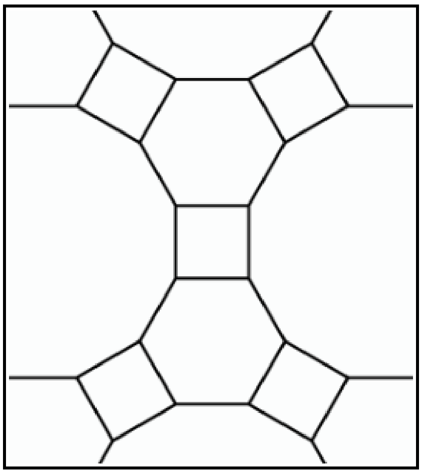
Example (4): A regular polygon has all its interior angles equal to 140°. How many sides does it have?

If the interior angles of the polygon are all equal to 140°, then its exterior angles must all be equal to (180-140)°, or 40°.

Since all the exterior angles of the polygon sum to 360°, it follows that the number of sides is equal to $\frac{360}{40}$ or 9.

(The name for such a polygon is a **regular nonagon**).





Example (5): A floor is tiled using regular polygonal tiles. The diagram on the right shows part of the pattern.

Two of the three types of polygon used are squares and regular hexagons.

Calculate the number of sides in the third type of polygon.

Three polygons meet at each corner of the pattern, namely a square, a regular hexagon and the unknown polygon.

The interior angle of the square is 90°, and the exterior angle of the hexagon is $\frac{360}{6}$ ° or 60°, so its interior angle is 120°.

Since the angles around each corner add to 360°, the third polygon has an interior angle of 360° - (90° + 120°), or 150°.

Hence one exterior angle of the third polygon is (180-150)° or 30°

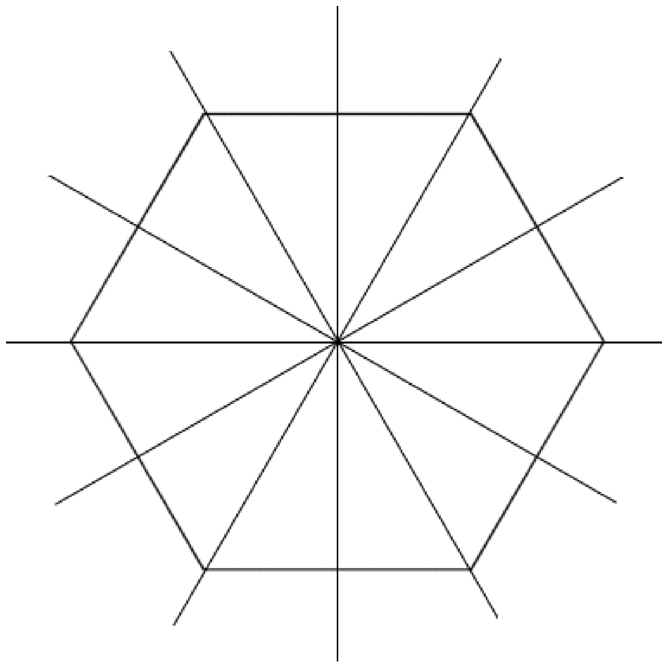
The number of sides in the third polygon is $\frac{360}{30}$ or 12, i.e. it is a dodecagon.

We have seen earlier how an equilateral triangle has 3 lines of symmetry and rotational symmetry of order 3; also, how a square has 4 lines of symmetry and rotational symmetry of order 4.

This same pattern occurs in **all** regular polygons – they have as many lines of symmetry as they have sides; likewise, their order of rotational symmetry is equal to the number of sides.

See the example of a regular hexagon below.

Regular hexagon

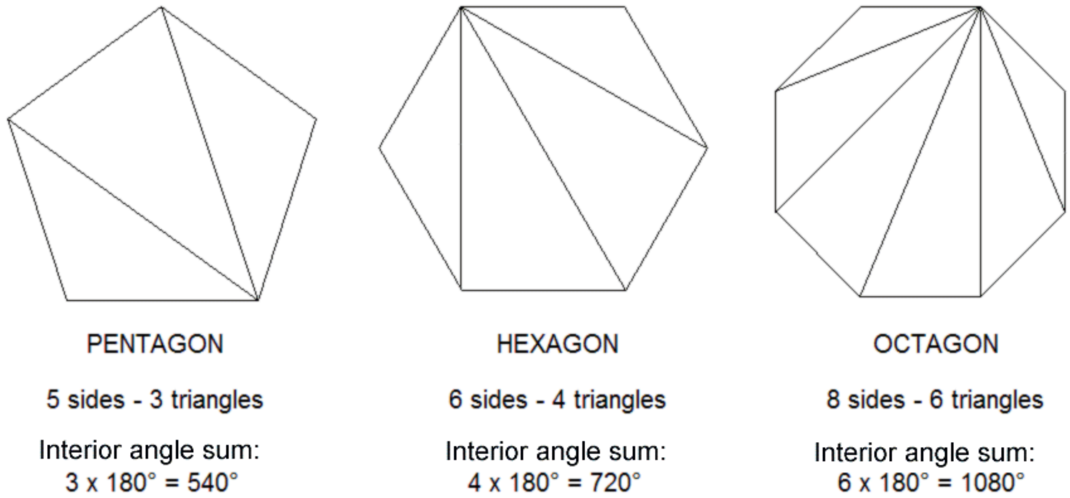


6 lines of symmetry
Rotational symmetry of order 6

The interior angle sum of a polygon.

This general fact applies to all polygons and not just regular ones.

Any polygon can similarly be split up onto triangles as shown below:



There is an obvious pattern here – any polygon with n sides can be split into n-2 triangles. Since the interior angle sum of a triangle is 180°, the sum of the interior angles of a polygon can be given by the formula

Angle sum = 180(n -2)° where n is the number of the sides in the polygon.

Example (6): A regular polygon has an interior angle sum of 1440°. How many sides does it have ?

We must solve the equation: $180(n - 2) = 1440$.

Dividing by 180 on both sides, we have $n - 2 = 8$ and hence $n = 10$.

The polygon in question is therefore 10-sided, i.e. a regular decagon.