5. Fibrations and Fibration Sequences.

In this section we show that one can use topology to study simplicial objects, such as simplicial sets, simplicial groups, simplicial rings, simplicial categories etc. A powerful tool is the fibration sequence. It plays a similar role in homotopy theory as the long exact homology sequence in homology theory.

1. Homotopies.

Definition. *Two maps* $g, f : (X, A) \to (Y, B)$ *are called* **homotopic** *(notation:* $f \simeq g$*) if there is a map*

$$H : (X \times I, A \times I) \to (Y, B)$$

such that $H|X \times 0 = f$ *and* $H|X \times 1 = g$.

Notation. One write $f \simeq g$ if f and g are homotopic.

It is easily verified that $\simeq$ defines an equivalence relation on the set of all functions $f : (X, A) \to (Y, B)$. Given this equivalence relation, define

Definition. *Let* X, Y *be two CW-complexes. Then denote*

$$[X, Y] := \text{set of homotopy classes of all maps } X \to Y.$$

Remark. For pairs of CW-complexes this set is denoted by $[\ (X, A), (Y, B)\ ]$.

Definition. *The* **wedge sum** $X \vee Y$ *of two pointed spaces* $(X, x), (Y, y)$ *is obtained from the disjoint union* $X \cup Y$ *by identifying* $x = y$, *i.e.*

$$X \vee Y = (X, x) \sqcup_{x=y} (Y, y).$$

The **wedge product** $X \wedge Y$ *(or:* **smash product***) of two pointed spaces* $(X, *)$ *and* $(Y, *)$ *is the quotient of the product* $X \times Y$ *under the identification* $(x, *) \sim (*, y)$, *for all* $x \in X, y \in Y$, *i.e.*

$$X \wedge Y := X \times Y / X \vee Y.$$

Definition. *Let* X *be a CW-complex. Then*

$$\Sigma X \cong X \wedge S^1$$

is the **suspension** *of* X *and*

$$\Omega X := [(S^1, *), (X, *)]$$

is the **path-space** *of* X *(with base point* $*$*).*

Theorem. *Let* X, Y *be CW-complexes. Then*

$$[\Sigma X, Y] = [X, \Omega Y]$$

Proof. This follows since CW-complexes are locally compact Hausdorff spaces, see e.g. [Munkres, Topology, p.287]. $\diamondsuit$

Remark. $\mathrm{Hom}(X \wedge A, Y) \cong \mathrm{Hom}(X, \mathrm{Hom}(A, Y))$ (note the analogy to $\mathrm{Hom}(V \otimes U, W) \cong \mathrm{Hom}(V, \mathrm{Hom}(U, W))$ for R-modules, so

smash product $\sim$ tensor product.

Remark. In the same way one proves $C(X \times I, Y) = C(I, Y^X)$, i.e. a homotopy between two maps $f_0, f_1 : X \to Y$ can be viewed as a continuous path f_t in the space $C(X, Y) = Y^X$ of continuous maps.

2. Fibrations and Homotopy.

Definition. *A map* $p : E \to B$ *has the* **homotopy lifting property** *with respect to a space* X *if, given a homotopy* $g_t : X \to B$ *and a map* $\tilde{g}_0 : X \to E$ *lifting* g_0 *(i.e.,* $p\tilde{g}_0 = g_0$*), then there is a homotopy* $\tilde{g}_t : X \to E$ *lifting* g*, i.e.*

$$\begin{array}{ccc} & & E \\ & \tilde{g}_t \nearrow & \downarrow p \\ X \times I & \xrightarrow{g_t} & B \end{array}$$

Definition.

(1) A **Hurewicz fibration** *is a map having the homotopy lifting property with respect to all spaces* X.

(2) A **Serre fibration** *is a map having the homotopy lifting property with respect to all discs* D^k.

Proposition. *For a fibration* $p : E \to B$ *the fibers*

$$F_b = p^{-1}(b)$$

over each path component of B *are all homotopy equivalent.*

Proof. [Hatcher, p. 405]

A path $\gamma : I \to B$ gives rise to a homotopy $g_t F_{\gamma(0)} \to B$ with $g_t(F_{\gamma(0)}) = \gamma(t)$. The inclusion $F_{\gamma(0)} \hookrightarrow E$ provides a lift $\tilde{g}_0$, so by the homotopy lifting prperty, we have a homotopy $\tilde{g} : F_{\gamma(0)} \to E$ with $\tilde{g}_t(F_{\gamma(0)}) \subset F_{\gamma(t)}$, for all t. In particular, $\tilde{g}_1$ gives a map $L_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$. The association

$$\gamma \mapsto L_\gamma$$

has the following basic properties:

(1) If $\gamma \simeq \gamma'$ rel ∂I, then $L_\gamma \simeq L_{\gamma'}$.

In particular, the homotopy class of L_γ is independent of the choice of the lifting $\tilde{g}_t$ of g_t.

(2) For a composition of paths $\gamma\gamma'$, $L_{\gamma\gamma'}$ is homotopic to the composition $L_{\gamma'} L_\gamma$.

From these statements it follows that L_γ is a homotopy equivalence with homotopy inverse $L_{\bar{\gamma}}$, where $\bar{\gamma}$ is the inverse path of γ.

Before proving (1) note that a fibration has the homotopy lifting property for pairs $(X \times I, X \times \partial I)$ since the pairs $(I \times I, I \times \{0\} \cup \partial I \times I)$ and $(I \times I, I \times \{0\})$ are homeomorphic, hence the same is true after taking products with X.

To prove (1), let $\gamma(s,t)$ be a homotopy from $\gamma(t)$ to $\gamma'(t)$, $(s,t) \in I \times I$. This determines a family $g_{st} : F_{\gamma(0)} \to B$ with $g_{st}(F_{\gamma(0)}) = \gamma(s,t)$. Let $\tilde{g}_{0,t}$ and $\tilde{g}_{1,t}$ be lifts defining L_γ and $L_{\gamma'}$, and let $\tilde{g}_{s,0}$ be the inclusion $F_{\gamma(0)} \hookrightarrow E$, for all s. Using the homotopy lifting property for the par $(F_{\gamma(0)} \times I, F_{\gamma(0)} \times \partial I)$, we can extend these lifts to lifts $\tilde{g}_{s,t}$ for $(s,t) \in I \times I$. Restricting to $t = 1$, then gives a homotopy $L_\gamma \simeq L_{\gamma'}$.

Property (2) holds since for the lifts $\tilde{g}_t$ nad $\tilde{G}'_t$ defining L_γ and $L_{\gamma'}$ we obtain a lift defining $L_{\gamma\gamma'}$ by taking $\tilde{g}_{2t}$ for $0 \le t \le \frac{1}{2}$ and $\tilde{g}'_{2t-1} L_\gamma$, for $\frac{1}{2} \le t \le 1$. $\diamondsuit$

Definition. *Let* $f : X \to Y$ *be a map. Then define*

$$E_f := \{\ (x,w) \in X \times Y^I \mid w(0) = f(x)\ \}.$$

Proposition. *The projection* $E_f \to Y$, $(x,w) \mapsto w(1)$ *is a fibration.*

Proof. $\diamondsuit$

Proposition. *Every map* $f : X \to Y$ *factors through a homotopy equivalence and a fibration as follows*

$$\begin{array}{ccccc} & & E_f & & \\ & \simeq\nearrow & & \searrow & = \text{fibration} \\ X & & \xrightarrow{f} & & Y \end{array}$$

Remark. Every map is a fibration, up to homotopy.

Definition. *Let* X *be a CW-complex. We denote*

$$\pi_n X := [\Sigma^n S^0, X].$$

$\pi_n X$ *is the* **n-th homotopy group** *of* X. *The composition* $\pi_n X \times \pi_n X \to \pi_n X$ *is given by concatenation.*

Definition. *Let*

$$f : X \to Y$$

be a map and let $p : E_f \to Y$ *be as above. Then*

$$p^{-1}(y), \quad y \in Y,$$

is called the **homotopy fiber** *of the map* f *(all hootopy fibers are homotopy equivalent).*

Theorem. [The Long Exact Homotopy Sequence for Serre Fibrations]
Suppose $p : E \to B$ *is a Serre fibration and let* $F = p^{-1}(b),\ b \in B$.
Then

(1) the map $p_* : \pi_n(E, F) \to \pi_n B$ *is an isomorphism, for all* $n \geq 1$,

(2) if B *is path-connected, there is a long exact sequence*

$$\ldots \to \pi_n F \to \pi_n E \xrightarrow{p_*} \pi_n B \to \pi_{n-1} F \to \ldots \to \pi_0 F \to \pi_0 E \to 0.$$

Remark. The sequence is the same as

$$\ldots \to [S^n, F] \to [S^n, E] \xrightarrow{p_*} [S^n, B] \to [S^{n-1}, F] \to \ldots \to [S^0, F] \to [S^0, E] \to 0.$$

Proof.

ad (1).
p_* **is surjective.**
Let $f : (I^n, \partial I^n) \to (B, *)$ represent an element from $\pi_n B$. The constant map $* : J^{n-1} \to *$ gives a commutative diagram

$$\begin{array}{ccc} J^{n-1} & \xrightarrow{*} & E \\ \downarrow & & \downarrow p \\ I^n & \xrightarrow{f} & B \end{array}$$

Since p is a fibration, we get a lift $\tilde{f} : I^n \to E$. This lift satisfies $\tilde{f}(\partial I^n) \subset F$ since $f(\partial I^n) = *$. Then $[\ \tilde{f}\ ] \in \pi_n(E, F)$ and $p_*[\tilde{f}] = [f]$ since $p\tilde{f} = f$.

p_* **is injective.**
Suppose $\tilde{f}_0,\ \tilde{f}_1 : (I^n, \partial I^{\cdot} J^{n-1}) \to (E, F)$ are maps with $p_*[\tilde{f}_0] = p_*[\tilde{f}_1]$. Let

$$G : (I^n \times I, \partial I^n \times I) \to (B, *)$$

be a homotopy from $p\tilde{f}_0$ to $p\tilde{f}_1$. We have a commutative diagram

$$\begin{array}{ccc} (I^n \times 0) \cup (J^{n-1} \times I) \cup (I^n \times 1) & \xrightarrow{\tilde{G}} & E \\ \downarrow & & \downarrow p \\ I^n \times I & \xrightarrow{G} & B \end{array}$$

where $\tilde{G}|I^n \times i = \tilde{f}_i$, $i = 0, 1$, and $\tilde{G}|J^{n-1} \times I$ the constant map to x_0. By the lifting property of p, we get a lift $\tilde{G} : I^n \times I \to E$. This gives a homotopy $\tilde{f}_t : (I^n, \partial I^n, J^{n-1}) \to (E, F, *)$ from $\tilde{f}_0$ to $\tilde{f}_1$. So p_* is injective.

ad (2).

In order to show (2) we plug $\pi_n B$ in for $\pi_n(E, F)$ in the long exact sequence

$$\ldots \to \pi_n F \to \pi_n E \to \pi_n(E, F) \xrightarrow{\partial} \pi_{n-1} F \to \pi_{n-1} E \to \ldots$$

for the pair (E, F). The map $\pi_n E \to \pi_n(E, F)$ becomes the composition

$$\pi_n E \to \pi_n(E, F) \xrightarrow{p_*} \pi_n B$$

and this is $p_* : \pi_n E \to \pi_n B$.

Finally, B is path-connected. Hence a path in E from an arbitrary point $x \in E$ to F can be obtained by lifting a path in E from $p(x)$ to $*$. This proves surjectivity of $\pi_0 F \to \pi_0 E$. $\diamondsuit$

Corollary. $\pi_n X = \pi_{n-1} \Omega X$**.**

Proof. Use the long exact homotopy equivalence for the fibration $\Omega X \to PX \to X$ and us the fact that $\pi_q PX = 0$, for all $q \neq 0$, since PX is contractible. Thus we have

$$\ldots \to \pi_q \Omega X \to 0 \to \pi_q X \to \pi_{q-1} \Omega X \to 0 \to \ldots$$

and the corollary follows from exactness. $\diamondsuit$

Corollary. $\pi_3 S^2 \cong \mathbb{Z}$**.**

Proof. Apply the long exact sequence for the fibration

$$S^1 \to S^3 \to S^2$$

to obtain

$$\to \pi_3 S^1 \to \pi_3 S^3 \to \pi_3 S^2 \to \pi_2 S^1 \to$$

Since $\pi_3 S^{=} 0 = \pi_2 S^1$, we have $\pi_3 S^2 \cong \pi_3 S^3 \cong \mathbb{Z}$. $\diamondsuit$

Proposition. *If*

$$F \to E \to B$$

is a fibration with E contractible, then there is a weak homotopy equivalence

$$F \to \Omega B.$$

Proof. [Hatcher,p.408]

If we compose a contraction of E with the projection $p : E \to B$ then we have for each point $x \in E$ a path γ_x in B from $p(x)$ to a basepoint $b_0 = p(x_0)$, where x_0 is the point to which E contracts. This yields a map

$$E \to PB,\ x \mapsto \bar{\gamma}_x,$$

whose composition with the fibration $PB \to B$ is p.

$$\begin{array}{ccccc} F & \to & E & \xrightarrow{p} & B \\ \downarrow & & \downarrow & & \downarrow = \\ \Omega B & \to & PB & \to & B \end{array}$$

By restriction this gives a map $F \to \Omega B$, where $F = p^{-1}(b_0)$, and the long exact sequence of homotopy groups for $F \to E \to B$ maps to the long exact sequence for $\Omega B \to PB \to B$. Since E and PB are contractible, the five lemma implies that the map $f : F \to \Omega B$ is a weak homotopy equivalence. $\diamondsuit$

3. Fibrations and Homology.

There are no long exact homology sequences for fibrations. This is the reason why it is so difficult to calculate the homotopy groups for fibrations. Here one needs spectral sequences as replacement for the absent long exact homology sequence.

Definition. *A* **spectral sequence** *is a collection*

$$E = (E^k, d^k) = (E^k_{r,s}, d^k_{r,s})$$

of modules $E^k_{r,s}$ *and maps* $d^k_{r,s}$ *(so called* **differentials***) such that*

(1) $d^k_{r,s} : E^k_{r,s} \to E^k_{r-k,s+k-1}$ *is a homomorphism with* $d^k_{r+1,s} \circ d^k_{r,s} = 0$.
(2) $E^{k+1} \cong H(E^k)$, *where* $H(E)$ *denotes the collection* $H_{s,t} = \mathrm{ker} d^k_{r,s} / \mathrm{im} d^k_{r-k,s-(k+1)}$ *of homology modules.*

Theorem. [Spectral Sequences for Fibrations]. *For a fibration*

$$F \to X \to B$$

with B *simply connected, there is a spectral sequence* $\{E^r_{pq}, d_r\}$ *with*

(1) $d_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$ *and* $E^{p,q}_{r+1} = \ker d_r / \mathrm{im}\ d_r =$ *homology at* $E^{p,q}_r$.

(2) there is a filtration $0 \subset F^n_n \subset \ldots \subset F^n_0 = H^n X$ *of* $H^n X$ *such that* $E^{p,n-p}_\infty \cong F^n_p / F^n_{p+1}$.

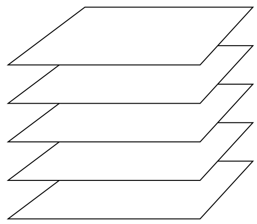
(3) $E^{p,q}_2 \cong H^p(B; H^q F))$.

Proof. [Hatcher] $\diamondsuit$

Remark. We make no attempt of proving this theorem, i.e., we will make no attempt to construct a spectral sequence. Usually one does not need to know how a spectral sequence is constructed. One only needs to know that it exists. All important information about spectral sequences can be deduced from its very definition. However, one needs to get used to spectral sequences.

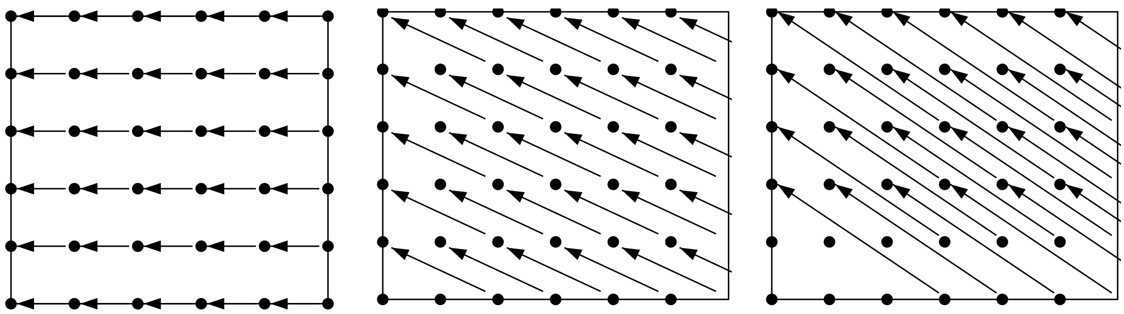
4. Spectral Sequences and Convergence.

A spectral sequence may be viewed as the integer lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ in Euclidean space which is organized in such a way so that it looks like a building. First, the lattice is organized in levels of lattices:



The building = it has floors

The levels in turn are made up of chains of arrows obeying a certain rule:



The floors = they have flights of rooms

Remarks. Thus a spectral sequence may be viewed as a building full of rooms (represented by dots) connected by one-way doors (represented by arrows) which allow to walk from one room to another.

Remark. So, schematically, all spectral sequences look the same. Given the first page of a spectral sequence, the modules of the second page are uniquely given. They are simply the homology groups of the chain complex of the first page. The problem is to set up the differentials of the second page. Whence they are given we are in business and we can form the next level and so on.

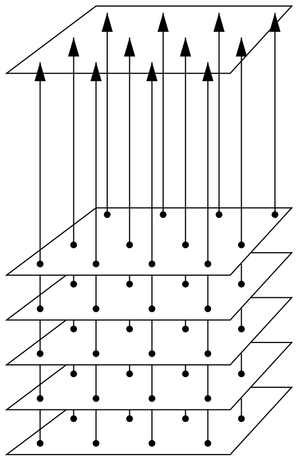
Remark Let us consider the distribution of arrows more closely. So let us be given a page full of chain complex. Thus instead of one chain complex (A_i, ∂_i) (for which we only need one index), we now begin with a sequence $(E_{r,s}, d_{r,s})$ of chain complexes (for which we need two indices). It is costumary to denote the modules $E_{r,s}$ in a spectral sequence by the letter E and the boundary maps $d_{r,s}$ by the letter d. Given this page of chain complexes we get a second page of chain complexes $(E^2_{r,s}, di^2_{r,s})$, i.e. we get a new module $E^2_{r,s}$ and a new differential $d_{r,s}$, for every location (r,s). This time, however, the boundary maps are called differentials. Note that we have introduces a third index (written as an upper index) to record the page in which the module and the differential is located. The differentials in the third page are even longer and even more tilted and so on. We will speak of a spectral sequence if the new modules and boundary maps are given according to the specific rules.

The Top Floor of Spectral Sequences. At the moment the definition of a spectral sequence is to general to be useful. In fact, with this definition one still cannot see the point of the construction.

What makes spectral sequences so useful in practice is that they satisfy some extra property (depending on the problem they are supposed to solve).

For us the extray property that our spectral sequences satisfy is that they are all so called **first quadrant spectral sequences**. This means that only the modules $E_{p,q}$ with $p,q \geq 0$ are non-trivial.

Given the above definition together with this extra property and something rather remarkable turns up as indicated by the following picture:



The top-floor = here sit all the chiefs

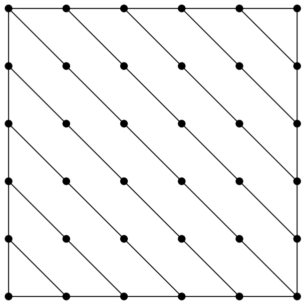
The vertical columns indicate the sequence

$$E^1_{p,q},\ E^2_{p,q},\ E^3_{p,q},\ \ldots$$

through the levels. Note that arrows get longer and longer. So with increasing r more and more arrows start or end in modules that are outside of the upper-right quadrant and that are therefore supposed to be 0. Thus more and more homology groups stavilize, i.e. they do not change anymore for large enough r. Of course, r must be larger the larger p,q is. So beginning with the origin we see the whole picture stablize. The vertical arrows all end in modules $E^r_{p,q} \to E^\infty_{p,q}$.

Convergence of Spectral Sequences. It turns out that the top floor has its own combinatorial structure

E∞



the order at the top = some chiefs are closer than others

The lines (p,q) with $p+q=n$ all have the same modules. They are not quite equal to $H_n(X)$, but they are equal to something close.

Remark. First quadrant spectral sequences are always convergent as we have seen. But there are other spectral that are also konvergent, namely the **bounded spectral sequences**. A spectral sequence is bounded if all lines $p+q=n$, for all n, have only finitely many non-zero terms. First quadrant are of course bounded but not vice versa. If a spectral sequence if not bounded then it is hard to predict what it calculates. However, we will only need first quadrant spectral sequences. So we do not worry about other spectral sequences, whether bounded or not.

The groups in the top floor will not all be the homology of E but close. We will see that for every group H_nE there will be a sequence of subgroups

$$0 = F_0H_nE \subset F_1H_nE \subset \ldots \subset F_nH_nE = H_nE$$

such that

$$E^\infty_{p,q} = F_pH_qE/F_{p-1}H_qE.$$

We will have to see how this sequence F_rH_qE is constructed and how one can make use of it.

If the above sequence exists one says that the spectral sequence converges and one denotes this with a special notation. More precisely, one writes

$$E^\infty_{p,q} \Rightarrow H_{p+q}$$

if the **spectral sequence converges** in the above sense.

Remark. We will not have to worry about the difference of convergent and non-convergent spectral sequences because our spectral sequences will al converge.

5. The Wang Exact Sequence.

The prove of the next theorem shows the spectral sequence method in action. The result will be needed e.g. in the calculation of the homotopy groups of spheres.

The Wang-Sequence Theorem. *Let* $f : E \to S^m, \ k \geq 2,$ *be a fibration. Then there is an exact sequence*

$$\ldots \to H_n E \to H_{n-k} F \xrightarrow{d^k} H_{n-1} F \to H_{n-1} E \to \ldots$$

Proof. By property (1) of the fibration theorem, we have

$$E^2_{p,q} \cong H_p(S^n; H_q F).$$

Now the base S^m is simply connected and

$$H_0 S^m \cong \mathbb{Z}, \ H_m S^m \cong \mathbb{Z} \text{ and } H_p S^m = 0, \ p \neq 0, m.$$

Thus we have

$$E^2_{m,q} \cong H_m(S^n; H_q F) = H_q F, \quad E^2_{0,q} \cong H_{0,q}(S^n; H_q F) \cong H_q F, \text{ and}$$
$$E^2_{p,q} = H_p(S^n; H_q F) = 0, \ p \neq 0, k.$$

So the E^2-floor has a rather simple appearance

floor E^2 :						
$q+1$	$\ldots$	0	$\ldots$	0	$\ldots$	0
q	$H_q F$	0	$\ldots$	0	$H_q F$	0
$\ldots$	$\ldots$	0	$\ldots$	0	$\ldots$	0
2	$H_2 F$	0	$\ldots$	0	$H_2 F$	0
1	$H_1 F$	0	$\ldots$	0	$H_1 F$	0
0	$H_0 F$	0	$\ldots$	0	$H_0 F$	0
	0	1	$\ldots$	$m-1$	m	$m+1$

By definition of the spectral sequence the abelian groups $E^3_{p,q}$ of the next floor E^3 above E^2 is given by the quotient of the incoming arrow for $E^2_{p,q}$ modulo the image of the outgoing arrow. Now, if the arrows are shorter than m, the all incoming arrows end in 0 and all the outgoing arrows start in 0. In this case the kernels of the outgoing arrows are equal to $H_q F$ and the images of the incoming arrows is also equal to 0. So $E^3_{p,q} = E^2_{p,q}$. By this reasoning we have

$$E^2 = E^3 = \ldots = E^m, \ E^{m+1} = E^{m+2} = \ldots = E^\infty$$

All the floors up to the floor E^m look like the bottom floor. All the floors above E^{m+1} look like E^{m+1}. Hence E^∞ also looks like E^{m+1}.

Now, look at the m-th column of the m-th floor E^m. We see differentials d^m going from the m-th column to the 0-th column. More precisely,

$$d^m : E^m_{m,q} \to E^m_{0,q+m-1}$$

Now, $E^{m+1}_{m,q} = \ker d^m / \mathrm{im}\ 0$, where 0 denotes the null-homomorphism $0 \to A$. This means

$$0 \to E^{m+1}_{m,q} \to E^m_{m,q} \xrightarrow{d^m} E^m_{0,q+m-1}$$

is exact. Moreover, $E^{m+1}_{0,q} = \ker 0 / \mathrm{im}\ d^m$, where this time 0 denotes the null-homomorphism $A \to 0$. But $\ker 0 = H_{q+m-1}F = E^m_{0,q+m-1}$. This means that

$$E^2_{m,q} \xrightarrow{d^m} E^{2,0+m-1} \to E^{m+1}_{0,q+m-1} \to 0$$

is exact. Putting the two exact sequences together and using $E^{m+1}_{p,q} = E^\infty_{p,q}$ yields the exact sequence

$$0 \to E^\infty_{m,q} \to E^2 m, q \xrightarrow{d^m} E^2_{0,q+m-1} \to E^\infty_{0,q+m-1} \to 0 \tag{1}$$

We claim (exercise) that there is yet another exact sequence

$$0 \to E^\infty_{0,n} \to H_n(E) \to E^\infty_{m,n-m} \to 0 \tag{2}$$

Now set $q = n - m$ and recall $E^2_{m,q} = E^2_{m,n-m} = H_{n-m}F$ and $E^2_{0,q+m-1} = E^2_{0,n-1} = H_{n-1}F$ (by property (1) of the theorem). Hence the exact sequence (1) becomes

$$0 \to E^\infty_{m,n-m} \to H_{n-m}F \to H_{n-1}F \to E^\infty_{0,n-1} \to 0$$

We next put this exact sequence together with (2) as follows

$$\begin{array}{ccccc}
 & H_nE & & & 0 \\
 & \downarrow & & & \downarrow \\
0 \to & E^\infty_{m,n-m} & \to & H_{n-m}F \to H_{n-1}F & \to & E^\infty_{0,n-1} \to 0 \\
 & \downarrow & & & \downarrow \\
 & 0 & & & H_{n-1}E
\end{array}$$

Composition of maps yields:

$$
\begin{tikzcd}
H_nE \arrow[dr] & & \\
& H_{n-m}F \arrow[r] & H_{n-1}F \arrow[dr] & \\
& & & H_{n-1}E
\end{tikzcd}
$$

i.e. we get an exact sequence, for every n. All those sequences in turn yield the Wang exact sequence from the theorem. $\diamondsuit$

For completeness we here mention yet another exact sequence associated to fibrations and that can be obtained from spectral sequences.

The Gysin-Sequence Theorem. *Let*

$$S^n \to E \xrightarrow{p} B$$

be a fibration with B simply connected and $n \neq 0$. Then there is an exact sequence

$$\ldots \to H_{q-n}B \to H_qE \xrightarrow{p} H_qB \xrightarrow{d^{n+1}} H_{q-n-1} \to H_{p-1}E \xrightarrow{p} \ldots$$

In particular, $H_qE \cong H_qB$, for $0 \leq q < n$.

Proof. The proof is similar to the proof of the Wang sequence. For more details see e.g. [Weibel]. $\diamondsuit$

6. Calculations for Loop Spaces.

Example. $H_q(\Omega S^1;\mathbb{Z}) \cong \begin{cases} \oplus\mathbb{Z}, & \text{for } q=0 \\ 0, & \text{else} \end{cases}.$

Example. $H_q(\Omega S^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q|(n-1), \\ 0, & \text{else} \end{cases}$

Example. The cohomology of ΩS^n has the following ring structure

$$H^*(\Omega S^m;\mathbb{Z}) \cong \begin{cases} \Gamma_{\mathbb{Z}}[a], & \text{if } m \text{ is odd} \\ \Lambda_{\mathbb{Z}}[a]\otimes\Gamma_{\mathbb{Z}}[b], & \text{if } m \text{ is even.} \end{cases}$$

where $\Gamma_{\mathbb{Z}}[a]$ is a divided polynomial algebra and where $\Lambda_{\mathbb{Z}}[a]$ is the exterior algebra, $|a| = m-1$, $|b| = 2m-2$.

More generally, we have

Theorem. *Let X be a simply connected space. Suppose $H^*(X,R)$ is a polynomial algebra on a generator of degree n (n is then necessarily even). Then the loop space ΩX is a cohomology $(n-1)$-sphere.*

Proof. Use Gysin sequence [Spanier, p. 513]. $\diamondsuit$

Theorem. *Suppose X is a cohomology n-sphere for some $n > 1$ odd. Then the cohomology ring $H^*\Omega X$ has a basis consisting of elements $\{1, u_1, u_2, \ldots, \}$ with degree $u_i = i(n-1)$ and $u_p \cup u_q = \frac{(p+q)!}{p!q!}\, u_{p+q}$.*

Proof. From the Wang exact sequence [Spanier, p. 514]. $\diamondsuit$

Theorem. *Suppose X is a simply connected space which is a rational cohomology n-sphere with $n > 1$ odd. Then*

$$H^*(\Omega X,\mathbb{Q})$$

is a polynomial algebra with one generator of degree $n-1$.

Proof. [Spanier] $\diamondsuit$

Example. $H^q(\Omega^2 S^n;\mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{for } q=0,n \\ 0, & \text{else} \end{cases}$ as a group.

By the universal coefficient theorem, we have for the integral group

Example. $H_q(\Omega^2 S^n;\mathbb{Z}) \cong \begin{cases} \text{torsion group}, & \text{if } 0 < q \neq n-2 \\ \text{torsion group} \oplus \mathbb{Z}, & \text{if } q = n-2 \end{cases}$

7. Calculations for Eilenberg-MacLane Spaces.

Let $K(\mathbb{Z},2)$ denote the Eilenberg-MacLane space whose only homotopy group is in degree 2. Then i

Example. $H_q(K(\mathbb{Z},1);\mathbb{Z}) = H_q S^1 = \begin{cases} \mathbb{Z}, & \text{for } q = 0,1 \\ 0, & \text{else} \end{cases}$.

Example. $H_q(K(\mathbb{Z},2);\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for } q \text{ even} \\ 0, & \text{for } q \text{ odd} \end{cases}$. [Hatcher, p. 9]

Example. $H^*(K(\mathbb{Z},2);\mathbb{Z}) \cong \mathbb{Z}[x]$.

Example. $H^q(K(\mathbb{Z},3);\mathbb{Z}) \cong \begin{cases} 0, & \text{for } q = 4 \\ 0, & \text{for } q = 5 \\ \mathbb{Z}_2, & \text{for } q = 6 \\ 0, & \text{for } q = 7 \\ \mathbb{Z}_3 y, & \text{for } q = 8 \\ \mathbb{Z}_2 x^3, & \text{for } q = 9 \\ \dots, & \dots \end{cases}$

This cannot be calculated forever without additional information. In rational coefficients we have

$$H^*(K(\mathbb{Z},n),\mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x], & \text{for } n \text{ even} \\ \Lambda_{\mathbb{Q}}[x], & \text{for } n \text{ odd.} \end{cases}$$

Appendix.

Theorem. *Let* $\{E_r^{p,q}, d_r\}$ *be a spectral sequence. Then there are bilinear products*

$$E_r^{p,q} \times E_r s,t \to E_r^{p+s,q+t}$$

such that the following holds:

(1) Each differential d_r *is a* **derivation** *in the sense that*

$$d(xy) = (dx)y + (-1)^{p+q} x(dy),\ x \in E_r^{pq}$$

(This implies that the product $E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t}$ *induces a product* $E_{r+1}1p,q \times E_{r+1}^{s,t} \to E_{r+1}^{p+s,q+t}$ *and this is the product for* E_{r+1}*. The product in* E_∞ *is the one induced from the products in* E_r *for finite* r*.)*

(2) The product $E_2^{p,q} \times E_2^{s,t} \to E_2^{p+s,q+t}$ *is* $(-1)^{qs}$ *times the standard cup product*

$$H^p(B,HqF)) \times H^2B,H_qF)) \to H^{p+s}(B,H^{q+t}F)).$$

(3) The cup product in H^*X *restricts to maps* $F_p^m \times F_s^n \to F_{p+s}^{m+n}$*. These induce quotient maps*

$$F_p^m/F_{p+1}^m \times F_s^m/F_{s+1}^m \to F_{p+s}^m/F_{p+s+1}^m$$

that coincide with the products $E_\infty^{p,m-p} \times E_\infty^{s,n-s} \to E_\infty^{p+s,m+n-p-s}$*.*

Proof. [Hatcher]. $\diamondsuit$

Literature.

E. H. Spanier, Algebraic Topology, McGraw-Hill (1966)

A. Hatcher, Spectral Sequences, Homepage A. Hatcher

P. May, The Geometry of Iterated Loop Spaces