UNIVERSITE D'AIX-MARSEILLE **Mathématiques fondamentales**
Groups and dynamics **2020-21**

Group actions

Exercice 1 Let X, Y be two locally compact metric spaces.

1. Show that the compact-open topology and uniform convergence on compact subsets coincide in $\mathcal{C}(X, Y)$.
2. Prove that Homeo(X) is a topological group, i.e., the maps $(g, h) \in \text{Homeo}(X) \times \text{Homeo}(X) \mapsto gh$ and $g \in \text{Homeo}(X) \mapsto g^{-1}$ are continuous with respect to the compact-open topology.

Exercice 2 Let G be a locally compact and σ-compact group that acts continuously on a locally compact space X.

1. If $H < G$ is a closed subgroup, then G/H is locally compact and the canonical projection $p : G \to G/H$ is an open map.
2. For all $x \in X$, Stab x is a closed subgroup.
3. Let $x \in X$; if Gx is closed then $g \in G \mapsto gx \in Gx$ induces a homeomorphism between i $G/\text{Stab}\ x$ and Gx.

Let G be a group acting on a topological space X by homeomorphisms. Let $\rho : G \to \text{Homeo}(X)$ be its representation.

1. The action is *discrete* if G is discrete in the group of homeomorphisms of X supplied with the compact open topology.
2. The action has *discrete orbits* if the orbit of any point is a discrete subset of X, i.e., if every point has a neighborhood so that its orbit has only finitely many points in it.
3. The action is *wandering* if every $x \in X$ admits a neighborhood V such that
$$\{g \in G,\ g(V) \cap V \neq \varnothing\}$$
is a finite set.
4. The action is *properly discontinuous* if, for any compact subsets K and L of X,
$$\{g \in G,\ g(K) \cap L \neq \varnothing\}$$
is finite.

Exercice 3 Let X be locally compact and G a topological group acting on X.

1. Show the following implications: a properly discontinuous action is wandering, a wandering action has discrete orbits, and an action with discrete orbits is discrete. Do the converse statements hold ? If not, produce counter-examples.
2. Are all actions of a discrete group discrete ? Does a discrete action come from a discrete group ?

Exercice 4 Show that if an action is wandering then the stabiliser of each point is finite.

Exercice 5 Show that an action of a countable group on a locally compact space is properly discontinuous if and only if the map
$$\begin{aligned} G \times X &\to X \times X \\ (g, x) &\mapsto (g(x), x) \end{aligned}$$
is proper, where we endow G with the discrete topology. Show then that $\text{Ker}\,\rho$ is finite.

Exercice 6 Let X be a locally compact metric space. The action of G on X is free and properly discontinuous if and only if X/G is a Hausdorff quotient and $X \to X/G$ is a covering map.

Exercice 7 Let us consider the action of $\mathbb{Z}$ on $\mathbb{R}^2 \setminus \{0\}$ defined by

$$g_n(x, y) = (2^n x, y/2^n)\,.$$

1. Show that the action is wandering.

2. Prove that the quotient is not Hausdorff.

Exercice 8 Prove that if a group G acts properly discontinuously on a locally compact space X then, for all $x \in X$, there is a neighborhood U of x such that $\mathrm{Stab}\, x = \{g \in G,\ g(U) \cap U \neq \varnothing\}$.

Exercice 9 A subgroup H has finite index in a group G if G admits a properly discontinuous action on a space X such that the induced action of H is cocompact, i.e., X/H is compact.

Exercice 10 Let (X, w) be a proper, geodesic hyperbolic space. If $r : \mathbb{R}_+ \to X$ is a ray, show that

$$\lim_{s,t\to\infty} (r(t)|r(s))_w = \infty\,.$$

Conversely, show that if (x_n) tends to infinity, then there exists a ray $r : \mathbb{R}_+ \to X$ such that $r(0) = w$ and $(r(n))_n \sim (x_n)_n$.

Exercice 11 Let X, Y be two compact metric spaces and let us consider a sequence $(X \xrightarrow{f_n} Y)_n$ of η-quasi-Möbius maps for some fixed distortion function η. We assume that there are two distinct points $y_1, y_2 \in Y$ such that the sequences $(f_n^{-1}(y_1))_n$ et $(f_n^{-1}(y_2))_n$ tend to a common point $x \in X$. Using the definition, prove that $(f_n)_n$ tends uniformly to a constant on the compact subsets of $X \setminus \{x\}$.

Exercice 12 Let G be a finitely generated group, S and S′ two finite generating sets and $f : \mathbb{N} \to \mathbb{R}_+$ a Floyd gauge. We denote by d and d' the word metrics with respect to S and S′, and d_f and d'_f the corresponding Floyd metrics.

1. Prove that $\mathrm{Id} : (G, d) \to (G, d')$ is bi-Lipschitz.

2. Let $C > 0$, $(x_n)_n$ and $(y_n)_n$ two sequences in G such that $d(x_n, y_n) \le C$ for all $n \ge 0$ and $\lim d_f(x_n, y_n) = 0$. Prove that $\lim d'_f(x_n, y_n) = 0$.

3. Give a sufficient condition on f that ensures that $\mathrm{Id} : (G, d_f) \to (G, d'_f)$ is uniformly continuous.