Relationships Among Unit Vectors

Recall that we could represent a point P in a particular system by just listing the 3 corresponding coordinates in triplet form:

(x, y, z) Cartesian

(r, θ, φ) Spherical

and that we could convert the point P's location from one coordinate system to another using coordinate transformations.

Cartesian → Spherical

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

Spherical → Cartesian

$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

Recall that we could represent a point P in a particular system using vectors:

$\langle x, y, z\rangle$ Cartesian

$\langle r, \theta, \varphi\rangle$ Spherical

or

$\mathbf{P} = a\hat{\mathbf{x}} + b\hat{\mathbf{y}} + c\hat{\mathbf{z}}$ Cartesian

$\mathbf{P} = a\hat{\mathbf{r}} + b\hat{\boldsymbol{\theta}} + c\hat{\boldsymbol{\varphi}}$ Spherical

NOTE: The Cartesian system is taken to be the default coordinate system by which all others are vector systems defined.

What happens if we want to convert vector information about a point P from one coordinate system to another?

What we need are relationships or transformations between the various unit vectors

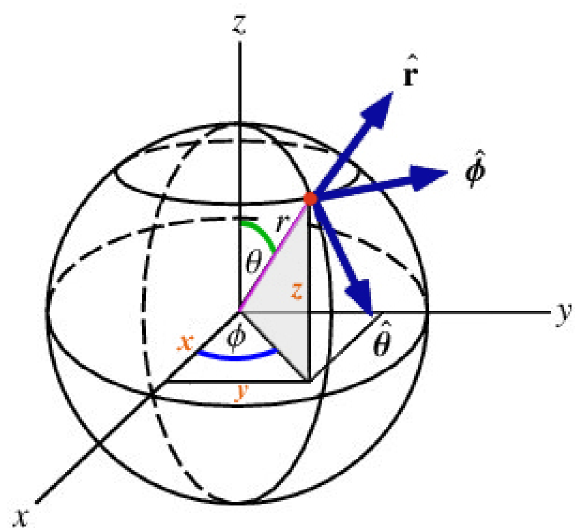
Consider a point P in spherical coordinates with the vector form:

$$\mathbf{P} = a\hat{\mathbf{r}} + b\hat{\boldsymbol{\theta}} + c\hat{\boldsymbol{\varphi}}$$

Since $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ for a orthogonal basis set as does $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$, we can write $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ with the appropriate transformations of the form:

$$\hat{\mathbf{r}} = a_1\hat{\mathbf{x}} + b_1\hat{\mathbf{y}} + c_1\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = a_2\hat{\mathbf{x}} + b_2\hat{\mathbf{y}} + c_2\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\varphi}} = a_3\hat{\mathbf{x}} + b_3\hat{\mathbf{y}} + c_3\hat{\mathbf{z}}$$

To determine what coefficients a_i, b_i and c_i are, we must take the dot product of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ with each of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$. To determine the value of the dot products, we can use the following figure and make use of the geometry between spherical and Cartesian coordinates:



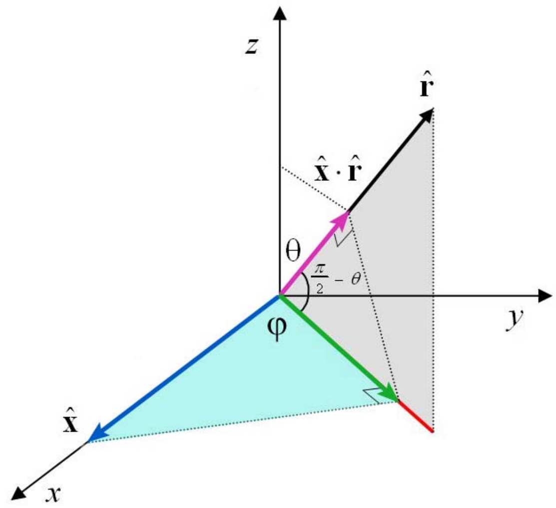
If we wanted to write $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, we would need to use the angles of θ and φ.

Ex.

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{r}} = a_1\hat{\mathbf{x}} + b_1\hat{\mathbf{y}} + c_1\hat{\mathbf{z}} = a_1$$

Note: a_1 is the projection of $\hat{\mathbf{x}}$ onto $\hat{\mathbf{r}}$. To find a_1 requires a two step process:

1) Project $\hat{\mathbf{x}}$ onto the line formed by $\hat{\mathbf{r}}$ and its projection onto the *xy* plane
2) Project (1) onto $\hat{\mathbf{r}}$



1. the projection length of $\hat{\mathbf{x}}$ onto line formed by $\hat{\mathbf{r}}$ and its projection onto the *xy* plane [green arrow] is given by:

$$\left|\hat{\mathbf{x}}\right|\cos\varphi$$

2. $\left|\hat{\mathbf{x}}\right|\cos\varphi$ projected onto $\hat{\mathbf{r}}$ [purple arrow] is given by:

$$\left|\hat{\mathbf{x}}\right|\cos\varphi\cos\left(\frac{\pi}{2}-\theta\right)=\left|\hat{\mathbf{x}}\right|\cos\varphi\sin\theta$$

Thus

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{r}}=a_1=\sin\theta\cos\varphi$$

By a similar process, the other Cartesian dot products with $\hat{\mathbf{r}}$ yield,

$$\hat{\mathbf{y}}\cdot\hat{\mathbf{r}}=b_1=\sin\theta\sin\varphi$$

$$\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}=c_1=\cos\theta$$

Finally, we collect all the different terms to find $\hat{\mathbf{r}}$ in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$:

$$\left\{\ \hat{\mathbf{r}}=\sin\theta\cos\varphi\hat{\mathbf{x}}+\sin\theta\sin\varphi\hat{\mathbf{y}}+\cos\theta\hat{\mathbf{z}}\right.$$

For the other spherical unit vectors, we have:

$$\left\{\begin{array}{l}\hat{\boldsymbol{\theta}}=\cos\theta\cos\varphi\hat{\mathbf{x}}+\cos\theta\sin\varphi\hat{\mathbf{y}}-\sin\theta\hat{\mathbf{z}}\\ \hat{\boldsymbol{\varphi}}=-\sin\varphi\hat{\mathbf{x}}+\cos\varphi\hat{\mathbf{y}}\end{array}\right.$$

NOTICE: Unlike $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$; $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ are **NOT** uniquely defined!

The game can be played in reverse to find $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$:

$$\begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\varphi\hat{\mathbf{r}} + \cos\theta\cos\varphi\hat{\boldsymbol{\theta}} - \sin\varphi\hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\varphi\hat{\mathbf{r}} + \cos\theta\sin\varphi\hat{\boldsymbol{\theta}} + \cos\varphi\hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \end{cases}$$

APPLICATION:

One of the most common vectors we will deal with is the position vector, **r**. In Cartesian form, it looks like this:

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

This form of the vector is measuring the displacement from the origin to some point P.

If the position vector is measuring the displacement from some starting location other than the origin, say (x_o, y_o, z_o), we can represent this new starting coordinate with the vector $\mathbf{r_o}$ relative to the origin. The new position vector would look like this:

$$\mathbf{r} - \mathbf{r_o} = (x - x_o)\hat{\mathbf{x}} + (y - y_o)\hat{\mathbf{y}} + (z - z_o)\hat{\mathbf{z}}$$

NOTE: *We will come back and use this form at a later time. In the mean time, all vectors will be measured with respect to the origin.*

Now that we have different unit vectors for different systems and a way to go back and forth between them, the position vector can be represented in various ways:

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$\mathbf{r} = r\hat{\mathbf{r}}$$

$$\mathbf{r} = \left(\sqrt{x^2 + y^2 + z^2}\right)\hat{\mathbf{r}}$$

$$\mathbf{r} = r\sin\theta\cos\varphi\hat{\mathbf{x}} + r\sin\theta\sin\varphi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}}$$

** Depending on the problem, select the vector form that will be the most useful and the easiest to work with.