ON SYNTHESIZING CROSS AMBIGUITY FUNCTIONS

Hao He[*], Petre Stoica[†] and Jian Li[*]

[*]Dept. of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611-6130, USA
Email: haohe@ufl.edu, li@dsp.ufl.edu
[†]Dept. of Information Technology, Uppsala University, PO Box 337, SE-75105 Uppsala, Sweden
Email: ps@it.uu.se

ABSTRACT

The cross ambiguity function (CAF) arises in many areas such as radar/sonar and communications when correlation processing is performed in the presence of a Doppler frequency shift. In this paper, the CAF synthesis problem is tackled: a pair of waveforms are jointly designed so that their CAF approximates a desired one. The so-generated waveforms have relatively low peak-to-average power ratios and in certain cases can be constant modulus. Numerical examples are provided to show the effectiveness of the proposed algorithm in synthesizing different types of CAF.

Index Terms— cross ambiguity function, probing waveform synthesis, receive filter design

1. INTRODUCTION

The ambiguity function (AF) is a two-dimensional function $\chi(\tau, f)$ that shows the response of a matched filter to the signal with time delay τ and Doppler frequency shift f. How to realize a desired ambiguity function using practical signals has been a classical problem in the waveform design area and there exists much literature on this topic; see, e.g., [1]–[8] and the references therein. Despite extensive literature, there has not been any existing method that can synthesize an arbitrary ambiguity function. In fact, matching only the zero-Doppler cut of an ambiguity function, or minimizing the sidelobes of the auto-correlation function, is already a difficult problem (e.g., see [9]).

In this paper we focus on the synthesis of the *cross* ambi-

guity function (CAF) that is defined as follows:

$$\chi(\tau, f) = \int_{-\infty}^{\infty} u(t)v^*(t+\tau)e^{j2\pi ft}dt \qquad (1)$$

where $u(t)$ and $v(t)$ are a pair of waveforms and * denotes the complex conjugate. The usage of CAF naturally appears when $u(t)$ is the transmit signal and $v(t)$ is the receive filter (see, e.g., [3]), or arises in the case of a MIMO radar (see, e.g., [8]). Note that the definition of CAF leads to more degrees of freedom compared to that of the conventional ambiguity function, a case where $v(t)$ equals $u(t)$.

In Sections 2 and 3, we propose an algorithm that designs $u(t)$ and $v(t)$ jointly so that their CAF $\chi(\tau, f)$ approximates a desired one. The desired CAF need not be a valid function calculable from a pair of underlying signals; it can be any 2D function that represents a generally desirable shape, such as a peak in the origin and low sidelobes in a certain area. Note that the volume of a CAF, which is defined as $V = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\tau, f)|^2 d\tau df$, is always equal to the product of the energy of $u(t)$ and that of $v(t)$. Such a property essentially prevents the synthesis of an ideal CAF (in fact, an ideal AF, too) that has a narrow high peak in the origin and zero sidelobes everywhere else. In Section 4 we demonstrate the synthesis of a CAF with a clear area close to the origin, as well as a CAF with a high diagonal ridge but low sidelobes in the rest of the (τ, f) plane.

Throughout the paper we use bold lowercase/uppercase letters to denote vectors/matrices, respectively. $(\cdot)^H$ indicates vector/matrix conjugate transpose and $(\cdot)^T$ indicates transpose. $\arg(x)$ denotes the phase of x and $\| \cdot \|$ denotes the vector Euclidean norm. $\mathbf{I}_N$ is the $N \times N$ identity matrix.

2. PROBLEM FORMULATION

Suppose that both $u(t)$ and $v(t)$ consist of N subpulses

$$u(t) = \sum_{k=1}^{N} x_k p_k(t), \quad v(t) = \sum_{l=1}^{N} y_l p_l(t) \qquad (2)$$

where $\{p_n(t)\}_{n=1}^N$ are pulse-shaping functions. For example, $p_n(t)$ can be the ideal rectangular pulse:

$$p_n(t) = \frac{1}{\sqrt{t_p}} \text{rect}\left(\frac{t - (n-1)t_p}{t_p}\right), \quad n = 1, \ldots, N \quad (3)$$

where t_p is the time duration of each subpulse and

$$\text{rect}(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & \text{elsewhere.} \end{cases} \quad (4)$$

Under the above waveform setting, the CAF in (1) becomes

$$\chi(\tau, f) = \int_{-\infty}^{\infty} \left(\sum_{k=1}^N x_k p_k(t)\right) \left(\sum_{l=1}^N y_l^* p_l^*(t + \tau)\right) e^{j2\pi ft} dt$$

$$= \sum_{k=1}^N \sum_{l=1}^N x_k y_l^* \int_{-\infty}^{\infty} p_k(t) p_l^*(t + \tau) e^{j2\pi ft} dt. \quad (5)$$

Let

$$\bar{\chi}_{kl}(\tau, f) = \int_{-\infty}^{\infty} p_k(t) p_l^*(t + \tau) e^{j2\pi ft} dt \quad (6)$$

denote the CAF of the pulse shaping functions and define

$$\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_N \end{bmatrix}^T, \; \mathbf{y} = \begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix}^T, \quad (7)$$

$$\mathbf{K}(\tau, f) = \begin{bmatrix} \bar{\chi}_{1,1}(\tau, f) & \cdots & \bar{\chi}_{N,1}(\tau, f) \\ \vdots & & \vdots \\ \bar{\chi}_{1,N}(\tau, f) & \cdots & \bar{\chi}_{N,N}(\tau, f) \end{bmatrix}. \quad (8)$$

Then the CAF in (5) can be written compactly as

$$\chi(\tau, f) = \mathbf{y}^H \mathbf{K}(\tau, f) \mathbf{x}. \quad (9)$$

Assume that the desired CAF modulus is given by $d(\tau, f)$ (real-valued and non-negative). The CAF phases are usually not of interest because of carrier phase incoherence. We aim to design a pair of signals $u(t)$ and $v(t)$ (more precisely $\mathbf{x}$ and $\mathbf{y}$, because the pulse shaping function is fixed), so that the modulus of their CAF approximates $d(\tau, f)$ as closely as possible. Such an optimization problem can be formulated as

$$\min_{\mathbf{x}, \mathbf{y}} g(\mathbf{x}, \mathbf{y}) \quad (10)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(\tau, f) \cdot \left[d(\tau, f) - |\mathbf{y}^H \mathbf{K}(\tau, f) \mathbf{x}|\right]^2 d\tau df$$

where $w(\tau, f)$ is the weighting function that specifies which area of CAF needs to be emphasized. For example, in the case of an X-band radar operating at the carrier frequency 10 GHz (i.e., the wavelength $\lambda = 0.03$ m), a target moving at the speed of $v = 150$ m/s can only induce a Doppler frequency shift of $\nu = 2v/\lambda = 10$ kHz. In contrast, the radar bandwidth is usually on the order of MHz. Therefore, the values of $\chi(\tau, f)$ for large f do not matter and can be safely ignored by choosing a zero weight in $w(\tau, f)$ correspondingly.

3. THE PROPOSED ALGORITHM

To solve (10), we first introduce auxiliary phases $\phi(\tau, f)$ and write the criterion as

$$\bar{g}(\mathbf{x}, \mathbf{y}, \phi) \quad (11)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(\tau, f) \left| d(\tau, f) e^{j\phi(\tau, f)} - \mathbf{y}^H \mathbf{K}(\tau, f) \mathbf{x} \right|^2 d\tau df.$$

It is not difficult to see that

$$\min_{\phi(\tau, f)} \bar{g}(\mathbf{x}, \mathbf{y}, \phi) = g(\mathbf{x}, \mathbf{y}) \quad (12)$$

and that the minimizer $\phi(\tau, f)$ is given by

$$\phi(\tau, f) = \arg\{\mathbf{y}^H \mathbf{K}(\tau, f) \mathbf{x}\}. \quad (13)$$

Therefore, introducing $\phi(\tau, f)$ does not change the value of the criterion. The benefit of doing so is to remove the operation of the absolute value in $|\mathbf{y}^H \mathbf{K}(\tau, f) \mathbf{x}|$ that is difficult to deal with mathematically. Such a technique was first introduced in [1] and also used for phase retrieval in optics applications; see [10] and also [11].

The minimization of (11) can be approached in a cyclic way: fix two variables out of $\{\mathbf{x}, \mathbf{y}, \phi(\tau, f)\}$ and minimize $\bar{g}$ with respect to the third variable. For fixed $\mathbf{x}$ and $\mathbf{y}$, the minimizer $\phi(\tau, f)$ is already given in (13). For fixed $\phi(\tau, f)$ and $\mathbf{y}$, the criterion $\bar{g}$ can be written as

$$\bar{g}(\mathbf{x}) = \mathbf{x}^H \mathbf{D}_1 \mathbf{x} - \mathbf{x}^H \mathbf{B} \mathbf{y} - \mathbf{y}^H \mathbf{B}^H \mathbf{x} \quad (14)$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(\tau, f) |d(\tau, f)|^2 d\tau df$$

$$= (\mathbf{x} - \mathbf{D}_1^{-1} \mathbf{B} \mathbf{y})^H \mathbf{D}_1 (\mathbf{x} - \mathbf{D}_1^{-1} \mathbf{B} \mathbf{y}) + \text{const}_1$$

where

$$\mathbf{D}_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(\tau, f) \mathbf{K}^H(\tau, f) \mathbf{y} \mathbf{y}^H \mathbf{K}(\tau, f) \, d\tau df, \quad (15)$$

$$\mathbf{B} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(\tau, f) d(\tau, f) e^{j\phi(\tau, f)} \mathbf{K}^H(\tau, f) \, d\tau df, \quad (16)$$

and const_1 is a term that does not depend on $\mathbf{x}$. It follows from (14) that the minimizer $\mathbf{x}$ is given by

$$\mathbf{x} = \mathbf{D}_1^{-1} \mathbf{B} \mathbf{y}. \quad (17)$$

Similarly, for fixed $\phi(\tau, f)$ and $\mathbf{x}$, the criterion $\bar{g}$ can be written as

$$\bar{g}(\mathbf{y}) = (\mathbf{y} - \mathbf{D}_2^{-1} \mathbf{B}^H \mathbf{x})^H \mathbf{D}_2 (\mathbf{y} - \mathbf{D}_2^{-1} \mathbf{B}^H \mathbf{x}) + \text{const}_2 \quad (18)$$

where

$$\mathbf{D}_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(\tau, f) \mathbf{K}(\tau, f) \mathbf{x} \mathbf{x}^H \mathbf{K}^H(\tau, f) \, d\tau df \quad (19)$$

and const$_2$ is a term that does not depend on **y**. The minimizer **y** is thus given by

$$\mathbf{y} = \mathbf{D}_2^{-1}\mathbf{B}^H\mathbf{x}. \tag{20}$$

The above steps are summarized in Table 1. Each step of the algorithm decreases the minimization criterion and thus the algorithm is guaranteed to converge to a local minimum. Note that $\phi(\tau, f)$ and $\mathbf{K}(\tau, f)$ are functions of the time delay τ and Doppler frequency shift f, both of which range from $-\infty$ to ∞ theoretically. In practice, however, the maximum time delay can be chosen as Nt_p (which is the time duration of the signal) and the maximum Doppler frequency shift can be chosen as the signal bandwidth.

When the rectangular pulse (3) is used, each element of $\mathbf{K}(\tau, f)$ has a closed-form formula. Furthermore, (3) is a type of orthogonal shaping pulse (i.e., $\int_{-\infty}^{\infty} p_k(t)p_l(t)dt$ equals 1 when $k = l$ and equals 0 otherwise), which leads to the fact that $\mathbf{D}_1 = \|\mathbf{y}\|^2\mathbf{I}_N$ and $\mathbf{D}_2 = \|\mathbf{x}\|^2\mathbf{I}_N$ if $w(\tau, f)$ is equal to 1 for all (τ, f). (The proof of these properties is omitted for the sake of brevity.) Such properties can be leveraged to greatly speed up the computation of $\mathbf{K}(\tau, f)$, $\mathbf{D}_1$ and $\mathbf{D}_2$, which generally requires numerical approximations with fine sampling.

In many applications, low peak-to-average power ratio (PAR) is desired for the transmit waveform **x** (but not so for the receive filter **y**). The PAR of **x** is defined as

$$\text{PAR}(\mathbf{x}) = \frac{\max_n |x_n|^2}{\frac{1}{N}\sum_{n=1}^{N} |x_n|^2} \tag{21}$$

and PAR $= 1$ (the lowest possible value) corresponds to a constant-modulus sequence. If such a property is required, the following operation

$$x_n \leftarrow e^{j\arg(x_n)}, \quad n = 1, \ldots, N \tag{22}$$

can be added after $\mathbf{x} = \mathbf{D}_1^{-1}\mathbf{B}\mathbf{y}$ in Step 2 of Table 1. Note that combined with (22), Step 2 does not necessarily decrease the criterion $\bar{g}$ unless $\mathbf{D}_1$ is proportional to $\mathbf{I}_N$ (a case mentioned in the previous paragraph). For the numerical examples in Section 4, (22) will be mentioned explicitly if used.

Table 1. The proposed algorithm to minimize (10)

0: Initialize **x** and **y** using randomly generated sequences.
Repeat the following three steps until a certain stop criterion is met (e.g., until $\|\mathbf{x}^{(i)} - \mathbf{x}^{(i+1)}\|^2 + \|\mathbf{y}^{(i)} - \mathbf{y}^{(i+1)}\|^2 < 10^{-3}$ where $(\cdot)^{(i)}$ indicates the ith iteration).
1. $\phi(\tau, f) = \arg\{\mathbf{y}^H\mathbf{K}(\tau, f)\mathbf{x}\}$
2. $\mathbf{x} = \mathbf{D}_1^{-1}\mathbf{B}\mathbf{y}$
3. $\mathbf{y} = \mathbf{D}_2^{-1}\mathbf{B}^H\mathbf{x}$
where the definitions of $\mathbf{D}_1$, $\mathbf{B}$ and $\mathbf{D}_2$ are given in (15)(16)(19), respectively.

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4. NUMERICAL EXAMPLES

Assume $N = 50$ subpulses with the rectangular pulse shaping (3). The time duration of each subpulse is t_p and that of the total waveform is $T = Nt_p$. In the simulations, the time delay τ is normalized by T and the Doppler frequency f is normalized by $1/T$, so the particular value of t_p does not affect the final sequences obtained from the algorithm. Randomly generated sequences are used in the initialization. The algorithm usually converges within a few hundred iterations and takes tens of minutes in an ordinary PC.
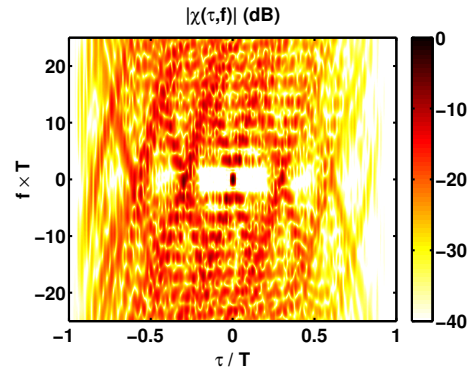

Fig. 1. The synthesized CAF when $d(\tau, f)$ and $w(\tau, f)$ are given in (23) and (24), respectively.
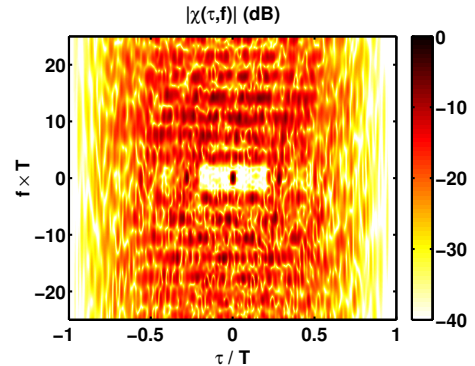

Fig. 2. The synthesized CAF obtained in the same setting as used in Fig. 1, except that (22) is used additionally in Step 2 of Table 1.

4.1. Example 1

Suppose that a thumbtack CAF is desired:

$$d(\tau, f) = \begin{cases} N, & (\tau, f) = (0, 0) \\ 0, & \text{elsewhere} \end{cases}. \tag{23}$$

The weighting function is selected as

$$w(\tau, f) = \begin{cases} 1, & (\tau, f) \in \Omega \ \& \ (\tau, f) \notin \Omega_m \\ 0, & \text{elsewhere} \end{cases} \tag{24}$$

where $\Omega = \{(\tau, f), \ |\tau| \le 10t_p \ \& \ |f| \le 2/T\}$ is the region of interest and $\Omega_m = \{(\tau, f), \ |\tau| \le t_p \ \& \ |f| \le 1/T \ \& \ \tau f \ne 0\}$ is the mainlobe area without the origin. Ω_m is excluded from the region of interest in order to compensate for the sharp change of $d(\tau, f)$ near the origin.

Fig. 1 shows the CAF of the so-obtained sequences $\mathbf{x}$ and $\mathbf{y}$. The white rectangular area close to the origin is the desired low sidelobe region. The peak-to-average power ratio (PAR) of $\mathbf{x}$ is 3.3 and that of $\mathbf{y}$ is 3.5. This relatively high PAR is due to the fact that the amplitude of $\mathbf{x}$ or $\mathbf{y}$ is not constrained in the algorithm. To ensure a constant-modulus $\mathbf{x}$, we can add (22) in Step 2 of Table 1. The resulting CAF is shown in Fig. 2, which has somewhat higher sidelobes in the region of interest than Fig. 1.

4.2. Example 2

The thumbtack-shaped CAF shown in Section 4.1 leads to good range and Doppler resolution within the clear area and it is often demanded in practice. In this example, however, we try to synthesize the CAF in Fig. 3(a) that has a diagonal ridge of height N and equals zero elsewhere. Such a CAF is tolerant to Doppler frequency shifts and is desired when a bank of filters is too expensive for different Doppler frequencies [12].

We set the weighting function $w(\tau, f) = 1$ for all (τ, f) and add (22) in Step 2 of Table 1 so that $\mathbf{x}$ has constant modulus. The so-obtained CAF is shown in Fig. 3(b), which approximates Fig. 3(a) well.

5. CONCLUSIONS

An algorithm was presented to synthesize a given cross ambiguity function (CAF) by minimizing the integrated square error between the given CAF and a realizable CAF. The given CAF can be any 2D function that illustrates the general shape of a desired CAF and is not required to be a valid ambiguity function. Weighting was introduced in the formulation that accounts for the emphasis on a certain region in the (τ, f) plane. The generated waveforms have relatively low peak-to-average power ratios. Two examples were provided showing the synthesis of a CAF with a clear area in the middle and with a diagonal ridge, respectively.

6. REFERENCES

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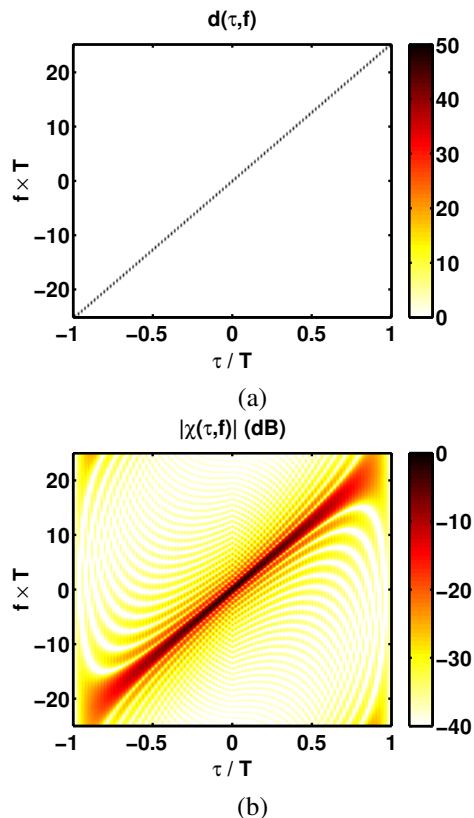
(a)

(b)

Fig. 3. (a) The desired CAF and (b) the synthesized CAF.

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