Lecture 10

Forward Kinematics

Alli Nilles

Modern Robotics Chapter 4
October 1, 2019

Admin

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- Homework 4 should be demonstrated individually to your TA in lab this week or next. Next week is the last week of Lab 3.
- Homework 5 is due this Friday.
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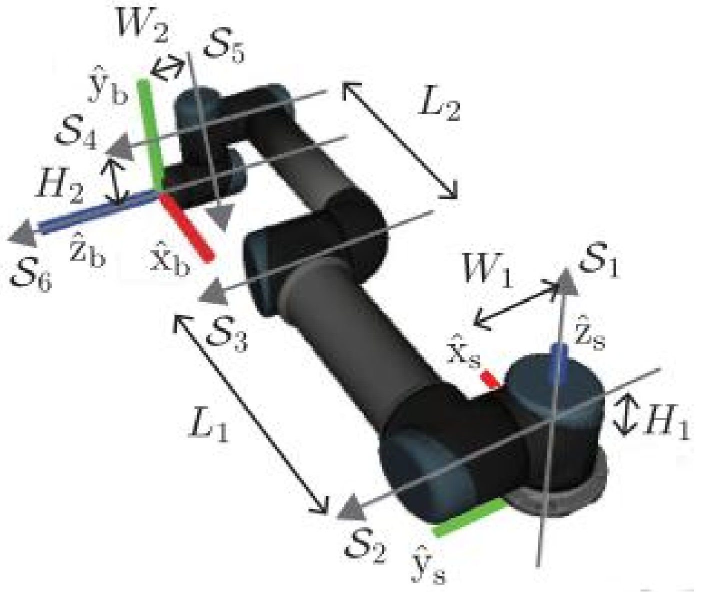

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- Our robot is a kinematic chain, made of rigid *links* connected by movable *joints*
- No branches or loops (will discuss later)
- All joints have one degree of freedom and are *revolute* or *prismatic*

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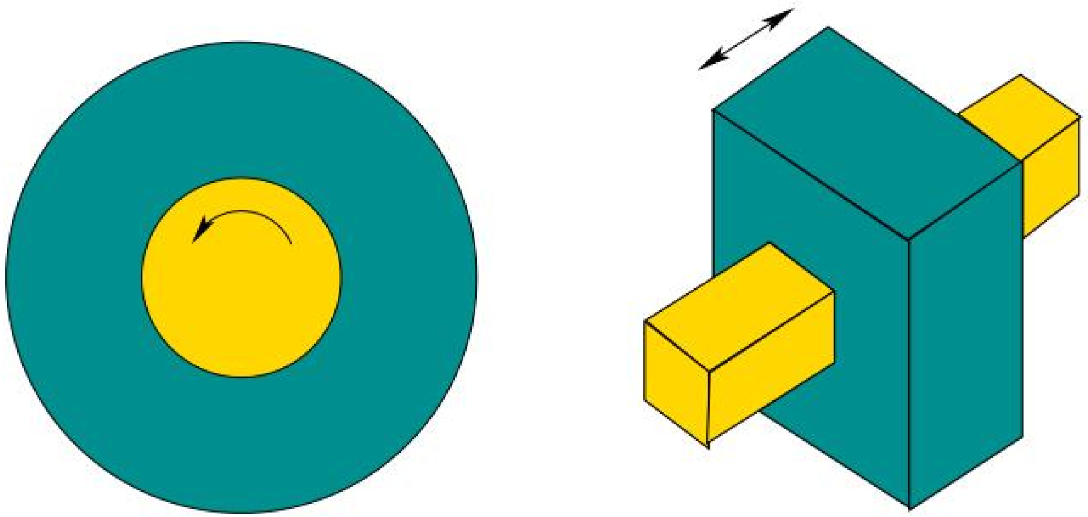

Revolute
1 Degree of Freedom

Prismatic
1 Degree of Freedom

Review on Screw Motions

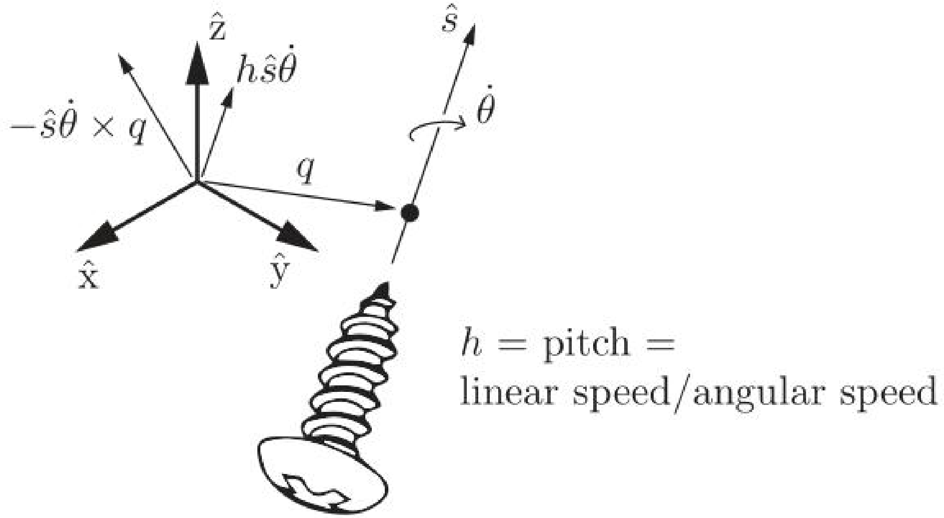

Figure 3.19: A screw axis $\mathcal{S}$ represented by a point q, a unit direction $\hat{s}$, and a pitch h.

Review on Screw Motions

Definition 3.24 from Modern Robotics: a screw axis $\mathcal{S}$ is written as

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

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  - where the pitch of the screw is infinite and the motion is a translation along the axis defined by v.

Screw Motions as Matrix Exponential

The screw axis $\mathcal{S}$ can be expressed in matrix form as

$$[\mathcal{S}_i] = \begin{bmatrix} [\omega_i] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

where $[\dots]$ is the skew symmetric form.

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You can think of the Lie algebra as related to a tangent space, so a screw vector gives us a description of instantaneous tangent motion, and the exponential function "integrates" this motion over a displacement θ.

Actual Form of Matrix Exponential

$$e^{[\mathcal{S}]\theta} = I + [\mathcal{S}]\theta + [\mathcal{S}]^2\frac{\theta^2}{2!} + [\mathcal{S}]^3\frac{\theta^3}{3!} + \cdots$$

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Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \qquad (3.88)$$

If $\omega = 0$ and $\|v\| = 1$, then

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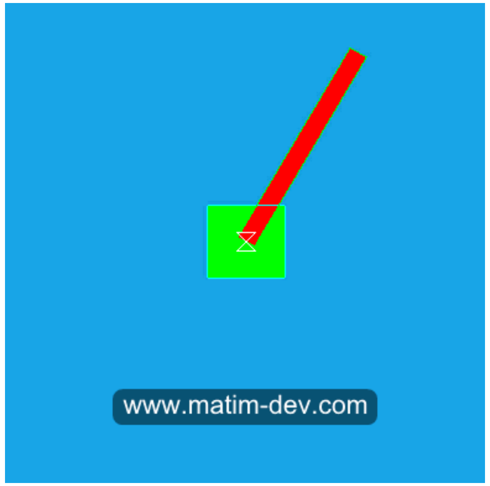
Modelling Robot Joints as Screw Motions

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

case 1:



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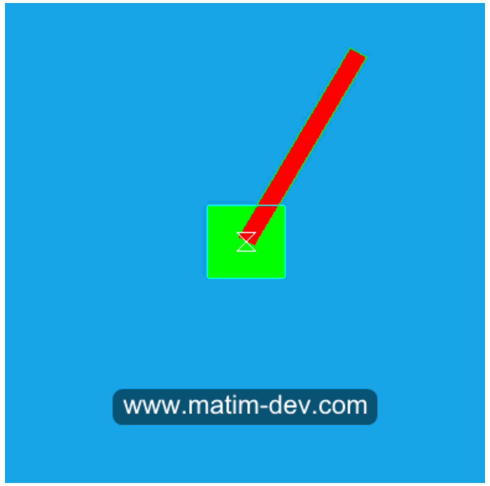

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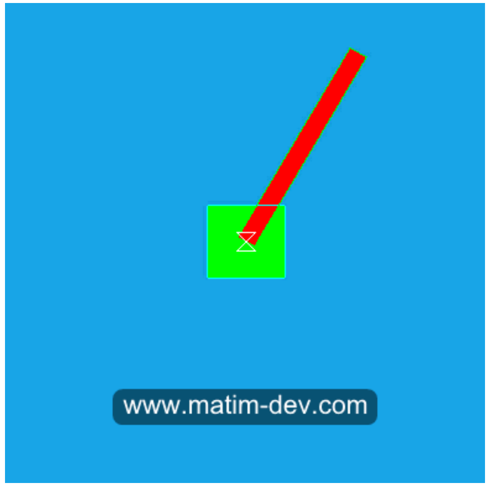

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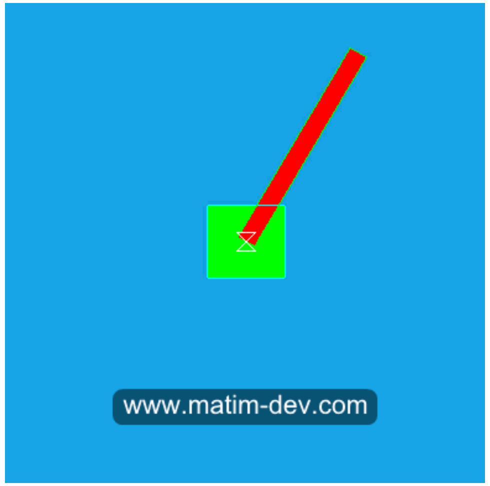

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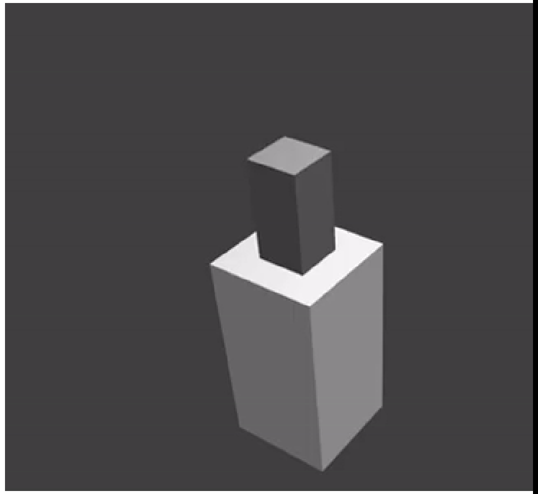

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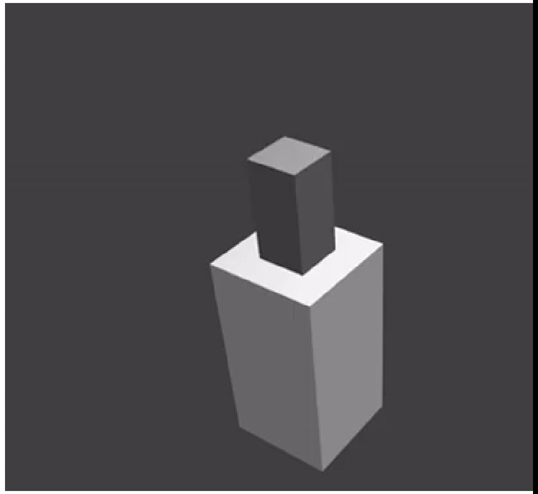

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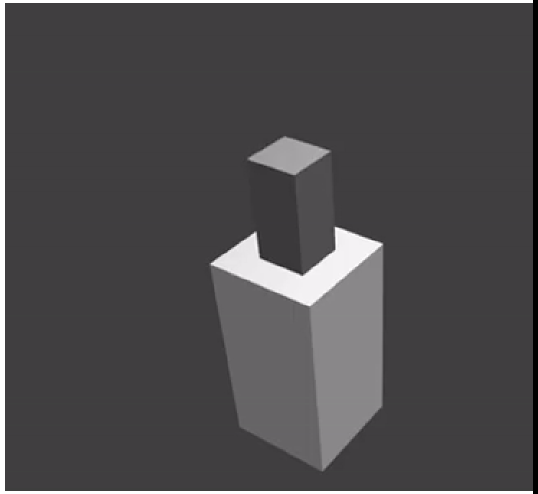

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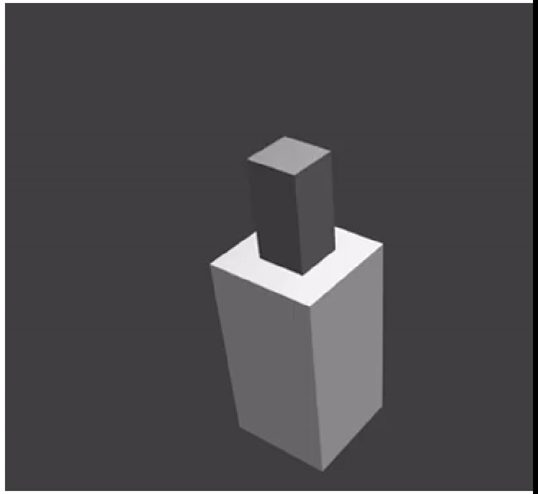

Product of Exponentials Approach

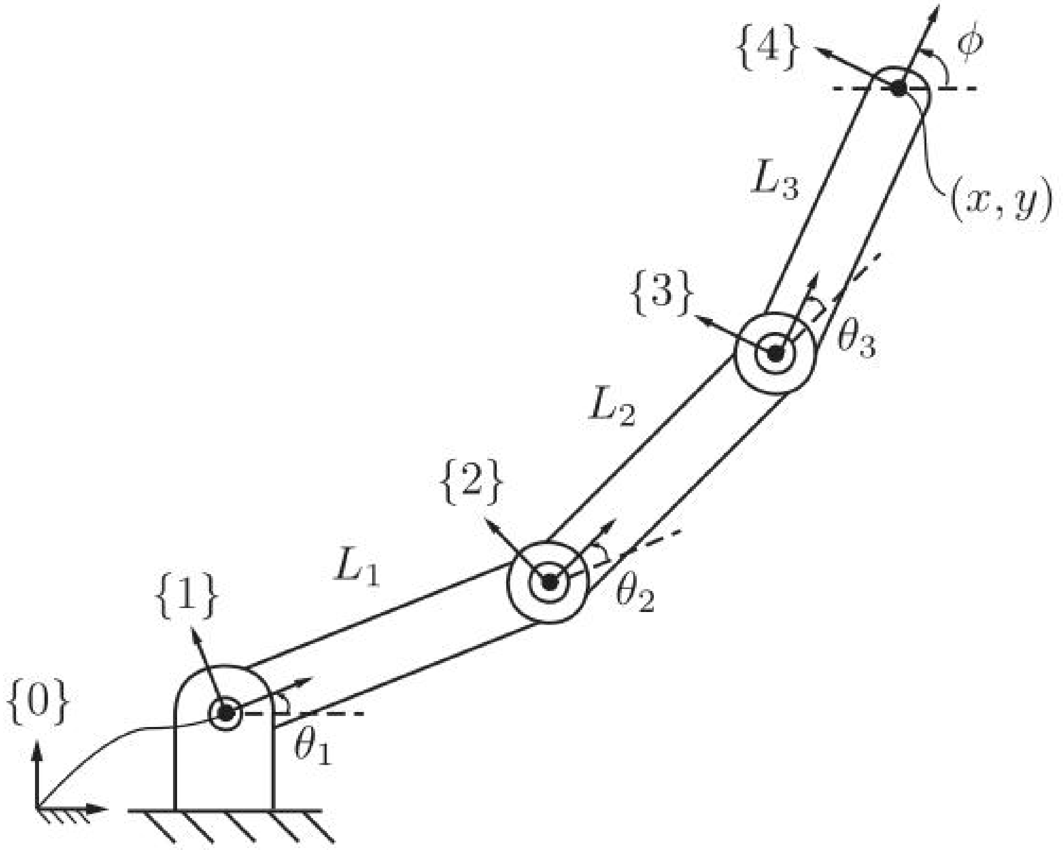

Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the x̂- and ŷ-axis is shown; the ẑ-axes are parallel and out of the page.

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- Put all joints in "zero position"
- Let $M \in SE(3)$ be the configuration of $\{b\}$ in the $\{s\}$ frame when robot is in zero position

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This form composes nicely through multiplication, giving us the Product of Exponentials (PoE) formula!

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

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Visualizing the Formula

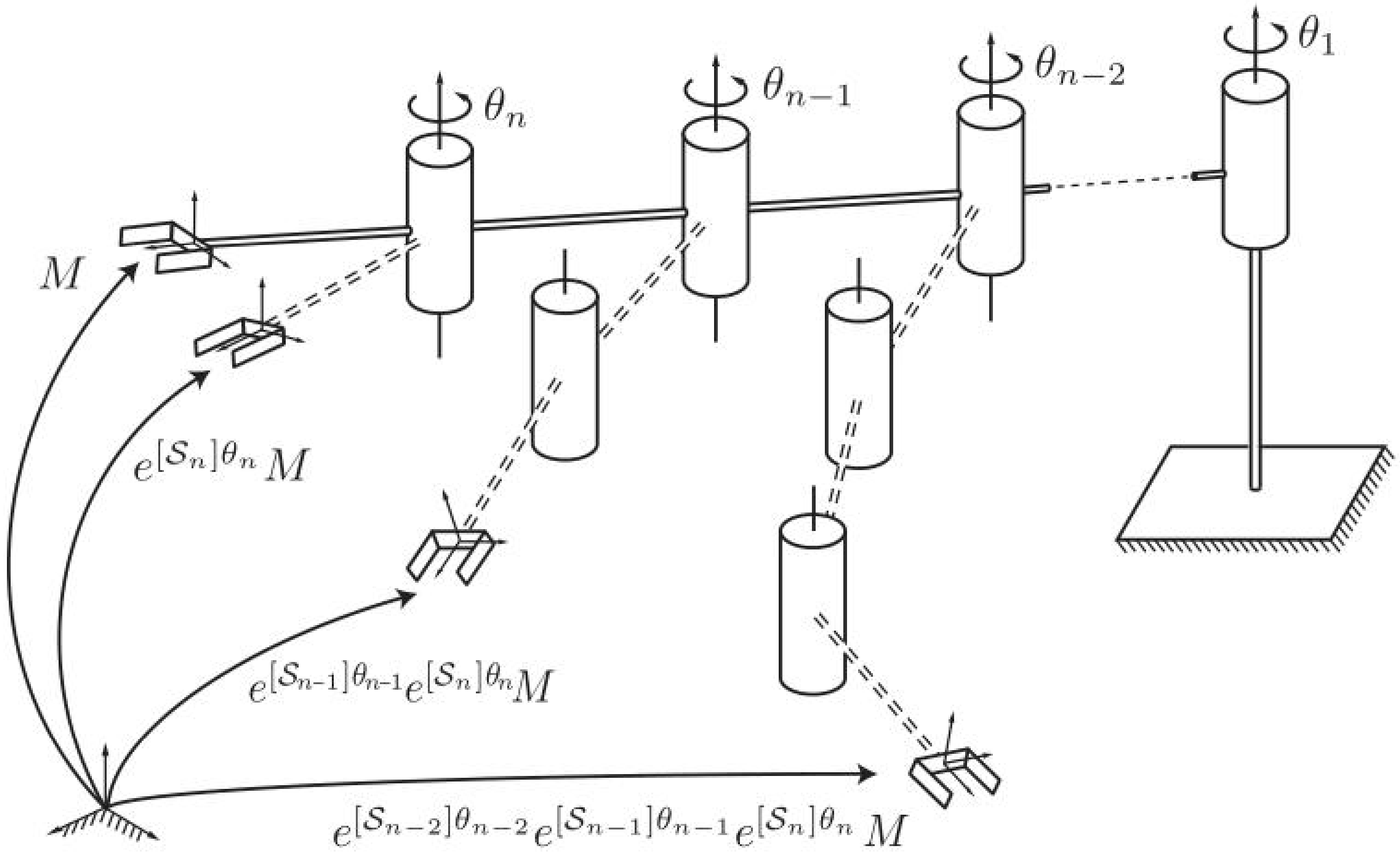

Figure 4.2: Illustration of the PoE formula for an n-link spatial open chain.

Example 1

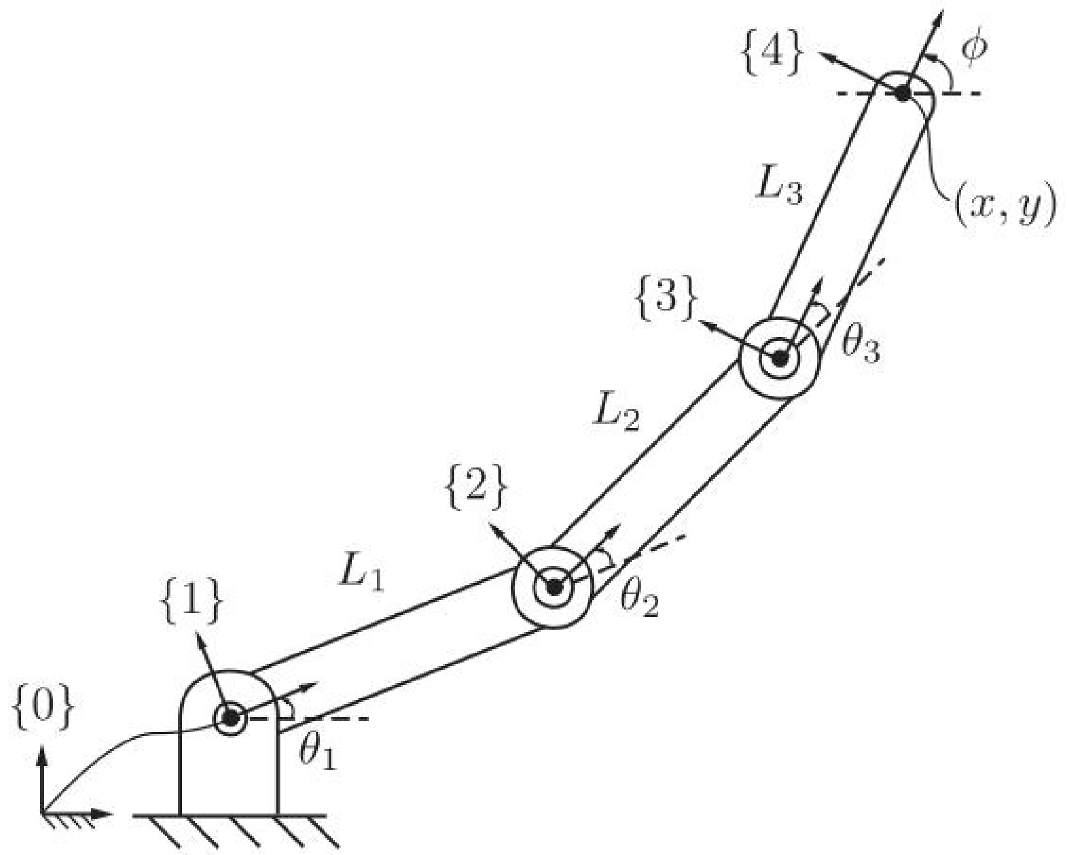

Figure 4.1: Forward kinematics of a 3R planar open chain. For each frame, the $\hat{x}$- and $\hat{y}$-axis is shown; the $\hat{z}$-axes are parallel and out of the page.

Example 1

$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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All axes:

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$$v_1 = (0, 0, 0)$$

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$$v_3 = (0, -(L_1 + L_2), 0)$$

Example 1

Form $e^{[\mathcal{S}_i]\theta}$ for each joint:

$$e^{[\mathcal{S}_i]\theta} = \begin{bmatrix} e^{[\omega_i]\theta} & (I\theta + (1 - \cos(\theta)[\omega_i] + (\theta - \sin(\theta)[\omega_i]^2)v_i \\ 0 & 1 \end{bmatrix}$$

And compose with M

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

Example 2

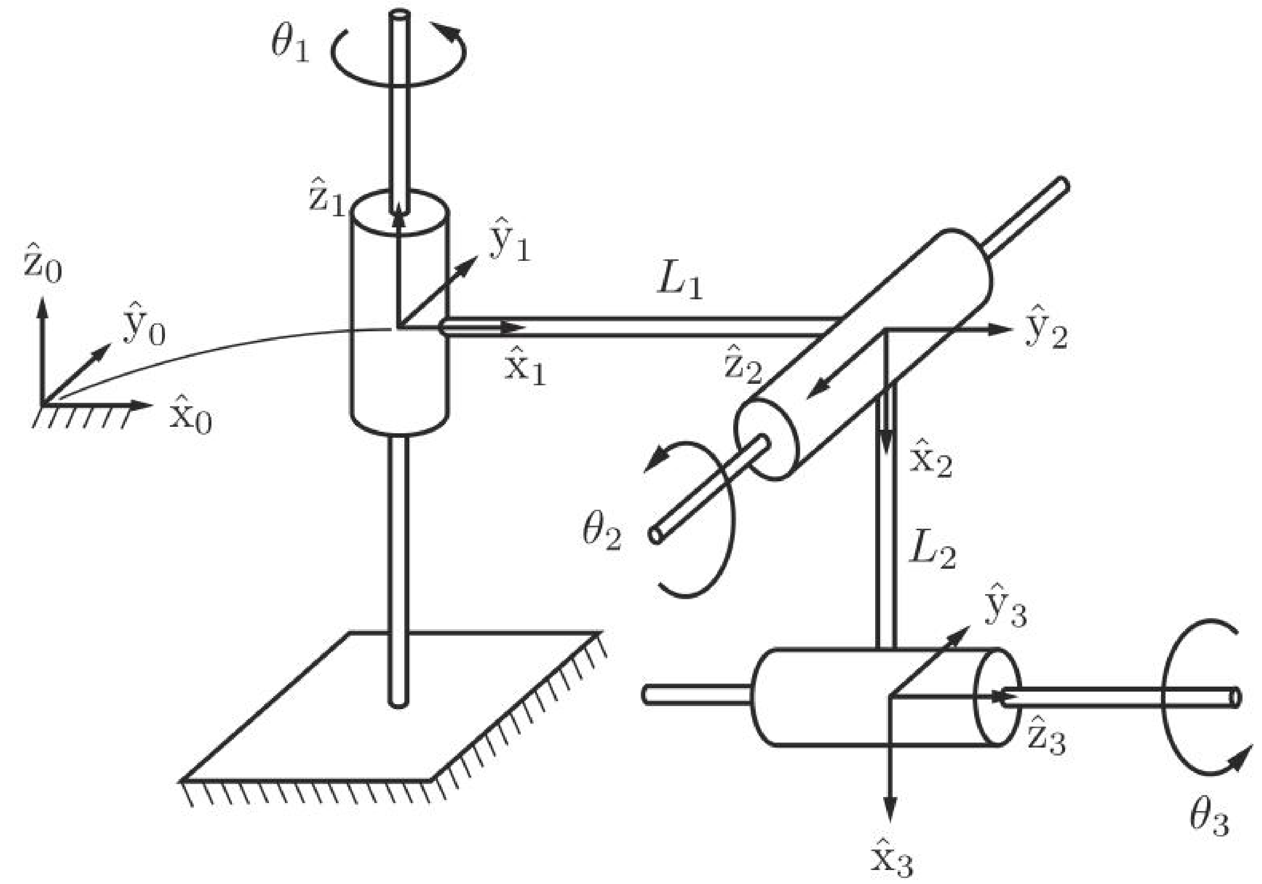

Figure 4.3: A 3R spatial open chain.

Example 2

First find M:

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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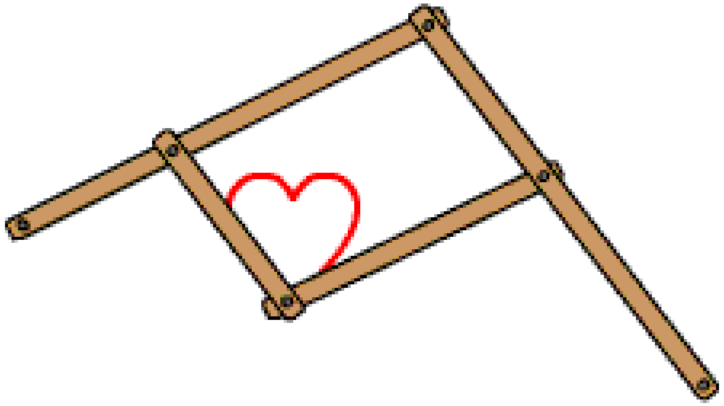
For joint 3: $\omega_3 = (1, 0, 0)$ $\quad q_3 = (0, 0, -L_2)$ $\quad v_3 = (0, -L_2, 0)$

Next Time



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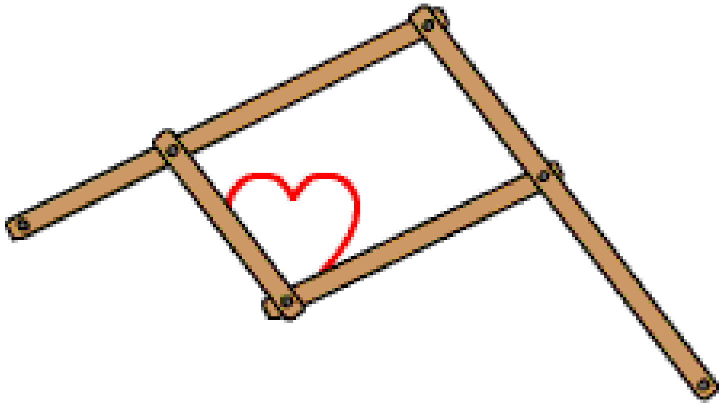
- Modelling robots in the Universal Robot Description Format
- Different kinds of joints
- What if my robot isn't a kinematic chain??



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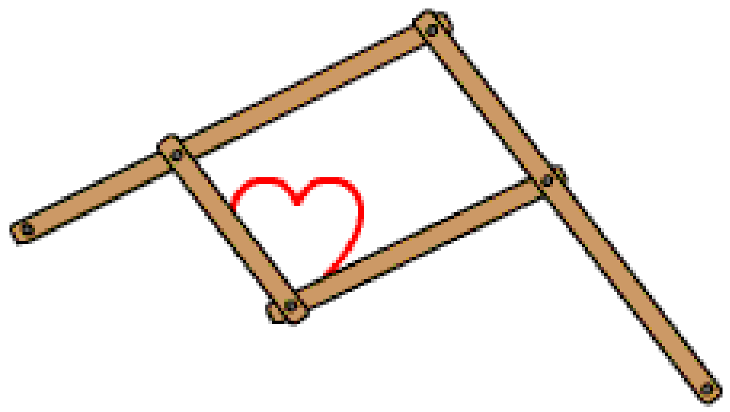

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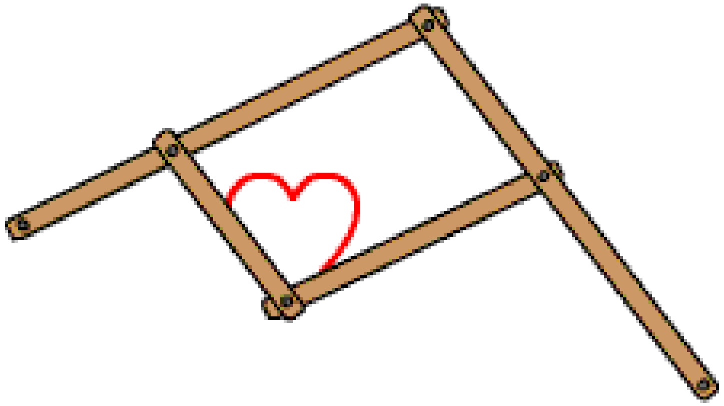

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