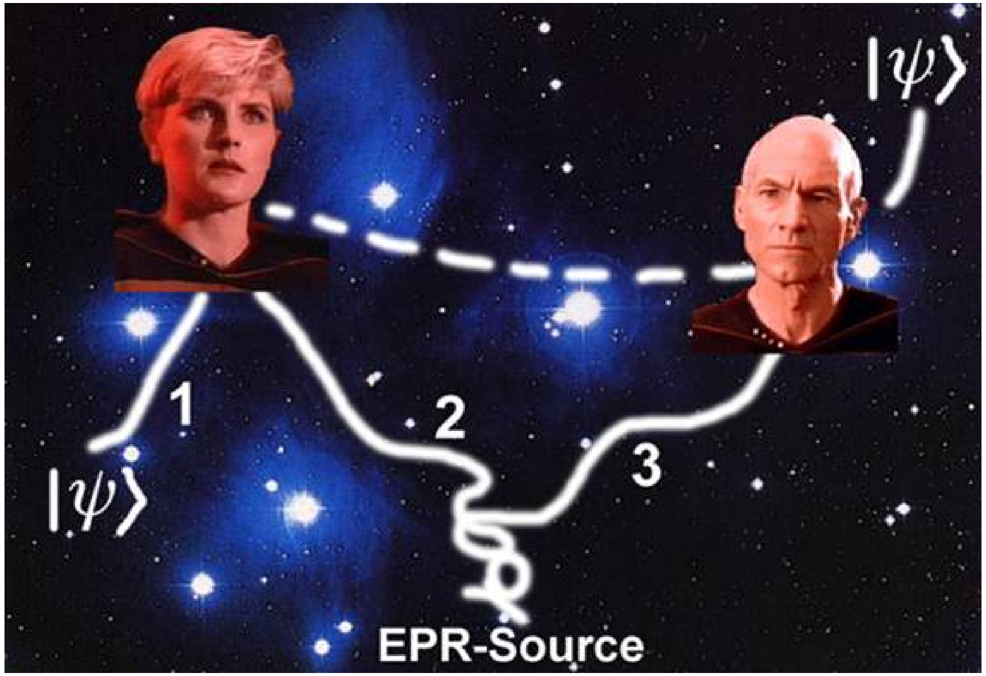
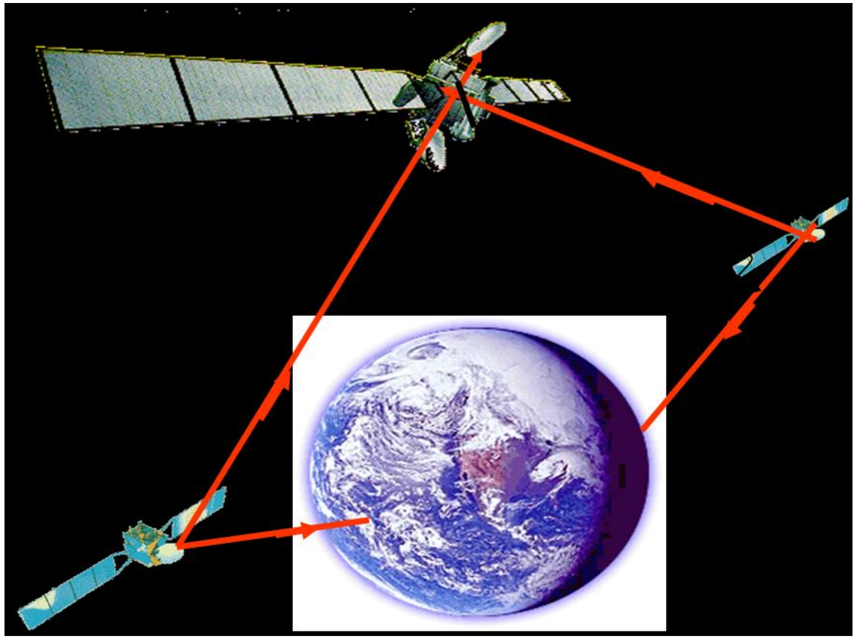
Lecture Note 4

Entanglement Purification

Jian-Wei Pan

Introduction(1)





Both long distance quantum teleportation or global quantum key distribution need to distribute a certain supply of pairs of particles in a maximally entangled state to two distant users.

Introduction(2)

However, distributed qubits will interact with the environment and decoherence will happen.

$$|0\rangle|E\rangle \xrightarrow{U(t)} |0\rangle|E_0(t)\rangle \qquad |1\rangle|E\rangle \xrightarrow{U(t)} |1\rangle|E_1(t)\rangle$$

Here $|0\rangle$, $|1\rangle$ represents the qubit state and $|E\rangle$ represents the environment initial state, $|U(t)\rangle$ is the joint unitary time evolution operator. For arbitrary qubit state:

$$(\alpha_0|0\rangle+\alpha_1|1\rangle)|E\rangle \xrightarrow{U(t)} \alpha_0|0\rangle|E_0(t)\rangle+\alpha_1|1\rangle|E_1(t)\rangle$$

$$\rho_q(t)=Tr_E\rho_{q+E}=\begin{bmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^*\langle E_1|E_0\rangle \\ \alpha_1\alpha_0^*\langle E_0|E_1\rangle & |\alpha_1|^2 \end{bmatrix}$$

The off-diagonal element of the qubit density matrix will drop down with the rate $\langle E_0(t)|E_1(t)\rangle=e^{-\Gamma t}$, $\Gamma(t)$ depends on the coupling between qubit and environment.

The maximally entangled state will be in some mixed state with a certain entanglement fidelity due to the process.

Introduction(3)

Solution to the decoherence

- Quantum Error Correction for Quantum computation
- Quantum Entanglement Purification for Quantum Communication
- Quantum Communication based on Decoherence Free Subspace

The basic idea of entanglement purification is to extract from a large ensemble of low-fidelity EPR pairs a small sub-ensemble with sufficiently high fidelity EPR pair.

- Entanglement Purification----improve purity of any kind of unknown mixed state
- Local filtering----improve entanglement quality of known state
- Entanglement Concentration---- improve entanglement quality for pure unknown state

Principle of Entanglement Purification

Model:

Suppose Alice want to share an ensembles of 2-qubit maximally entangled states $|\Psi^-\rangle = (|H\rangle|V\rangle - |V\rangle|H\rangle)/\sqrt{2}$ with Bob via a noise channel. After the transmission, the state has been changed into a general mixed state M, the purity of M can be expressed as $F = \langle\Psi^-|M|\Psi^-\rangle$.

Several ingredients in the Entanglement Purification:

(1) Bell states: $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle \pm |1\rangle|0\rangle)$, $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle)$;

(2) Werner state: $W_F = F|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{1-F}{3}|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-|$;

(3) Pauli rotation: σ_x , σ_y, σ_z;

(4) CNOT gate: $|x\rangle|y\rangle \rightarrow |x\rangle|x \oplus y\rangle$

$$|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle, \quad |1\rangle|0\rangle \rightarrow |1\rangle|1\rangle$$
$$|0\rangle|1\rangle \rightarrow |0\rangle|1\rangle, \quad |1\rangle|1\rangle \rightarrow |1\rangle|0\rangle$$

Principle of Entanglement Purification

Steps of Entanglement Purification:

(1) Random Bilateral Pauli Rotation on each photon in the states. This step can change arbitrary mixed state into Werner state:

$$W_F = F|\Psi^-\rangle\langle\Psi^-| + \frac{1-F}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{1-F}{3}|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}|\Phi^-\rangle\langle\Phi^-|$$

(2) A Unilateral σ_y Rotations converting the states from mostly $|\Psi^-\rangle$ Werner states to the analogous mostly $|\Phi^+\rangle$ states, (σ_y maps $|\Psi^\pm\rangle \to |\Phi^\mu\rangle$) ,

(3) Bilateral CNOT operations on two photon pairs in the Werner state.

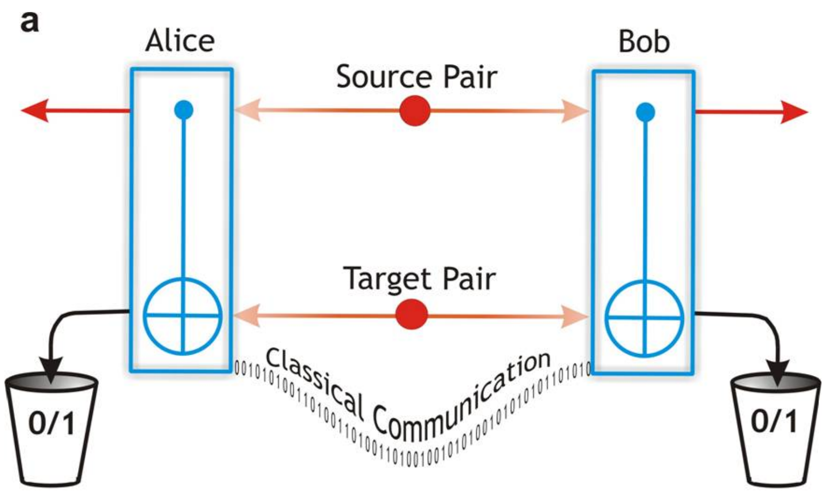


Figure 1a

[C. Bennett, et al., PRL 76, 772 (1996)]

Bilateral CNOT operations will convert the Bell states as the form:

PROBABLITY	BEFORE		AFTER	
	Source	Target	Source	Target
F^2	Φ^+	Φ^+	Φ^+	Φ^+
F(1-F)/3	Φ^-	Φ^+	Φ^-	Φ^+
F(1-F)/3	Ψ^+	Φ^+	Ψ^+	Ψ^+
F(1-F)/3	Ψ^-	Φ^+	Ψ^-	Ψ^+
$(1-F)^2/9$	Ψ^+	Ψ^+	Ψ^+	Φ^+
$(1-F)^2/9$	Ψ^-	Ψ^+	Ψ^-	Φ^+
F(1-F)/3	Φ^+	Ψ^+	Φ^+	Ψ^+
$(1-F)^2/9$	Φ^-	Ψ^+	Φ^-	Ψ^+
F(1-F)/3	Φ^+	Φ^-	Φ^-	Φ^-
$(1-F)^2/9$	Φ^-	Φ^-	Φ^+	Φ^-
$(1-F)^2/9$	Ψ^+	Φ^-	Ψ^-	Ψ^-
$(1-F)^2/9$	Ψ^-	Φ^-	Ψ^+	Ψ^-
$(1-F)^2/9$	Ψ^+	Ψ^-	Ψ^-	Φ^-
$(1-F)^2/9$	Ψ^-	Ψ^-	Ψ^+	Φ^-
F(1-F)/3	Φ^+	Ψ^-	Φ^-	Ψ^-
$(1-F)^2/9$	Φ^-	Ψ^-	Φ^+	Ψ^-

For example:

F(1-F)/3 Probability, we have $\left|\Psi^+\right\rangle_S \left|\Phi^+\right\rangle_T$

$$CNOT(\left|\Psi^+\right\rangle_S \left|\Phi^+\right\rangle_T) = \frac{1}{2} CNOT[(\left|0\right\rangle_{SA}\left|1\right\rangle_{SB} + \left|1\right\rangle_{SA}\left|0\right\rangle_{SB})(\left|0\right\rangle_{TA}\left|0\right\rangle_{TB} + \left|1\right\rangle_{TA}\left|1\right\rangle_{TB})]$$

$$= \frac{1}{2} CNOT(\left|0\right\rangle_{SA}\left|0\right\rangle_{TA}\left|1\right\rangle_{SB}\left|0\right\rangle_{TB} + \left|0\right\rangle_{SA}\left|1\right\rangle_{TA}\left|1\right\rangle_{SB}\left|1\right\rangle_{TB} + \left|1\right\rangle_{SA}\left|0\right\rangle_{TA}\left|0\right\rangle_{SB}\left|0\right\rangle_{TB} + \left|1\right\rangle_{SA}\left|1\right\rangle_{TA}\left|0\right\rangle_{SB}\left|1\right\rangle_{TB})$$

$$= \frac{1}{2} (\left|0\right\rangle_{SA}\left|0\right\rangle_{TA}\left|1\right\rangle_{SB}\left|1\right\rangle_{TB} + \left|0\right\rangle_{SA}\left|1\right\rangle_{TA}\left|1\right\rangle_{SB}\left|0\right\rangle_{TB} + \left|1\right\rangle_{SA}\left|1\right\rangle_{TA}\left|0\right\rangle_{SB}\left|0\right\rangle_{TB} + \left|1\right\rangle_{SA}\left|0\right\rangle_{TA}\left|0\right\rangle_{SB}\left|1\right\rangle_{TB})$$

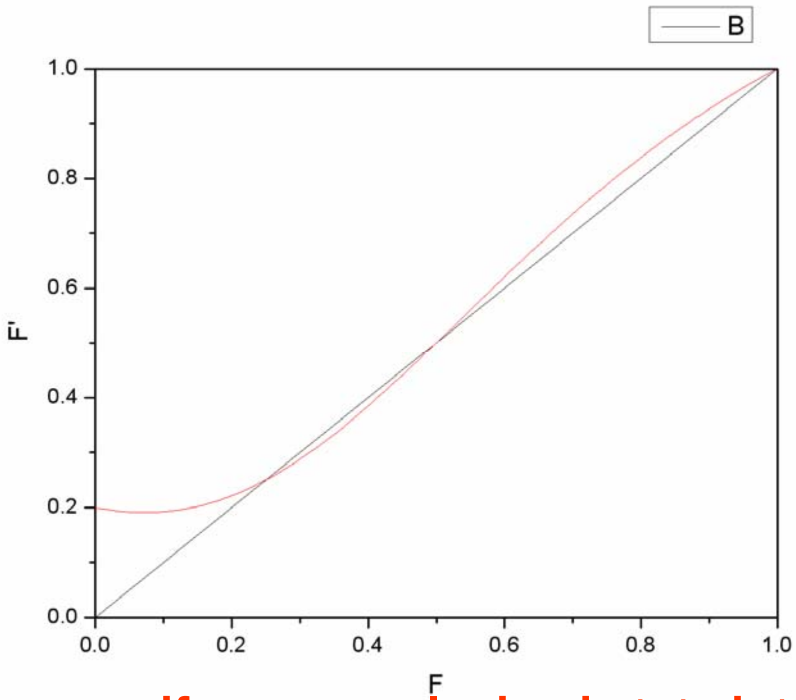
$$= \left|\Psi^+\right\rangle_S \left|\Psi^+\right\rangle_T$$

Principle of Entanglement Purification

(4) Measuring the target pair in z basis, if the result is parallel, keep the source pair, if not, discard the source pairs. After the protocol, the purity of the source pair has been improved:

$$F' = \frac{F^2 + \frac{1}{9}(1-F)^2}{F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2}.$$

If $F > 1/2$, then $F' > F$



Via several this kind processes, we can purify a general mixed state into a highly entangled state.

[C. Bennett, et al., PRL 76, 772 (1996)]

Experiment of Entanglement Purification

- The theoretical proposal needs a C-Not gate which requires high non-linearities.
- Unluckily, no efficient C-Not gate exists at the moment

A better solution for experiments

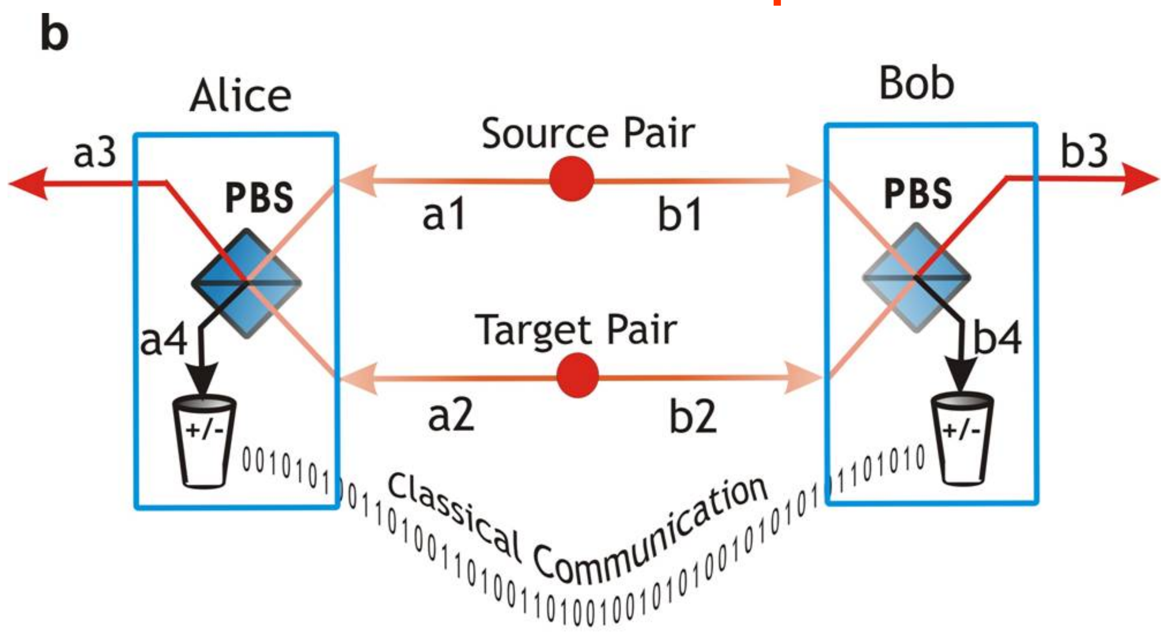


Figure 1b

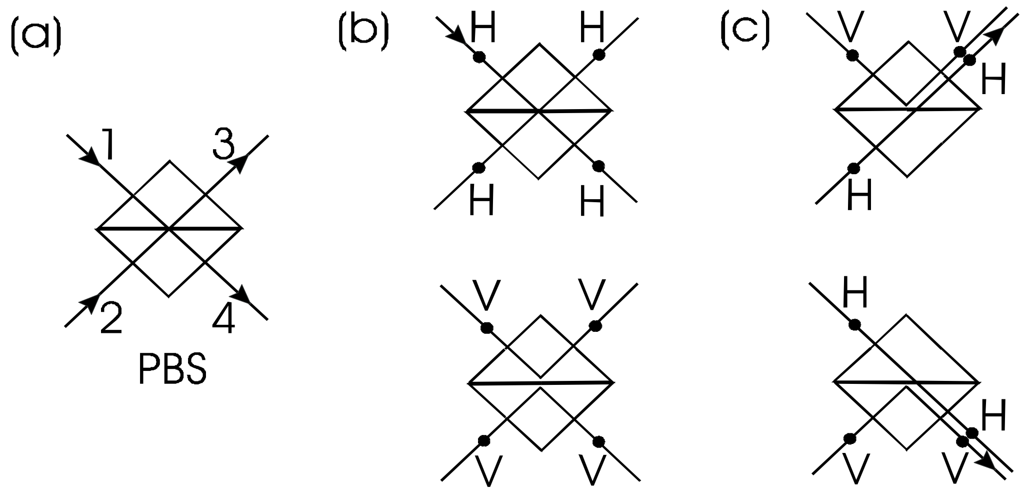
[J.-W. Pan et al., Nature 410, 1067 (2001)]

Initial State: $\rho_{ab} = F\,|\Phi^+\rangle_{ab}\langle\Phi^+| + (1-F)\,|\Psi^+\rangle_{ab}\langle\Psi^+|$

Purified State: $\rho'_{ab} = F'\,|\Phi^+\rangle_{ab}\langle\Phi^+| + (1-F')\,|\Psi^+\rangle_{ab}\langle\Psi^+|$

$$F' = \frac{F^2}{F^2 + (1-F)^2} > F(\text{if } F > 1/2)$$

F^2	$\lvert\Phi_s^{\,+}\rangle_{ab}\cdot\lvert\Phi_t^{\,+}\rangle_{ab}$
$F(1-F)$	$\lvert\Phi_s^{\,+}\rangle_{ab}\cdot\lvert\Psi_t^{\,+}\rangle_{ab}$
$(1-F)F$	$\lvert\Psi_s^{\,+}\rangle_{ab}\cdot\lvert\Phi_t^{\,+}\rangle_{ab}$
$(1-F)^2$	$\lvert\Psi_s^{\,+}\rangle_{ab}\cdot\lvert\Psi_t^{\,+}\rangle_{ab}$



For the first case,

$$|\Phi^+\rangle_{a1b1}\cdot|\Phi^+\rangle_{a2b2} = \tfrac{1}{2}(|H\rangle_{a1}|H\rangle_{b1}+|V\rangle_{a1}|V\rangle_{b1})\cdot(|H\rangle_{a2}|H\rangle_{b2}+|V\rangle_{a2}|V\rangle_{b2})$$

$H_{a1}H_{a2}H_{b1}H_{b2}$

$V_{a1}V_{a2}V_{b1}V_{b2}$

↓

Four-fold events

$H_{a1}V_{a2}H_{b1}V_{b2}$

$V_{a1}H_{a2}V_{b1}H_{b2}$

↓

No four-fold events

50%

$$\frac{1}{\sqrt{2}}(|H\rangle_{a3}|H\rangle_{a4}|H\rangle_{b3}|H\rangle_{b4}+|V\rangle_{a3}|V\rangle_{a4}|V\rangle_{b3}|V\rangle_{b4})$$

↓ probability of $F^2/2$

$$|\Phi^+>_{a3b3}=\frac{1}{\sqrt{2}}(|H>_{a3}|H>_{b3}+|V>_{a3}|V>_{b3})$$

Similarly,

$$|\Psi^+\rangle_{a1b1} \cdot |\Psi^+\rangle_{a2b2}$$

$$|\Phi^+\rangle_{a1b1} \cdot |\Psi^+\rangle_{a2b2}$$

$$|\Psi^+\rangle_{a1b1} \cdot |\Phi^+\rangle_{a2b2}$$

50%

No four-fold events

$$\frac{1}{\sqrt{2}}(|H\rangle_{a3}|H\rangle_{a4}|V\rangle_{b3}|V\rangle_{b4} + |V\rangle_{a3}|V\rangle_{a4}|H\rangle_{b3}|H\rangle_{b4})$$

probability of $(1-F)^2/2$

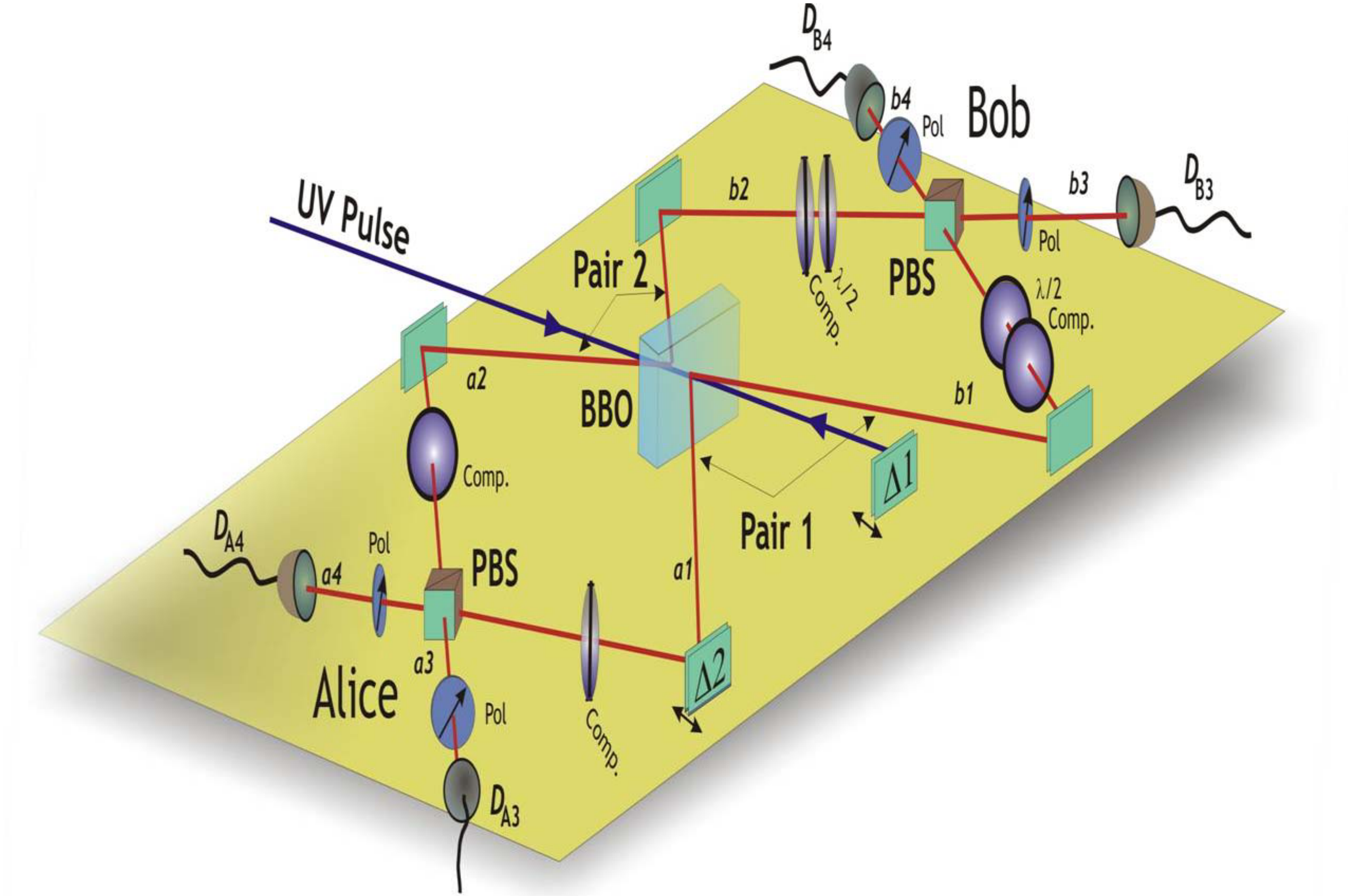
$$|\Psi^+\rangle_{a3b3} = \frac{1}{\sqrt{2}}(|H\rangle_{a3}|V\rangle_{b3} + |V\rangle_{a3}|H\rangle_{b3})$$

In this way…

$$\rho'_{ab} = F'|\Phi^+\rangle_{ab}\langle\Phi^+| + (1-F')|\Psi^+\rangle_{ab}\langle\Psi^+|$$

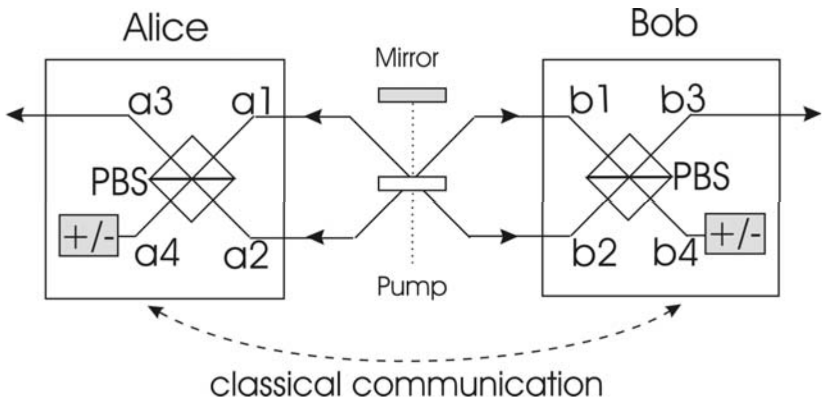
$$F' = \frac{F^2}{F^2 + (1-F)^2} > F \ (\text{if } F > 1/2)$$

Experimental Realization



[J.-W. Pan et al., Nature 423, 417 (2003)]

Four-fold contribution from double pairs emission



a1b1 contribution $V_{a3}H_{a4}V_{b3}H_{b4}$

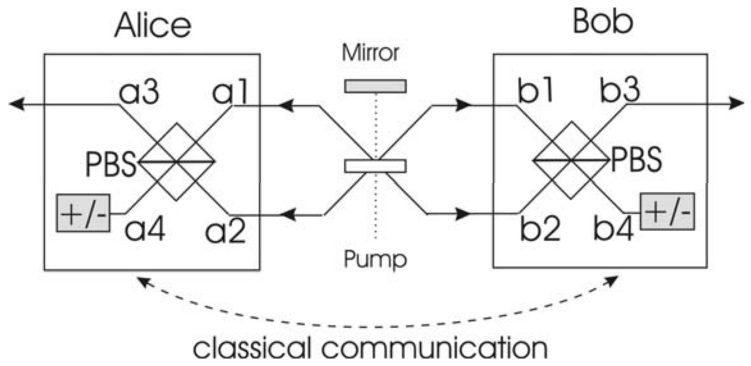
a2b2 contribution $H_{a3}V_{a4}H_{b3}V_{b4}$

$$|\Phi^{+}>_{a3b3}=\frac{1}{\sqrt{2}}(|H>_{a3}|H>_{b3}+|V>_{a3}|V>_{b3})$$

[J.-W. Pan et al., Nature 423, 417 (2003)]

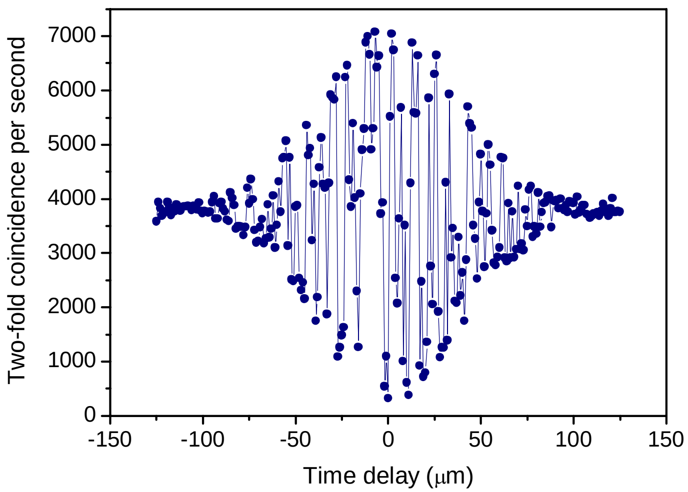
To keep the phase stable,

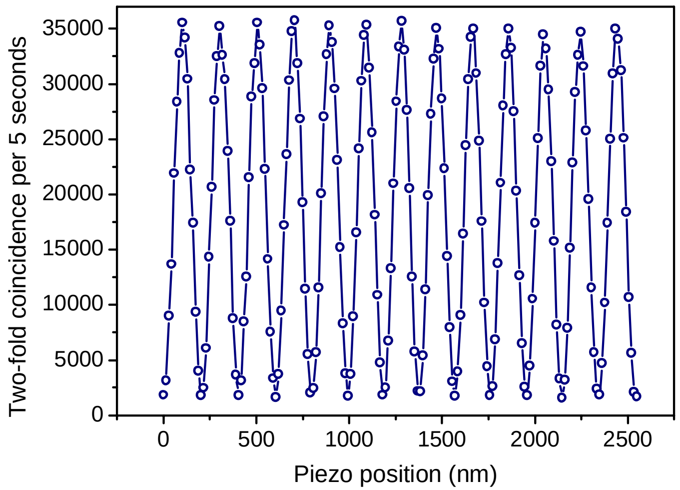
we use the polarization-spatial entanglement



$$\left(|H\rangle|H\rangle+|V\rangle|V\rangle\right)\left(|a_1\rangle|b_1\rangle+e^{i\phi}|a_2\rangle|b_2\rangle\right)$$

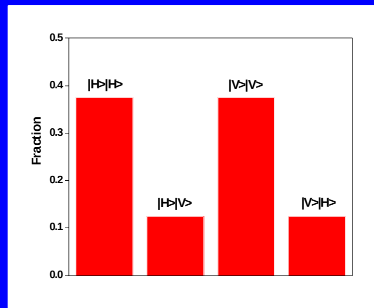
$$\rightarrow\left(|H\rangle|H\rangle+e^{i\phi}|V\rangle|V\rangle\right)\left(|a_3\rangle|b_3\rangle+|a_4\rangle|b_4\rangle\right)$$

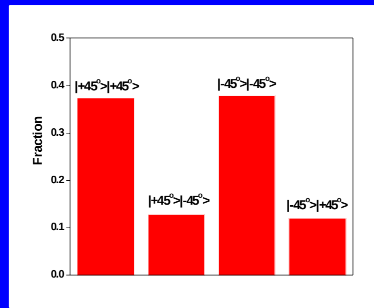




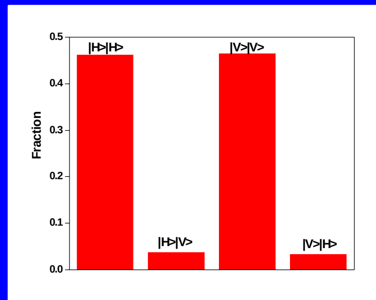
Experimental Result

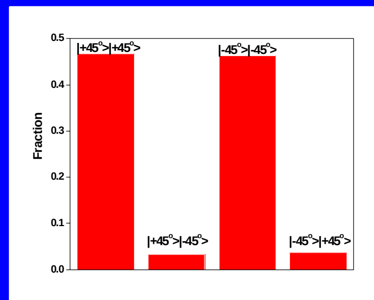
Before purification, F=3/4





After purification, F=13/14





[J.-W. Pan et al., Nature 423, 417 (2003)]

Local filtering (Procrustean protocol)

- The purification protocol is such a waste for the photon pair sources.

- When we have known some information of the state, there is some more efficient method to improve the purity.

- For example, $|\Phi\rangle_{\varepsilon} = (|00\rangle + \varepsilon|11\rangle)/\sqrt{1+\varepsilon^2}$ ($\varepsilon \neq 1$) is non-maximally entangled state, can be converted into the maximally entangled state $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ by subjecting one of the qubits to a generalized measurement filtering process: $|0\rangle \to \varepsilon|0\rangle$, $|1\rangle \to |1\rangle$. This process is called local filtering. It can only be used in known state and can only improve the entanglement degree.

[P. Kwiat et al., Nature 409, 1014 (2003)]

Local filtering

- Any inseparable two spin-1/2 system Matrices Can Be Distilled to a Singlet Form with **local filtering** and **entanglement purification**

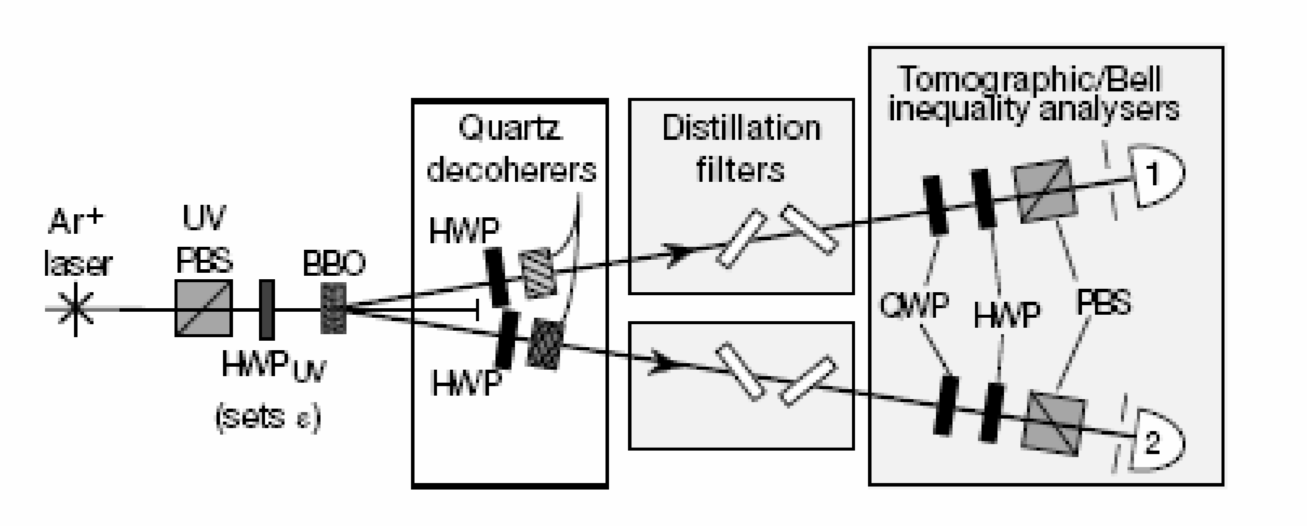
- A quantum system is called inseparable if its density matrix cannot be written as a mixture of product states:

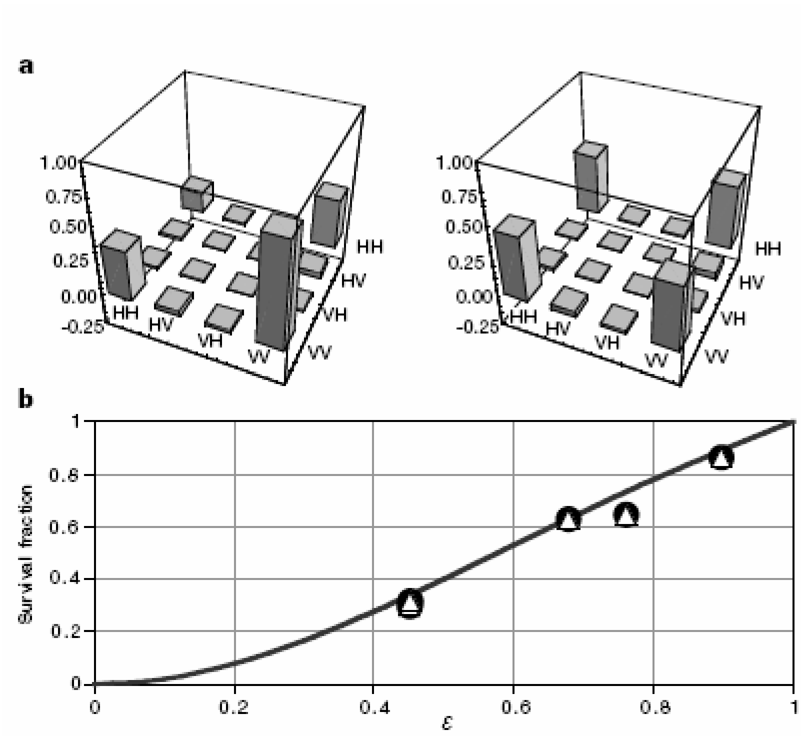
$$\varrho = \sum_{i=1}^{k} p_i \varrho_i \otimes \tilde{\varrho}_i ,$$

where ϱ_i and $\tilde{\varrho}_i$ are states of the subsystems

$$\sum_{i=1}^{k} p_i = 1$$

[M. Horodecki, et al., PRL 78, 574 (1997)]





[P. Kwiat et al., Nature 409, 1014 (2003)]

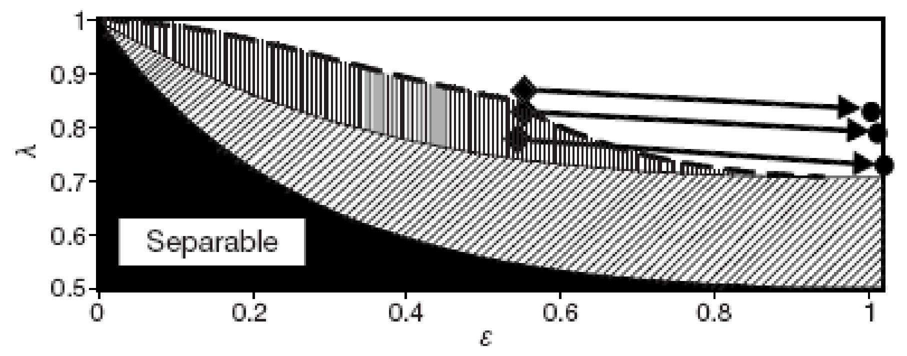
Local Filtering and Hidden Non-locality

- For the state $\rho_{\varepsilon,\lambda} = \lambda|\Phi_\varepsilon\rangle\langle\Phi_\varepsilon| + \frac{(1-\lambda)}{2}(|HV\rangle\langle HV| + |VH\rangle\langle VH|)$ sometimes it can not violate Bell inequality. After the Local Filtering process, the state is changed into the state $\frac{2\varepsilon^2\lambda}{1+\varepsilon^2}|\Phi^+\rangle\langle\Phi^+| + \frac{\varepsilon(1-\lambda)}{2}(|HV\rangle\langle HV| + |VH\rangle\langle VH|)$ which can violate the Bell inequality.

Table 1 Summary of distillation data for testing Bell's inequalities

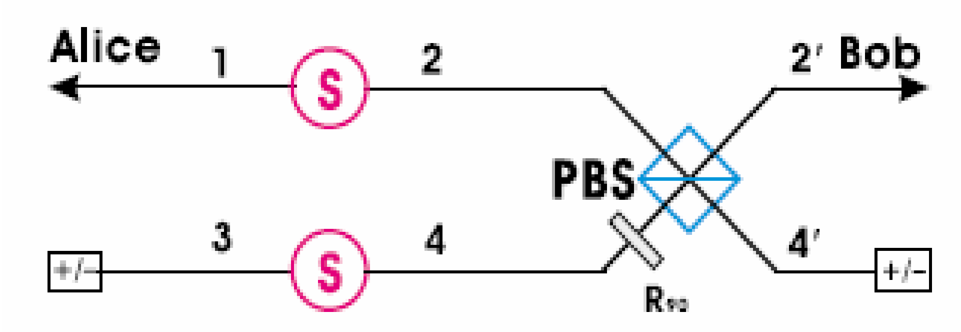
Trial	ϵ_{non}	λ_{non}	S^{calc}_{non}	S^{exp}_{non}	$\epsilon_{filtered}$	$\lambda_{filtered}$	$S^{calc}_{filtered}$	$S^{exp}_{filtered}$
1	0.46	0.78	1.63 ± 0.04	1.60 ± 0.02	1.02	0.73	2.02 ± 0.04	2.02 ± 0.03
2	0.47	0.83	1.81 ± 0.04	1.82 ± 0.02	1.01	0.79	2.20 ± 0.04	2.22 ± 0.03
3	0.47	0.87	1.96 ± 0.04	1.94 ± 0.02	0.99	0.83	2.31 ± 0.04	2.34 ± 0.02

The table shows the measured ϵ and λ parameters of the non-filtered and filtered states, and the calculated and experimentally measured S values. The errors in the latter are calculated from Poisson statistics. The calculated predictions for S were obtained by averaging S over an ensemble of density matrices slightly deviated from the ideal state corresponding to ϵ and λ.



[P. Kwiat et al., Nature 409, 1014 (2003)]

Entanglement Concentration---Scheme



$$|\psi\rangle_T = |\psi\rangle_{12} \otimes |\psi\rangle_{34}$$

$$= (\alpha H_1 V_2 + \beta V_1 H_2) \otimes (\alpha H_3 V_4 + \beta V_3 H_4)$$

$$\xrightarrow{R_{90}} (\alpha H_1 V_2 + \beta V_1 H_2) \otimes (\alpha V_3 H_4 + \beta H_3 V_4)$$

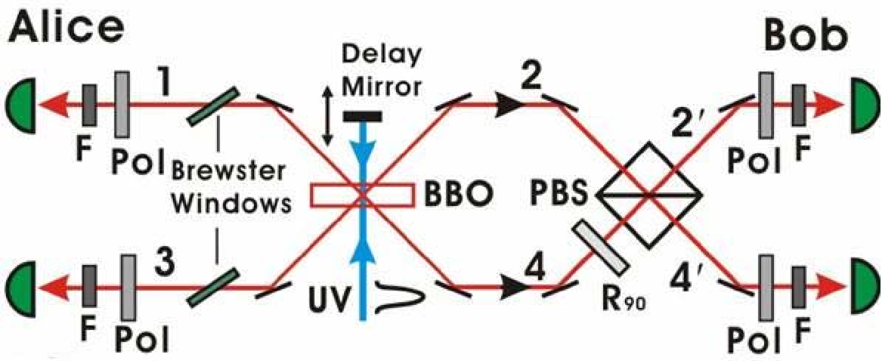
$$= \sqrt{2}\alpha\beta.\frac{1}{\sqrt{2}}(H_1 V_2 H_3 V_4 + V_1 H_2 V_3 H_4)$$

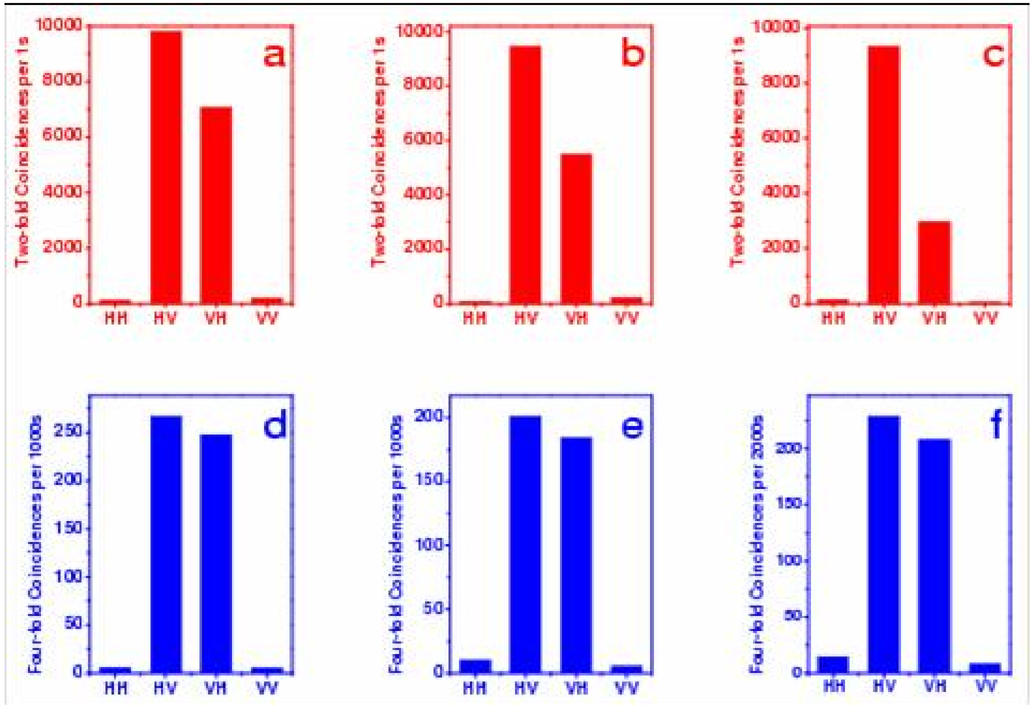
$$+ \alpha^2 H_1 V_2 V_3 H_4 + \beta^2 V_1 H_2 H_3 V_4$$

$$\xrightarrow{PBS} \sqrt{2}\alpha\beta.\frac{1}{\sqrt{2}}(H_1 V_2 H_3 V_4 + V_1 H_2 V_3 H_4)$$

[C. H. Bennett et al., Phys. Rev. A 53, 2046 (1996)]
[Z. Zhao et al., Phys. Rev. A64, 014301 (2001)]
[T. Yamamoto et al., Phys. Rev. A64, 012304 (2001)]

Experiment Realization





[Z. Zhao et al, Phys. Rev. Lett. 90, 207901(2003).]
[T. Yamamoto et al., Nature 421, 343 (2003)]

Difficulties in Long-distance Quantum Communication

Due to the noisy quantum channel

(1) absorption → **photon loss**

(2) decoherence → **degrading entanglement quality**

Solution to problem (1):

Entanglement swapping !

[N. Gisin et al., Rev. Mod. Phys. 74, 145 (2002)]

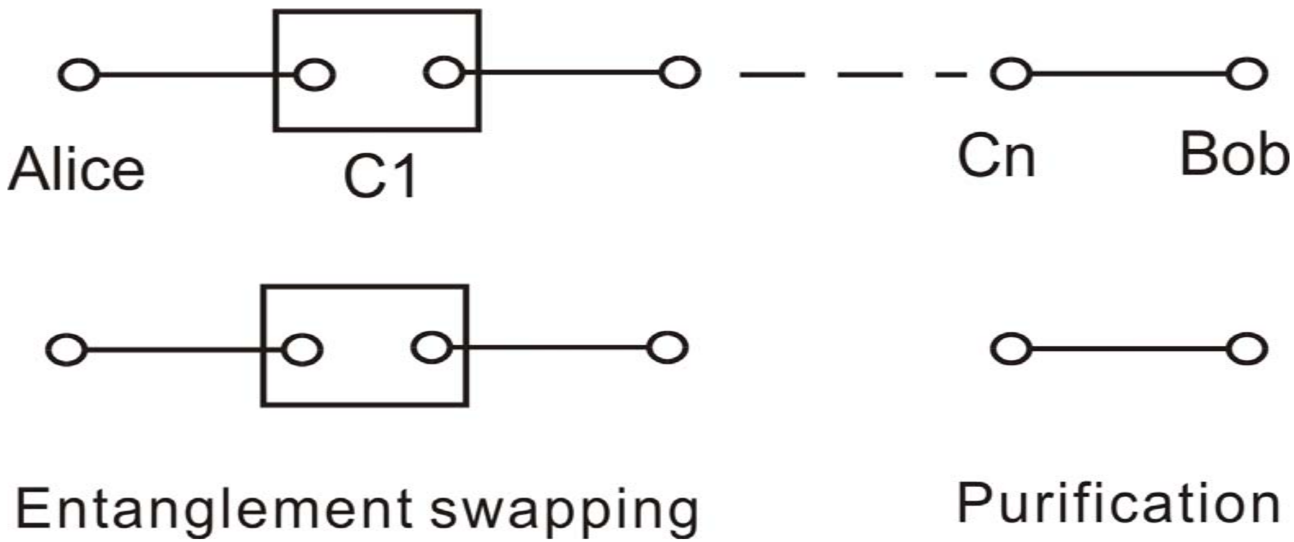
Solution to problem (2):

Entanglement purification!

[C. H. Bennett et al., Phys. Rev. Lett. 76, 722 (1996)]
[D. Deutsch et al., Phys. Rev. Lett. 77, 2818 (1996)]

The Kernel Device for Long Distance Quantum Communication

Quantum repeaters:

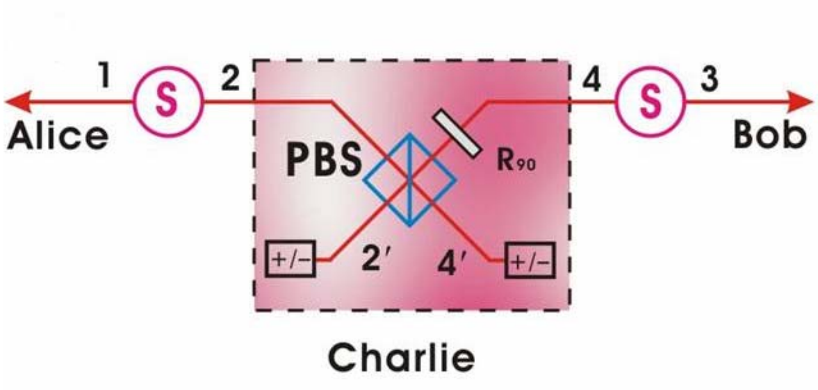


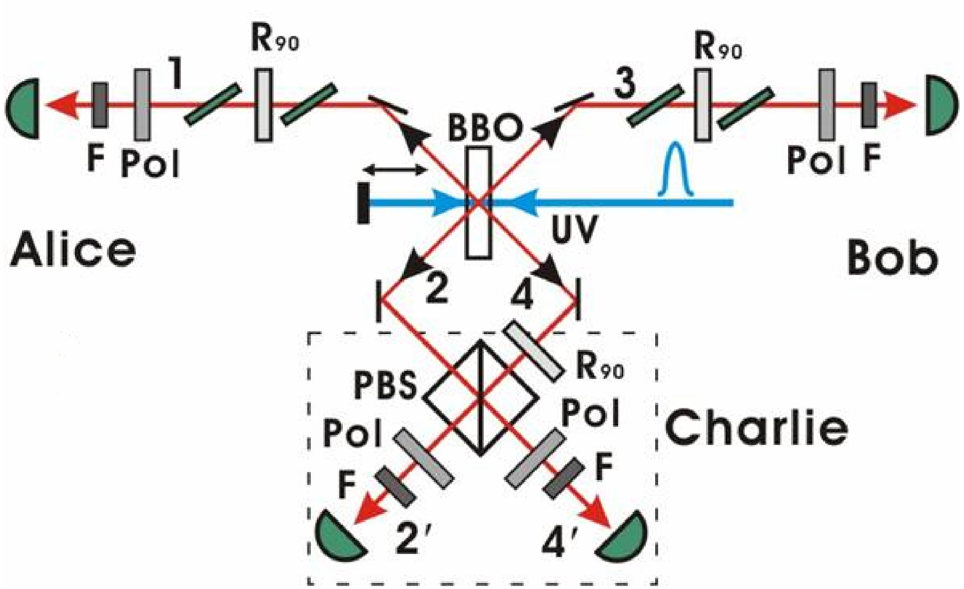
[H. Briegel et al., Phys. Rev. Lett. 81, 5932(1998)]

Require

- **entanglement swapping with high precision**
- **entanglement purification with high precision**
- **quantum memory**

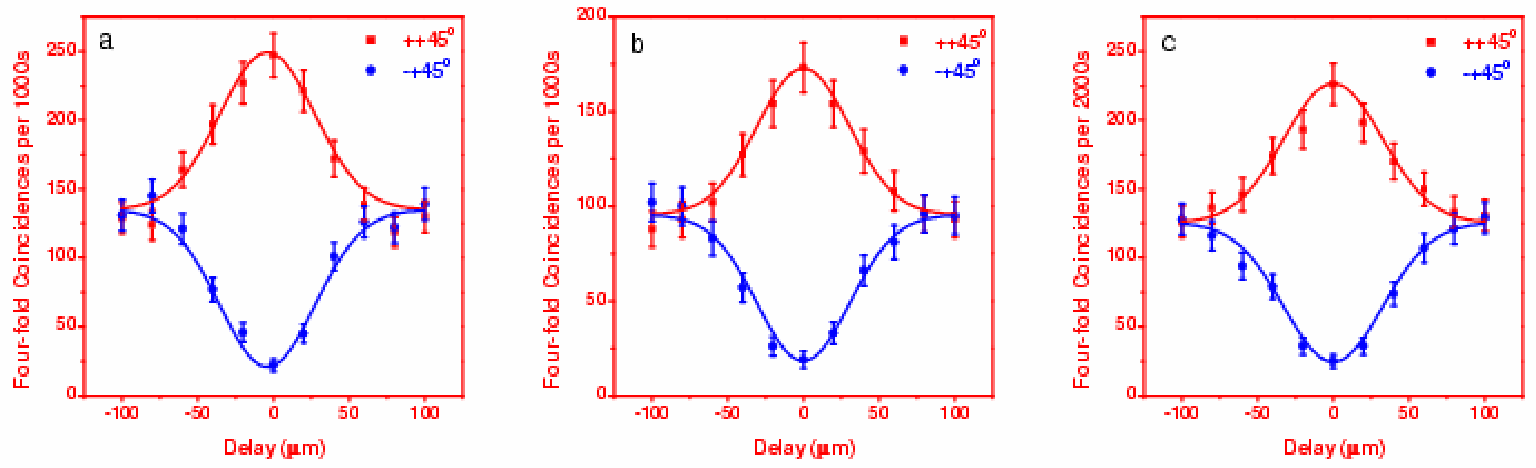
A proof-in-principle demonstration of a quantum repeater





[Z. Zhao et al, Phys. Rev. Lett. 90, 207901(2003).]

Results for Repeater



$V_1 = 0.83 \pm 0.04$ | $V_2 = 0.80 \pm 0.05$ | $V_2 = 0.80 \pm 0.04$

$S = 2.58 \pm 0.07$ | $|S| = 2.43 \pm 0.08$ | $|S| = 2.42 \pm 0.08$

8.3 Standard Deviation | 5.4 Standard Deviation | 5.4 Standard Deviation

$$Fidelity - 0.96 \pm 0.04 - - - - 0.93 \pm 0.04 - - - - - - - 0.93 \pm 0.04$$

[Z. Zhao et al, Phys. Rev. Lett. 90, 207901(2003).]

Drawback in Former Experiments

- Absence of quantum memory
- Probabilistic entangled photon source
- Probabilistic entanglement purification
- Huge photon loss in fiber

Quantum memory

Free-space entanglement distribution

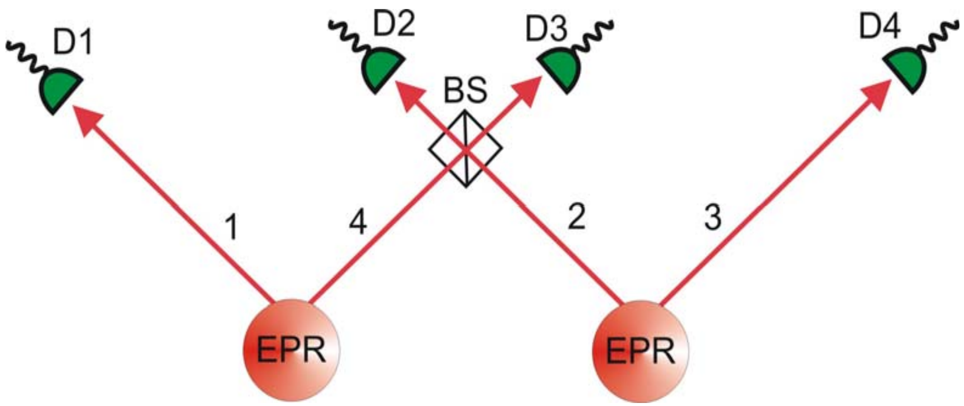
- we are working on 20km and 500km scale…

- Synchronization of independent lasers

- we are working on entanglement swapping…

[T. Yang et al., PRL 96, 110501 (2006)]

Drawback in Former Experiments



$$P \longleftrightarrow P$$

$$2/P^2$$

- In multi-stage experiments, the cost of resource is proportional to

  N/P^N thus not scalable

- If one knows when the photon pair is created and the entangled pair can be stored as demanded, the total cost is then

$$N/P$$

Another Solution----
Quantum Communication based on Decoherence free Subspace

- For special noise, we can utilize some entanglement subspace to directly implement quantum communication.
- For example, the phase flip error channel:

$$|0\rangle \rightarrow |0\rangle \text{ , } |1\rangle \rightarrow e^{i\phi}|1\rangle$$

- The Bell state $|\psi^+\rangle, |\psi^-\rangle$ are immune to the noise:

$$(|0\rangle|1\rangle + |1\rangle|0\rangle)/\sqrt{2} \rightarrow (|0\rangle e^{i\Phi}|1\rangle + e^{i\phi}|1\rangle|0\rangle)/\sqrt{2} = e^{i\Phi}(|0\rangle|1\rangle + |1\rangle|0\rangle)/\sqrt{2}$$
$$(|0\rangle|1\rangle - |1\rangle|0\rangle)/\sqrt{2} \rightarrow (|0\rangle e^{i\Phi}|1\rangle - e^{i\phi}|1\rangle|0\rangle)/\sqrt{2} = e^{i\Phi}(|0\rangle|1\rangle - |1\rangle|0\rangle)/\sqrt{2}$$

- We can take $|\bar{0}\rangle = |\Psi^+\rangle$ and $|\bar{1}\rangle = |\Psi^-\rangle$, Each state combined by the two basis can be used for decoherence free communication. The subspace is called decoherence free subspace.

[Q. Zhang et al., PRA 73. 030201(R) (2006)]

[T.-Y. Chen et al., PRL 96, 150504 (2006)]