Reference transversely isotropic medium approximating a given generally anisotropic medium

Luděk Klimeš

Department of Geophysics, Faculty of Mathematics and Physics, Charles University in Prague, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, http://sw3d.cz/staff/klimes.htm

Summary

For a given stiffness tensor (tensor of elastic moduli) of a generally anisotropic medium, we determine the reference transversely isotropic (uniaxial) medium which approximates the given generally anisotropic medium.

Keywords

Elastic anisotropy, stiffness tensor, elastic moduli, transverse isotropy, approximate transverse isotropy, reference symmetry axis.

1. Introduction

For a given stiffness tensor (tensor of elastic moduli) of a generally anisotropic medium, we can estimate to which extent the medium is transversely isotropic and determine the direction of its reference symmetry axis using the method of Klimeš (2015). If we find that the medium is approximately transversely isotropic (approximately uniaxial), we may wish to determine the reference transversely isotropic (uniaxial) medium which approximates the given generally anisotropic medium.

The stiffness tensor of a transversely isotropic medium is independent of the rotation around the symmetry axis. We thus take the reference symmetry axis determined using the method of Klimeš (2015), rotate the given stiffness tensor about this reference symmetry axis, and determine the reference transversely isotropic stiffness tensor as the average of the rotated stiffness tensor over all angles of rotation.

The lower–case Roman indices take values 1, 2 and 3. The Einstein summation over repetitive lower–case Roman indices is used throughout the paper.

Seismic Waves in Complex 3–D Structures, **26** (2016), 155–158 (ISSN 2336–3827, online at http://sw3d.cz)

2. Reference transversely isotropic stiffness tensor

We denote the density–reduced stiffness tensor of a given generally anisotropic medium by a_{ijkl}. The unit reference symmetry vector t_i in the direction of the reference symmetry axis can be obtained using the method of Klimeš (2015).

The projection matrix onto the reference symmetry vector is

$$Z_{ia} = t_i t_a \quad . \tag{1}$$

The projection matrix onto the plane perpendicular to the reference symmetry vector is

$$C_{ia} = \delta_{ia} - t_i t_a \quad , \tag{2}$$

where Kronecker delta δ_{in} represents the elements of the identity matrix. Minus the generator matrix of the rotation about the reference symmetry vector is

$$S_{ia} = \varepsilon_{iar} t_r \quad , \tag{3}$$

where ε_{ijk} is the Levi–Civita symbol.

Then the transformation matrix corresponding to the rotation of vectors about the given reference symmetry vector t_i by angle φ reads

$$R_{ia}(\varphi) = Z_{ia} + C_{ia} \cos(\varphi) - S_{ia} \sin(\varphi) \quad . \tag{4}$$

The rotated stiffness tensor reads

$$\tilde{a}_{ijkl}(\varphi) = R_{ia}(\varphi)\, R_{jb}(\varphi)\, R_{kc}(\varphi)\, R_{ld}(\varphi)\, a_{abcd} \quad , \tag{5}$$

where a_{pqrs} without argument φ is the given non–rotated tensor.

We define the stiffness tensor $\overline{a}_{ijkl}$ of the reference transversely isotropic medium as the average of the rotated stiffness tensor,

$$\overline{a}_{ijkl} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi\, \tilde{a}_{ijkl}(\varphi) \quad . \tag{6}$$

When inserting rotated stiffness tensor (5) with transformation matrix (4) into definition (6), the reference stiffness tensor is composed of the terms proportional to integrals

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi = 1 \quad , \tag{7}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi\, \cos(\varphi) = 0 \quad , \tag{8}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi\, [\cos(\varphi)]^2 = \frac{1}{2} \quad , \tag{9}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi\, [\cos(\varphi)]^3 = 0 \quad , \tag{10}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi\, [\cos(\varphi)]^4 = \frac{3}{8} \quad , \tag{11}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi\, \sin(\varphi) = 0 \quad , \tag{12}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi\, \cos(\varphi)\, \sin(\varphi) = 0 \quad , \tag{13}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\ [\cos(\varphi)]^2 \sin(\varphi) = 0 \quad , \tag{14}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\ [\cos(\varphi)]^3\ \sin(\varphi) = 0 \quad , \tag{15}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\ [\sin(\varphi)]^2 = \frac{1}{2} \quad , \tag{16}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\ \cos(\varphi)\ [\sin(\varphi)]^2 = 0 \quad , \tag{17}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\ [\cos(\varphi)]^2\ [\sin(\varphi)]^2 = \frac{1}{8} \quad , \tag{18}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\ [\sin(\varphi)]^3 = 0 \quad , \tag{19}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\ \cos(\varphi)\ [\sin(\varphi)]^3 = 0 \quad , \tag{20}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi\ [\sin(\varphi)]^4 = \frac{3}{8} \quad , \tag{21}$$

Considering these integrals, the stiffness tensor of the reference transversely isotropic medium reads

$$\begin{aligned}\overline{a}_{ijkl} = \big\{ & Z_{ia}Z_{jb}Z_{kc}Z_{ld} \\ & +\tfrac{1}{2}[Z_{ia}Z_{jb}C_{kc}C_{ld} + Z_{ia}C_{jb}Z_{kc}C_{ld} + Z_{ia}C_{jb}C_{kc}Z_{ld} \\ & \quad +C_{ia}Z_{jb}Z_{kc}C_{ld} + C_{ia}Z_{jb}C_{kc}Z_{ld} + C_{ia}C_{jb}Z_{kc}Z_{ld}] \\ & +\tfrac{1}{2}[Z_{ia}Z_{jb}S_{kc}S_{ld} + Z_{ia}S_{jb}Z_{kc}S_{ld} + Z_{ia}S_{jb}S_{kc}Z_{ld} \\ & \quad +S_{ia}Z_{jb}Z_{kc}S_{ld} + S_{ia}Z_{jb}S_{kc}Z_{ld} + S_{ia}S_{jb}Z_{kc}Z_{ld}] \\ & +\tfrac{1}{8}[C_{ia}C_{jb}S_{kc}S_{ld} + C_{ia}S_{jb}C_{kc}S_{ld} + C_{ia}S_{jb}S_{kc}C_{ld} \\ & \quad +S_{ia}C_{jb}C_{kc}S_{ld} + S_{ia}C_{jb}S_{kc}C_{ld} + S_{ia}S_{jb}C_{kc}C_{ld}] \\ & +\tfrac{3}{8}[C_{ia}C_{jb}C_{kc}C_{ld} + S_{ia}S_{jb}S_{kc}S_{ld}]\big\}\, a_{abcd} \quad . \end{aligned} \tag{22}$$

3. Conclusions

The proposed method of determining the reference transversely isotropic stiffness tensor for the given stiffness tensor of a generally anisotropic medium has been coded as a new option of program `tiaxis.for` of software package FORMS (Bucha & Bulant, 2016).

Acknowledgements

The research has been supported by the Grant Agency of the Czech Republic under contract 16-05237S, by the Ministry of Education, Youth and Sports of the Czech Republic within research project CzechGeo/EPOS LM2015079, and by the members of the consortium “Seismic Waves in Complex 3–D Structures” (see “http://sw3d.cz”).

References

Bucha, V. & Bulant, P. (eds.) (2016): SW3D–CD–20 (DVD–ROM). *Seismic Waves in Complex 3–D Structures*, **26**, 183–184, online at “http://sw3d.cz”.

Klimeš, L. (2015): Determination of the reference symmetry axis of a generally anisotropic medium which is approximately transversely isotropic. *Seismic Waves in Complex 3–D Structures*, **25**, 177–185, online at “http://sw3d.cz”.