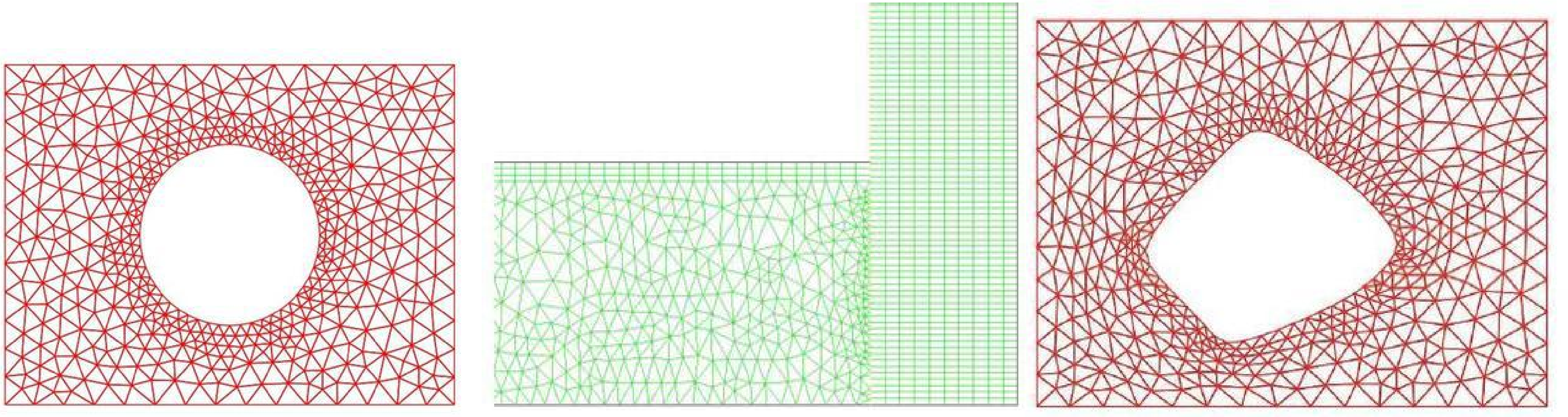
3.2 微分方程的有限差分法离散



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3.2.1 什么是有限差分？

- 常微分和偏微分的引入

$$\frac{du}{dx} \Leftarrow \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$\left.\frac{\partial u}{\partial x}\right|_{y} \Leftarrow \lim_{\substack{\Delta x \to 0 \\ y=\text{常值}}} \frac{\Delta u}{\Delta x}$$

有 限 差 分

$$\frac{\Delta u}{\Delta x}=\frac{u(x+\Delta x)-u(x)}{\Delta x}\underset{\Delta x\text{充分小}}{\overset{\text{离散}x}{\Longleftarrow}}\frac{du}{dx}$$

$$\left.\frac{\Delta u}{\Delta x}\right|_{y=\text{常值}}=\left.\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}\right|_{y=\text{常值}}\underset{\substack{\Delta x\text{充分小}\\ y=\text{常值}}}{\overset{\text{离散}x}{\Longleftarrow}}\left.\frac{\partial u}{\partial x}\right|_{y}$$

3.2.2 差分算子和微分算子

- 算子：代表一种数学运算

+ - x /

$$\frac{d}{dx} \qquad \left.\frac{\partial}{\partial x}\right|_y$$

1 移位算子

$$E_x u_j^n = u_{j+1}^n, \quad E_t^{-1} u_j^n = u_j^{n-1}$$

2 前差算子

$$\Delta_x u_j^n = u_{j+1}^n - u_j^n, \ \Delta_t u_j^n = u_j^{n+1} - u_j^n$$

3 后差算子

$$\nabla_x u_j^n = u_j^n - u_{j-1}^n, \ \nabla_t u_j^{n+1} = u_j^{n+1} - u_j^n$$

4 一步中心差算子

$$\delta_x u_j^n = u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n$$

5 二阶中心差算子

$$\delta_x^2 u_j^n = \delta_x[\delta_x(u_j^n)] = \delta_x(u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n) = u_{j+1}^n - 2u_j^n + u_{j-1}^n$$

6 平均算子

一步平均：

$$\mu_x u_j^n = \frac{1}{2}\left(u^n_{j+\frac{1}{2}} + u^n_{j-\frac{1}{2}}\right)$$

两步平均：

$$\mu_{2x} u_j^n = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right)$$

7 两步中心差算子

$$(\mu\delta)_x u_j^n = \mu_x(\delta_x u_j^n) = \mu_x(u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n) = \frac{1}{2}(u_{j+1}^n - u_{j-1}^n)$$

8 微分算子

$$\mathrm{D_x} = \frac{\partial}{\partial x} = \frac{\partial}{h_x}, \quad \mathrm{D}_t = \frac{\partial}{\partial t} = \frac{\partial}{h_t}$$

各个差分算子之间的换算关系

3.2.3 基于 Taylor 展开的有限差分离散

前差差分格式

$$u_{j+1}^n = u_j^n + \left(\frac{\partial u}{\partial x}\right)_j^n \Delta x + \left(\frac{\partial^2 u}{\partial x^2}\right)_j^n \frac{\Delta x^2}{2!} + \left(\frac{\partial^3 u}{\partial x^3}\right)_j^n \frac{\Delta x^3}{3!} + \cdots$$

$$\left(\frac{\partial u}{\partial x}\right)_j^n = \boxed{\frac{u_{j+1}^n - u_j^n}{\Delta x}} - \left(\frac{\partial^2 u}{\partial x^2}\right)_j^n \frac{\Delta x}{2!} - \left(\frac{\partial^3 u}{\partial x^3}\right)_j^n \frac{\Delta x^2}{3!} + \cdots = \boxed{\frac{u_{j+1}^n - u_j^n}{\Delta x}} + O(\Delta x)$$

后差差分格式

$$u_{j-1}^{n} = u_{j}^{n} - \left(\frac{\partial u}{\partial x}\right)_{j}^{n} \Delta x + \left(\frac{\partial^{2} u}{\partial x^{2}}\right)_{j}^{n} \frac{\Delta x^{2}}{2!} - \left(\frac{\partial^{3} u}{\partial x^{3}}\right)_{j}^{n} \frac{\Delta x^{3}}{3!} + \cdots$$

$$\left(\frac{\partial u}{\partial x}\right)_{j}^{n} = \boxed{\frac{u_{j}^{n} - u_{j-1}^{n}}{\Delta x}} + \left(\frac{\partial^{2} u}{\partial x^{2}}\right)_{j}^{n} \frac{\Delta x}{2!} - \left(\frac{\partial^{3} u}{\partial x^{3}}\right)_{j}^{n} \frac{\Delta x^{2}}{3!} + \cdots = \boxed{\frac{u_{j}^{n} - u_{j-1}^{n}}{\Delta x}} + O(\Delta x)$$

两步中心差分格式

$$\begin{cases} u_{j+1}^n = u_j^n + \left(\dfrac{\partial u}{\partial x}\right)_j^n \Delta x + \left(\dfrac{\partial^2 u}{\partial x^2}\right)_j^n \dfrac{\Delta x^2}{2!} + \left(\dfrac{\partial^3 u}{\partial x^3}\right)_j^n \dfrac{\Delta x^3}{3!} + \cdots \\ u_{j-1}^n = u_j^n - \left(\dfrac{\partial u}{\partial x}\right)_j^n \Delta x + \left(\dfrac{\partial^2 u}{\partial x^2}\right)_j^n \dfrac{\Delta x^2}{2!} - \left(\dfrac{\partial^3 u}{\partial x^3}\right)_j^n \dfrac{\Delta x^3}{3!} + \cdots \end{cases}$$

$$\left(\frac{\partial u}{\partial x}\right)_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + O(\Delta x^2)$$

一步中心差分格式

$$\left(\frac{\partial u}{\partial x}\right)_j^n = \frac{u_{j+1/2}^n - u_{j-1/2}^n}{\Delta x} + O(\Delta x^2)$$

二阶中心差分格式

$$\begin{cases} u_{j+1}^n = u_j^n + \left(\dfrac{\partial u}{\partial x}\right)_j^n \Delta x + \left(\dfrac{\partial^2 u}{\partial x^2}\right)_j^n \dfrac{\Delta x^2}{2!} + \left(\dfrac{\partial^3 u}{\partial x^3}\right)_j^n \dfrac{\Delta x^3}{3!} + \cdots \\ u_{j-1}^n = u_j^n - \left(\dfrac{\partial u}{\partial x}\right)_j^n \Delta x + \left(\dfrac{\partial^2 u}{\partial x^2}\right)_j^n \dfrac{\Delta x^2}{2!} - \left(\dfrac{\partial^3 u}{\partial x^3}\right)_j^n \dfrac{\Delta x^3}{3!} + \cdots \end{cases}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + O(\Delta x^2)$$

一阶导数、二阶导数、交错导数的常用差分表达式

表3-2

表3-3