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Equilibrium in a Stackelberg duopoly

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Abstract. *Theory of the games offers perfect tools for modelling imperfect competition specific processes, manifested in relation with product quantity (Cournot/Stackelberg type), product price (Bertrand type) or quality. The equilibrium solution in terms of output is highlighted in a Cournot situation, whilst the price equilibrium solution can be revealed in a Bertrand scenario. Despite the different strategy type based on, the common denominator of these two models is given by the fact that strategic choices are made simultaneously. The Stackelberg model instead, represents a perfect information sequential game – firms advocating for quantity competition – having both theoretical and practical applicability. In the simplest possible case, with two players moving in two stages, the leader will always choose a certain output level, and the follower observes this decision and then establish his action path accordingly. Present paper's main goal is to analyze a duopoly market with players adopting a Stackelberg behavior. Regardless the analyzed scenario, both firms are expected to survive and a stable equilibrium will manifest (the Subgame Perfect Equilibrium). The price will be invariable at market demand curve slope, whilst player's choosed quantities and also gained profits level will be in an inverse dependence relation with it. The leader's chosen output and also registered profit levels will be double vs the follower's profit.*

Keywords: Stackelberg equilibrium, Stackelberg model, Cournot model, oligopoly, stability.

JEL Classification: C72, D01, D43, L13.

1. Introduction

As one of the fundamental oligopoly games illustration, monopoly theory can be traced back two centuries ago, at the date when Antoine Augustin Cournot first put forward the mathematical model of duopoly competition (1838). Cournot model became a starting point for oligopoly theory analysis – duopoly scenario, with firms producing homogeneous products, choosing to compete in terms of quantities, taking simultaneously individual decisions regarding production level.

After almost a century, another duopoly market model has been developed, with players competing also in quantities, but this time the decisions were taken consecutively (Stackelberg duopoly). Known also as Stackelberg competition and being an imperfect competition model based on a non-cooperative game, it actually represents an extension of the Cournot's model. It was developed by Heinrich von Stackelberg (1934) in his book "Market Structure and Equilibrium" and represent a breaking point in the market structure analysis, mainly in duopoly scenario. Based on different starting hypothesis and offering different conclusions than those of the Cournot's and Bertrand's models, the Stackelberg duopoly model is a sequential game with perfect information (unlike the Cournot's model, which is a simultaneous one).

As previously anticipated, the model has a real theoretical importance but also a practical one. It can be efficiently used in industrial organizations, to study the market structure determinants and other connected issues like market entry and entry preemption (Mueller, 1986; Sutton, 2007 and Berry and Reiss, 2007). Stackelberg model is also a perfect tool for hierarchical structure scenario's analysis. Zhang and Zhang (2009) used such a game to model the problem of spectrum allocation in Cognitive Radio Networks. Another Stackelberg game-based approach has also been used to model the efficient bandwidth allocation problem in the cloud-based wireless networks, where desktop users watching the same channel may be willing to share their live-streaming with the nearby mobile users (Nan et al., 2014). Stackelberg models have been widely used in the security domain to illustrate the attacker-defender models (Pita et al., 2009 – protection against terrorists at Los Angeles International Airport; Michael and Scheffer, 2011 – adversial learning modelling in the setup when the opponent tries to manipulate the data miner's data to reduce the accuracy of the classifier.; Trejo et al., 2015; Clempner and Poznyak, 2015, etc.). Concluding, theoretical Stackelberg game models have been widely used to model different situations in various real market areas.

We further investigate the influence of market demand curve slope, on Stackelberg static equilibrium model, emphasising aspects such as firm stability and demand curve slope impact on the perfect subgame equilibrium theory. The principles of the related mathematic model are also described below.

2. The model

The background used is one with two firms, which sell homogeneous goods, subject to the same demand and cost functions. One of them (called "the leader") has the right to make the first move, thanks to certain potential advantages as market power, size, reputation, historical precedence, sophistication, information, innovation and so forth. Stackelberg assumes the fact that this duopolist is sufficiently sophisticated to recognise that his rival acts on the Cournot assumption. This recognition allows him to determine his competitor's reaction curve and include it in his own profit function, acting as a monopolist in an attempt to maximise payoff. The other one (called "the follower") observes his strategy and decides about its own accordingly. His profit depends on the output level choosed by the leader which is predetermined in his opinion, therefore will be considered an invariable information.

It is worth mentioning that the leader's action is irreversible as he knows ex ante that the follower observes his actions, establishing his own action path accordingly. The first mover advantage is undeniable, triggering the idea that the leader yields a higher payoff than the follower does.

An example of such leadership may be Microsoft's software markets dominance. Although Microsoft can make decisions first, other smaller companies can only react by making their own decisions. The followers action, in turn, affect Microsoft. Another potential Stackelberg leadership's scenario is highlighted in the aircraft industry – Boeing and Airbus competition (Waldman and Jensen, 2016).

Let's consider a general price function P(Q), better expressed as $P(q_1 + q_2)$, giving the existing duopoly scenario. q_1 and q_2 below represent the leader/the follower output level and Q represents the aggregate market demand:

$$P(Q) = P(q_1 + q_2) \tag{1}$$

We also assume that firm *i* cost structure is $C_i(q_i), i = \overline{1,2}$.

To solve the model and find the subgame perfect equilibrium, we need to use backward induction, as in any sequential game. The leader anticipate the follower's best reaction, more precisely the way this will respond once it has observed his decision. After that, choose his maximizing payoff quantity q_1, to which the follower reacts by choosing the expected quantity q_2. We should first determine the follower's best response function.

The profit function of the follower will be:

$$\pi_2 = P(q_1 + q_2)q_2 - C_2(q_2) \tag{2}$$

First order derivate expression can be found below:

$$\frac{\delta \pi_2}{\delta q_2} = \frac{\delta P(q_1 + q_2)}{\delta q_2} q_2 + P(q_1 + q_2) - \frac{\delta C_2(q_2)}{\delta q_2} \tag{3}$$

whilst setting to zero value the marginal profit expression, opens the path for finding out the follower best reply function.

We are looking forward now to the leader's best reply function:

$$\pi_1 = P(q_1 + q_2(q_1))q_1 - C_1(q_1) \tag{4}$$

where $q_2(q_1)$ represent the follower's quantity as a strictly dependent function of the leader's output, as we have previously agreed. The leader marginal profit expression, who's leading to its best reply function, can be described as follows:

$$\frac{\delta\pi_1}{\delta q_1} = \frac{\delta P(q_1+q_2)}{\delta q_2}\frac{\delta q_2(q_1)}{\delta q_1}q_1 + \frac{\delta P(q_1+q_2)}{\delta q_1}q_1 +$$

$$P(q_1+q_2(q_1)) - \frac{\delta C_1(q_1)}{\delta q_1} \tag{5}$$

Let's further consider a downward sloping linear demand curve scenario, with price dependence described as follows:

$$P(q_1+q_2) = a - b(q_1+q_2) \tag{6}$$

where a > 0, b > 0, whilst P represents the price paid by consumers for purchasing required product amount. The prior mentioned inverse demand function is getting close to the second products particular case of the formula used by Kresemir Zigic (2012) in his analyze, which presents a differentiated products Stackelberg scenario (with b ∈(0,1) reflecting the degree of product differentiation or substitutability)

Adjusting previously mentioned formulas to the current hypothesis, the follower's profit function expression becomes:

$$\pi_2 = [a - b(q_1+q_2)]q_2 - C_2(q_2) \tag{7}$$

Marginal profit expressions represent the starting point in the revealing of the follower reaction function (see Appendix A):

$$q_2 = \frac{a - bq_1 - \frac{\delta C_2(q_2)}{\delta q_2}}{2b} \tag{8}$$

The spring of further equilibrium values, is represented by the leader profit function:

$$\pi_1 = \left[a - b\left(q_1+q_2(q_1)\right)\right]q_1 - C_1(q_1) \tag{9}$$

and all other mathematical calculation (related in Appendix A) leads to:

$$q_1^* = \frac{a + \frac{\delta C_2(q_2)}{\delta q_2} - 2\frac{\delta C_1(q_1)}{\delta q_1}}{2b} \tag{10}$$

$$q_2^* = \frac{a - 3\frac{\delta C_2(q_2)}{\delta q_2} + 2\frac{\delta C_1(q_1)}{\delta q_1}}{4b} \tag{11}$$

where q_1^* – the leader's best response to the follower's reaction, and q_2^* – the follower's reaction function. That means the market demand level in the equilibrium situation is:

$$Q^* = q_1^* + q_2^* = \frac{3a - \frac{\delta C_2(q_2)}{\delta q_2} - 2\frac{\delta C_1(q_1)}{\delta q_1}}{4b} \tag{12}$$

and further, the equilibrium price

$$p^* = \frac{a + \frac{\delta C_2(q_2)}{\delta q_2} + 2\frac{\delta C_1(q_1)}{\delta q_1}}{4} \tag{13}$$

Referring now to the cost function, for the simplicity of calculation but also for a better understanding, we can impose some mathematical restrictions

$$\frac{\delta^2 C_i(q_i)}{\delta q_i\, \delta q_j} = 0;\ \frac{\delta C_i(q_i)}{\delta q_j} = 0;\ \ i,j = \overline{1,2} \tag{14}$$

and from all types of possible functions, we pick the linear one $C_i(q_i) = c_i q_i$. Including also this last hypothesis in the model, the Stackelberg perfect subgame equilibrium values become:

$$q_1^* = \frac{a-c}{2b} \tag{15}$$

$$q_2^* = \frac{a-c}{4b} \tag{16}$$

$$p^* = \frac{a+3c}{4} \tag{17}$$

$$\pi_1^* = \frac{(a-c)^2}{8b} \tag{18}$$

$$\pi_2^* = \frac{(a-c)^2}{16b} \tag{19}$$

The results obtained lead to the following conclusions:

- $q_1^* > q_2^*$, meaning the leader produce more (better said, the leader's output is twice as much the follower does);
- $p^* > c$, confirming that both players have the possibility of making profits;
- $\pi_1^* > \pi_2^*$, the leader register higher (double) profit, so there is a real advantage to move first. There are two main reasons: the leader knows that by increasing his output will force the follower to reduce his own and this decision is irreversible (undoing its action we would reach the Cournot scenario).
- $Q^* > Q_{COURNOT} \rightarrow p^* < p_{COURNOT}$. The Stackelberg game leads to a more competitive equilibrium than the Cournot one does.

We are now treating the b = 1 situation – perfectly substitutes products. Therefore, p = a-q1-q2 highlighting the simplest possible form for price – output mathematical relation. Thus p + Q = a, meaning their sum remaining constant, equalizing a parameter value. The quantity offered by the leader will be q_1 = (a-c)/2 whilst his follower's response is q_1 = (a-c)/4. The price value suffered no modification p = (a+3c)/4 as it isn't affected by the parameter's b variation; looking further, we can note that the leader/follower profit level become π_1= (a-c)〗^2/8, respectively π_1= 〖(a-c)〗^2/16.

We further analyze the quantity/profit sensitivity to the changes in the level of parameter b, in a subgame equilibrium scenario (the price is not related with b parameter, being constant at any a and c hypothetical value pairs.). All the mathematical calculations representing graphical analyze basis below, are reflected by Appendix B, whilst in our simulation, we customize parameters a and c, as follows: a = 100 EUR; c = 40 EUR. Considering these assumptions, we start the parameter b gradual increasing, with a convenient ratio of 0.1, from the initial 0.1 value, up to 3.0 final value.

Figure 1. *Nash equilibrium quantity evolution*

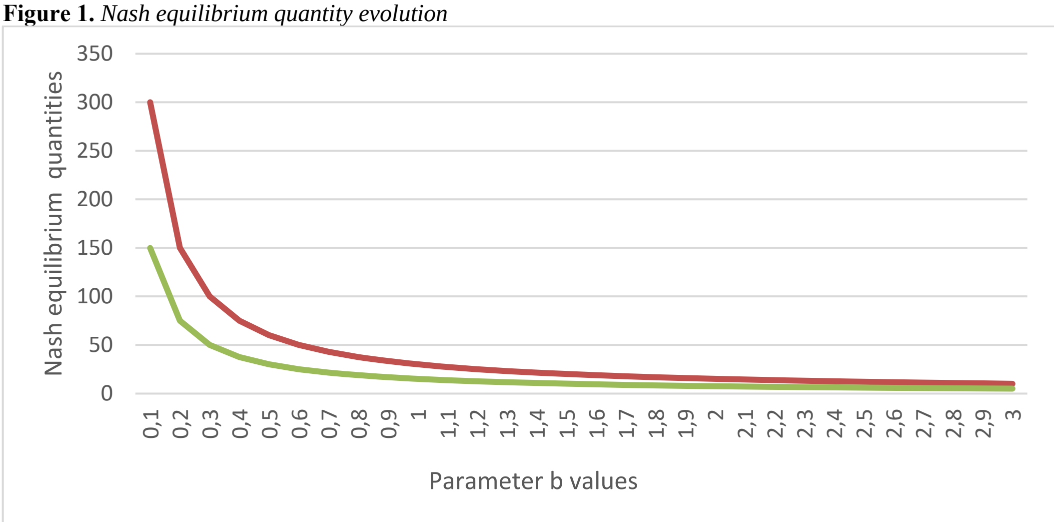

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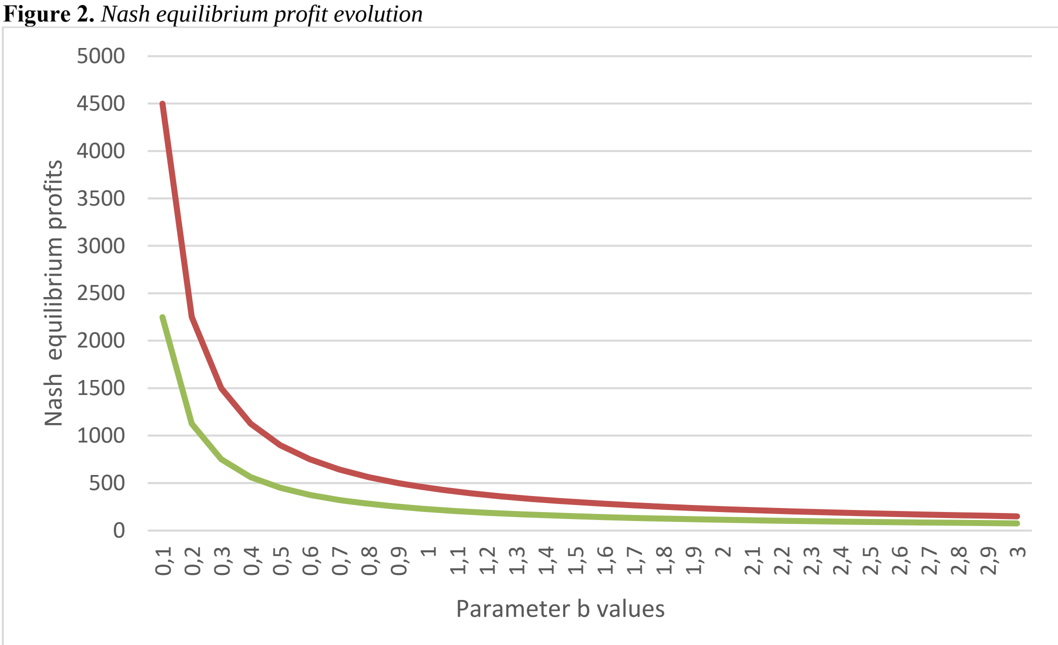
Figure 2. *Nash equilibrium profit evolution*



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Conclusions

Despite the fact that b parameter value is continuously changing, we can easily observe that the equilibrium price remain constant, with its mathematical expression depending only on a and c parameters. More precisely, regardless b level growth from 0.1 to 3, the equilibrium price keeps its initial 55 EUR value.

As for the quantity triggering the equilibrium scenario, a downward trend is observed, starting with 5*(a-c) (leader case)/ 2.5*(a-c) (follower case). The explanation is also mathematical, deriving from the fact that $q_1{}^{*\prime} = -\frac{a-c}{2b^2}, q_2{}^{*\prime} = -\frac{a-c}{4b^2}$ are negative expressions – kind of monotony specific for decreasing functions. Going further, $q_1{}^{*\prime\prime} = \frac{a-c}{b^3}, q_2{}^{*\prime\prime} = \frac{a-c}{2b^3}$, strictly positive second order derivates provoking the graph's convexity. Referring at figures, equilibrium quantity level follows a decreasing trend from its initial value of 300 kg (leader)/150 kg (follower), down to zero value (close to, but not tangible, because y=0 and x=0 represents horizontal/vertical asymptotes, in fact).

In profit equilibrium scenario a downward trend can be highlighted as well, starting from 1.25 $(a-c)^2$ (leader) / 0.625 $(a-c)^2$ (follower) down to zero, value which would also never been reached. One more time, math principles offer the key, as $\pi_1{}^{*\prime} = -\frac{(a-c)^2}{8b^2}$, $\pi_2{}^{*\prime} = -\frac{(a-c)^2}{16b^2}$, strictly negative expressions being specific for decreasing functions. For the same aforementioned reasons (second order positive derivates), we face also a function convexity scenario. Previously hypothesis being given, a downward profit trend can be observed, starting with 4.500 EUR (leader) / 2.250 EUR (follower) down to minimum rentability level (zero profit – not tangible, having y = 0 also a horizontal asymptote).

3. Graphic approach

In the next paragraphs, we will try to explain the Stackelberg behaviour, making use by the graphical method, based on duopolist's reaction functions. First of all, we have to deduce the leader isoprofit curve's general expression, and looking forward, his competitor's best response:

$\pi^1(q_1, q_2) = [a - b(q_1 + q_2) - c]q_1$,

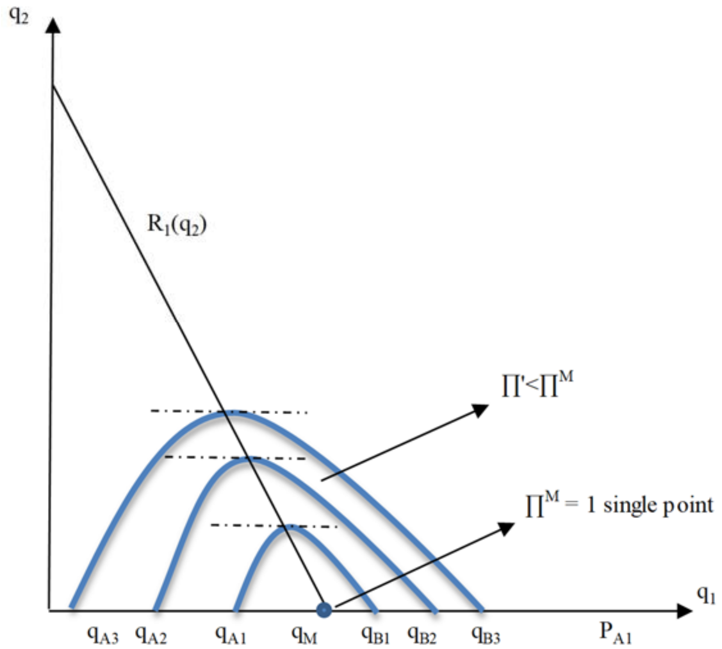
then $\bar{\pi} = [a - b(q_1 + q_2) - c]q_1 = aq_1 - bq_1^2 - bq_1q_2 - cq_1$

$$\rightarrow bq_1q_2 = (a-c)q_1 - bq_1^2 - \bar{\pi} \ \rightarrow q_2 = \frac{a-c}{b} - q_1 - \frac{\bar{\pi}}{bq_1}$$

Each isoprofit curve reflects a constant level of profit that could be obtained by a certain player at different output levels choosed by him and his rival. The follower's first order derivate expression highlights the isoprofits curves trend (ascending/descending), whilst the one related to the second order offers very important informations regarding the concavity related to the axes:

$$\frac{dq_2}{dq_1} = -1 + \frac{\bar{\pi}}{bq_1^2} \rightarrow \frac{d^2q_2}{dq_1^2} = -\frac{2\bar{\pi}}{bq_1^2} < 0$$

Figure 3. *Leader's isoprofit and best reply functions*



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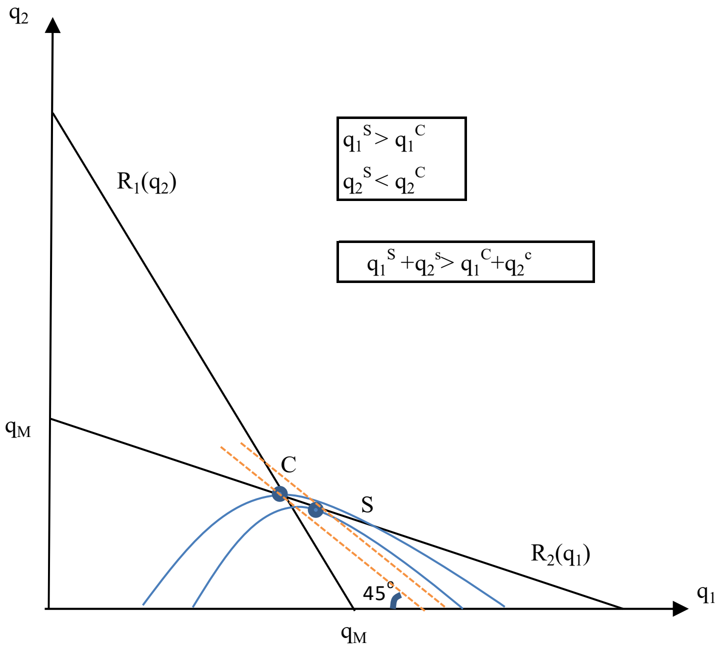
First player (the leader) will always choose its best response, highlighted by the isoprofit curve that corresponds to the maximum profit, at a q_2 given level (Figure 3).

The intersection point of the isoprofit curves with the reaction function, has the mathematical zero slope property (Machado, 2008).

$R_1(q_2) = argmax\ \pi^1(q_1, q_2) \rightarrow \pi_1^1(R_1(q_2), q_2) = 0.$

Besides, we already know that $\pi^1(q_1, q_2) = \bar{\pi} \rightarrow \pi_1^1 dq_1 + \pi_2^1 dq_2 = 0 \rightarrow \frac{dq_2}{dq_1} = -\frac{\pi_1^1}{\pi_2^1}$, resulting the derivate $\frac{dq_2}{dq_1}$ should be null in leader's best response scenario $q_1 = R_1(q_2)$

Figure 4. *Stackelberg equilibrium vs. Cournot equilibrium*



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The leader's optimal behavior is reached in the tangency point S of his isoprofit curve with the reaction curve of the follower (second player), whilst C represents the Cournot equilibrium, where the reaction curves cross and where dq2/dq1 = 0 (as we have previously mentioned). All three above mentioned relations (see Figure 4) can be easily proved either by comparing the specific equilibrium values of Stackelberg and Cournot models (see formulas (21)-(25)) or by a simple figure analyse.

$$q_1^S = \frac{a-c}{2b} = \frac{3}{2}\frac{a-c}{3b} = \frac{3}{2}q_1^C > q_1^C \tag{20}$$

$$q_2^S = \frac{a-c}{4b} = \frac{3}{4}\frac{a-c}{3b} = \frac{3}{4}q_2^C < q_2^C \tag{21}$$

$$Q^S = q_1^S + q_1^S = \frac{3(a-c)}{4b} > \frac{2(a-c)}{3b} = q_1^C + q_1^C = Q^C \tag{22}$$

$$p^S = \frac{a+3c}{4} < \frac{a+2c}{3} = p^C \xleftarrow{a>c} \frac{a+3c}{4} < \frac{a+2c}{3} \to$$

$$\to 3a + 9c < 4a + 8c \to c < a\ (A) \tag{23}$$

$$\pi_1^S = \frac{(a-c)^2}{8b} = \frac{9}{8}\frac{(a-c)^2}{9b} = \frac{9}{8}\pi_1^C > \pi_1^C \tag{24}$$

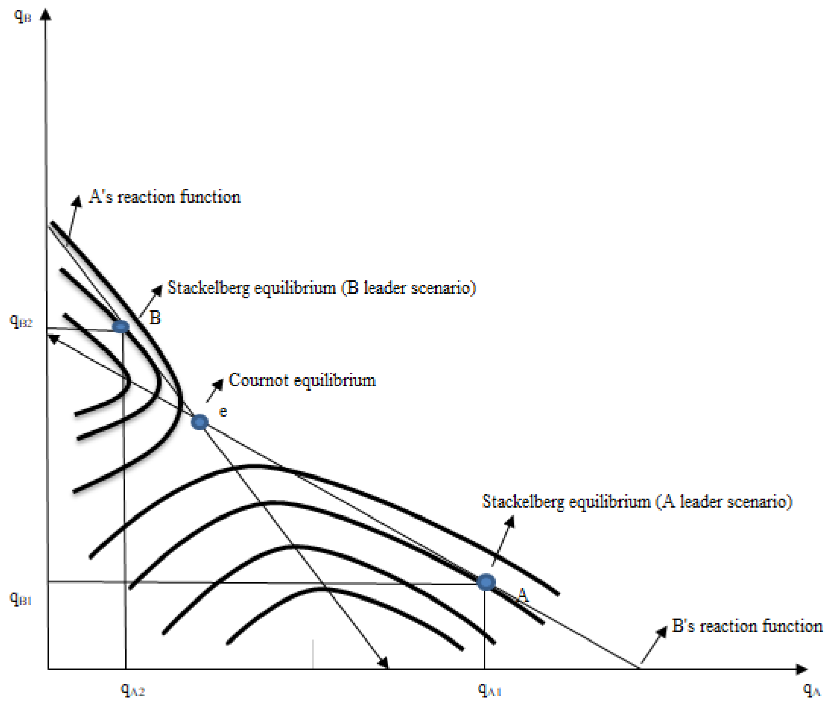
$$\pi_2^S = \frac{(a-c)^2}{16b} = \frac{9}{16}\frac{(a-c)^2}{9b} = \frac{9}{16}\pi_2^C < \pi_2^C \tag{25}$$

Conclusions: in the symmetric firms scenario (with matching costs), the Stackelberg solution is better than Cournot's one (higher aggregate output, lower price, higher aggregate profits). On the other side, leader's profit level should be no lower than in Cournot scenario because he could have always obtain the Cournot profits level by simply choosing the Cournot quantity q_1^C, to which his rival would have replied with its Cournot quantity $q_2^C = R_2(q_1^C)$, since the follower reaction curve in Cournot is the same as in Stackelberg.

We can further expand our analysis, referring less at the mathematic principles and focusing almost exclusively on the graphical approach. Moreover by offering additional details, we intend to facilitate a better understanding of the previous conclusions while also formulating new ones. Maintaining the direction, we are highlighting any possible scenario that can be found in the real market and also explaining behavioral patterns which can be rationally adopted by duopolists.

The isoprofit curves (concave to the axes, measuring players outputs) and also the duopolists reaction functions are presented in Figure 5. Assuming first player as the leader, it will consider that his competitor will always act after a stringent observation of its own reaction curve. Because of this assumption, the leader can afford to set its own output level in order to maximize its own profit. The level we are referring at, is represented by point A, situated on the lowest possible isoprofit leader's curve, highlighting the maximum profit this one can achieve given the follower's reaction curve.

Figure 5. *Duopolists reaction function and also isoprofit curves*



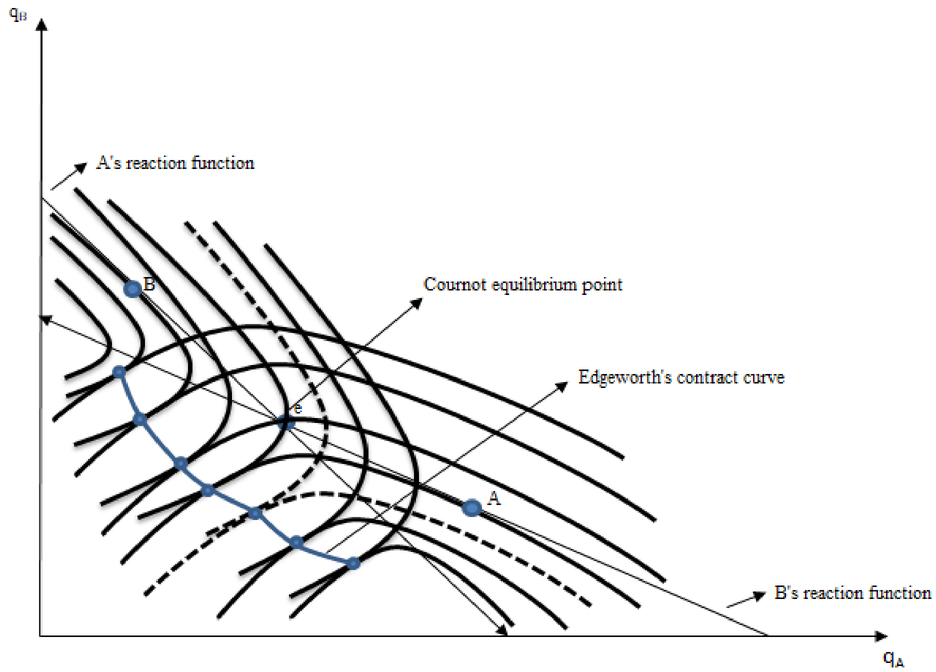
Source: own processing.

Acting as a monopolist, first player will take care to incorporate the follower's reaction function in his profit-maximizing estimations. He will choose to produce q_{A1}, and the second player will react by producing q_{B1} according to its reaction curve. The leader's market advantage is rewarding him, because in this scenario, he reaches an isoprofit curve closer to his axis than in the situation of behaving with the same naivete as the follower. The same scenario proved to be worse for the second player, comparing with the Cournot equilibrium case, since this output level allows him to reach an isoprofit curve further away from his axis.

In we consider a scenario with the second player as leader instead, his output producing decision being q_{B2}, whilst his competitor immediate reaction q_{A2}. The graphic corresponding point will be B this time, lied on first player's reaction function, measuring the largest profit level that player B can achieve, based on his isoprofit map and first player's reaction function. The actual leader register a higher profit whilst the first player has a lower profit as compared with the Cournot equilibrium scenario.

To conclude, with only one firm sophisticated on the market, emerging as the leader, a stable equilibrium will manifest and the naive firm will always act as a follower.

Figure 6. *Rational player equilibrium and Edgeworth's contract curve*



Source: own processing.

In the real market, duopoly' scenarios can be found, where both players having comparable market shares, sophistication, size, reputation, etc. In this situation, each of them will feel rightful to act as leader, because acting accordingly, will register a higher profit level. Such a behavior, with both having the pretence that the other one strategy being in strictly dependence of his own, will finally drive the market to instability. The situation is better known as Stackelberg's disequilibrium and there exists only two possible scenarios to get out of it: either a price war triggers, until one player surrenders and accepts to act as follower, or a collusion is targeted. Latest option will assume that firms abandons their naive reaction functions and moves to a point closer to the Edgeworth contract curve, higher profit levels being reached by each. If the final equilibrium lies precisely on the Edgeworth contract curve, the joint industry profits are maximised (Figure 6).

The last possibility is that both firms desire to be followers. Obviously, their expectations are not materialized, since each duopolist assumes that his competitor will act as a leader, so they have to revise them. Two behavioural patterns are possible. If each duopolist aware that his rival wants to be also a follower, the Cournot equilibrium is attained. Otherwise, one player should alter his behaviour and start acting as a leader before equilibrium is reached.

Stackelberg's model analyze drives to some interesting conclusions. First, it shows clearly that naive behaviour does not work. The players should admit their interdependence. Perfect awareness of his rival's reactions allows each duopolist to increase its profit level. If both players recognise their mutual interdependence, each starts considering his rival's profits and reactions and worrying about it. In return, if they continue to ignore each other, a price war will be implacable, its final result being worse off.

On the other side, the model highlights that a bargaining procedure and a collusive agreement will be advantageous for both. In such a scenario, the players may reach a point lied on the Edgeworth contract curve, then joint profit will be maximised.

To conclude our analyze, it could be useful to mention Stackleberg's model weaknesses, loudly criticized by some experts. The Stackelberg solution successfully correlate the duopoly issue to a family of related market structures. Unfortunately, the theory is focused on the use of reaction functions, highlighting individual profit maximisation for given values of the competitor's variable. This undermine theory's practical importance, by excluding the problem of coordination and collusion between duopolists.

1. The exclusion of the collusion aspect leads to unlikely results. There isn't any doubt regarding that the leadership equilibrium (one leader and one follower scenario) includes collusion or spontaneous coordination elements. However, they represent an arbitrary coordination form when leadership is resumed only at selecting a point along a traditional kind reaction curve. Such type of equilibria carry small meaning reported at joint profit maximisation.
2. The intersection-point equilibrium is based on arbitrary and wrong notions regarding the way of the competitor's behaves. They rest on the assumption that competitor's variable value is given regardless of the duopolist's own moves. The intersection-point equilibrium emerge from a mutual attempt to follow the rival's leadership. But the selected point on the reaction function by one duopolist does not play any part in shaping the policies of his rival, meaning this analysis basical assumption is arbitrary and wrong.
3. Resulting from the leadership attempts of both duopolists, the Stackerlberg disequilibrium is also based on wrong rationality and arbitrary assumptions. It may arise from the assumption that the competitor moves along a reaction curve which does not actually exist for him. Other possible explanation is based on the argument that each

competitor is forced to react along a curve which does not exist for him, being mandatory for him to act as a follower.

This way, the reaction curves of the Stackelberg problem, based on mere hypothesis, have made his theory poor and unrealistic. Despite these weaknesses, the Stackelberg model highlights the importance of mutual interdependence between duopolists. If they admit it, will be able to earn profits, but if they decide to ignore it, both will be losers. On the other way, the entrance into a collusive agreement, could maximise their jointly profits.

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Appendix A

$$\frac{\delta\pi_2}{\delta q_2} = 0 \to \frac{\delta[a - b(q_1 + q_2)]}{\delta q_2} q_2 + a - b(q_1 + q_2) - \frac{\delta C_2(q_2)}{\delta q_2} =$$

$$= -bq_2 + a - b(q_1 + q_2) - \frac{\delta C_2(q_2)}{\delta q_2} = 0$$

$$-2bq_2 = \frac{\delta C_2(q_2)}{\delta q_2} - a + bq_1 \;\to\; q_2 = \frac{a - bq_1 - \frac{\delta C_2(q_2)}{\delta q_2}}{2b}$$

$$\pi_1 = \left[a - b\left(q_1 + q_2(q_1)\right)\right] q_1 - C_1(q_1) =$$

$$= \left(a - b\left(q_1 + \frac{a - bq_1 - \frac{\delta C_2(q_2)}{\delta q_2}}{2b} \right) \right) q_1 - C_1(q_1) =$$

$$= \left(a - bq_1 - \frac{a - bq_1 - \frac{\delta C_2(q_2)}{\delta q_2}}{2} \right) q_1 - C_1(q_1) =$$

$$= \frac{a - bq_1 + \frac{\delta C_2(q_2)}{\delta q_2}}{2} q_1 - C_1(q_1)$$

$$\frac{\delta\pi_1}{\delta q_1} = 0 \to \frac{a - bq_1 + \frac{\delta C_2(q_2)}{\delta q_2}}{2} - \frac{bq_1}{2} - \frac{\delta C_1(q_1)}{\delta q_1} =$$

$$= \frac{a - 2bq_1 + \frac{\delta C_2(q_2)}{\delta q_2}}{2} - \frac{\delta C_1(q_1)}{\delta q_1} = 0$$

$$-bq_1 + \frac{a + \frac{\delta C_2(q_2)}{\delta q_2}}{2} - \frac{\delta C_1(q_1)}{\delta q_1} = 0 \to q_1^* = \frac{a + \frac{\delta C_2(q_2)}{\delta q_2} - 2\frac{\delta C_1(q_1)}{\delta q_1}}{2b}$$

$$q_2^* = \frac{a - b\dfrac{a + \dfrac{\delta C_2(q_2)}{\delta q_2} - 2\dfrac{\delta C_1(q_1)}{\delta q_1}}{2b} - \dfrac{\delta C_2(q_2)}{\delta q_2}}{2b} \rightarrow$$

$$\rightarrow\ q_2^* = \frac{a - 3\dfrac{\delta C_2(q_2)}{\delta q_2} + 2\dfrac{\delta C_1(q_1)}{\delta q_1}}{4b}$$

$$q^* = q_1^* + q_2^* = \frac{a + \dfrac{\delta C_2(q_2)}{\delta q_2} - 2\dfrac{\delta C_1(q_1)}{\delta q_1}}{2b} + \frac{a - 3\dfrac{\delta C_2(q_2)}{\delta q_2} + 2\dfrac{\delta C_1(q_1)}{\delta q_1}}{4b} =$$

$$= \frac{3a - \dfrac{\delta C_2(q_2)}{\delta q_2} - 2\dfrac{\delta C_1(q_1)}{\delta q_1}}{4b}$$

$$p^* = a - b(q_1^* + q_2^*) = a - b\frac{3a - \dfrac{\delta C_2(q_2)}{\delta q_2} - 2\dfrac{\delta C_1(q_1)}{\delta q_1}}{4b} \rightarrow$$

$$\rightarrow p^* = \frac{a + \dfrac{\delta C_2(q_2)}{\delta q_2} + 2\dfrac{\delta C_1(q_1)}{\delta q_1}}{4}$$

$$C_i(q_i) = cq_i \rightarrow \frac{\delta C_i(q_i)}{\delta q_i} = c_i(q_i) = c, (\forall)\ i = \overline{1,2}$$

$$q_1^* = \frac{a-c}{2b} \qquad q_2^* = \frac{a-c}{4b} \qquad p^* = \frac{a+3c}{4}$$

$$\pi_1^* = (p^* - c)q_1^* = \left(\frac{a+3c}{4} - c\right)\frac{a-c}{2b} = \frac{(a-c)^2}{8b}$$

$$\pi_2^* = (p^* - c)q_2^* = \left(\frac{a+3c}{4} - c\right)\frac{a-c}{4b} = \frac{(a-c)^2}{16b}$$

$$q_1^* = 2q_2^* \qquad \pi_1^* = 2\pi_2^*$$

Appendix B

Table 1. *Simulation of price, quantity and profit evolution*

b	p	q_1	q_2	π_1	π_2
0.1	0.25*a+0.75*c	5.000000*(a-c)	2.5*(a-c)	1.250000*(a-c)^2	0.625000*(a-c)^2
0.2	0.25*a+0.75*c	2.500000*(a-c)	1.25*(a-c)	0.625000*(a-c)^2	0.312500*(a-c)^2
0.3	0.25*a+0.75*c	1.666667*(a-c)	0.833333*(a-c)	0.416667*(a-c)^2	0.208333*(a-c)^2
0.4	0.25*a+0.75*c	1.250000*(a-c)	0.625000*(a-c)	0.312500*(a-c)^2	0.156250*(a-c)^2
0.5	0.25*a+0.75*c	a-c	0.500000*(a-c)	0.250000*(a-c)^2	0.125000*(a-c)^2
0.6	0.25*a+0.75*c	0.833333*(a-c)	0.416667*(a-c)	0.208333*(a-c)^2	0.104167*(a-c)^2
0.7	0.25*a+0.75*c	0.714286*(a-c)	0.357143*(a-c)	0.178571*(a-c)^2	0.089286*(a-c)^2
0.8	0.25*a+0.75*c	0.625*(a-c)	0.312500*(a-c)	0.156250*(a-c)^2	0.078125*(a-c)^2
0.9	0.25*a+0.75*c	0.555556*(a-c)	0.277778*(a-c)	0.138889*(a-c)^2	0.069444*(a-c)^2
1.0	0.25*a+0.75*c	0.5*(a-c)	0.250000*(a-c)	0.125000*(a-c)^2	0.062500*(a-c)^2
1.1	0.25*a+0.75*c	0.454545*(a-c)	0.227273*(a-c)	0.113636*(a-c)^2	0.056818*(a-c)^2
1.2	0.25*a+0.75*c	0.416667*(a-c)	0.208333*(a-c)	0.104167*(a-c)^2	0.052083*(a-c)^2
1.3	0.25*a+0.75*c	0.384615*(a-c)	0.192308*(a-c)	0.096154*(a-c)^2	0.048077*(a-c)^2
1.4	0.25*a+0.75*c	0.357143*(a-c)	0.178571*(a-c)	0.089286*(a-c)^2	0.044643*(a-c)^2
1.5	0.25*a+0.75*c	0.333333*(a-c)	0.166667*(a-c)	0.083333*(a-c)^2	0.041667*(a-c)^2
1.6	0.25*a+0.75*c	0.312500*(a-c)	0.156250*(a-c)	0.078125*(a-c)^2	0.039063 (a-c)^2
1.7	0.25*a+0.75*c	0.294118*(a-c)	0.147059*(a-c)	0.073529*(a-c)^2	0.036765*(a-c)^2
1.8	0.25*a+0.75*c	0.277778*(a-c)	0.138889*(a-c)	0.069444*(a-c)^2	0.034722*(a-c)^2
1.9	0.25*a+0.75*c	0.263158*(a-c)	0.131579*(a-c)	0.065789*(a-c)^2	0.032895*(a-c)^2
2.0	0.25*a+0.75*c	0.250000*(a-c)	0.125000*(a-c)	0.062500*(a-c)	0.031250*(a-c)
2.1	0.25*a+0.75*c	0.238095*(a-c)	0.119048*(a-c)	0.059524*(a-c)	0.029762*(a-c)
2.2	0.25*a+0.75*c	0.227273*(a-c)	0.113636*(a-c)	0.056818*(a-c)	0.028409*(a-c)
2.3	0.25*a+0.75*c	0.217391*(a-c)	0.108696*(a-c)	0.054348*(a-c)	0.027174*(a-c)
2.4	0.25*a+0.75*c	0.208333*(a-c)	0.104167*(a-c)	0.052083*(a-c)^2	0.026042*(a-c)^2
2.5	0.25*a+0.75*c	0.200000*(a-c)	0.100000*(a-c)	0.050000*(a-c)	0.025000*(a-c)
2.6	0.25*a+0.75*c	0.192308*(a-c)	0.096154*(a-c)	0.048077*(a-c)	0.024038*(a-c)
2.7	0.25*a+0.75*c	0.185185*(a-c)	0.092593*(a-c)	0.046296*(a-c)	0.023148*(a-c)
2.8	0.25*a+0.75*c	0.178571*(a-c)	0.089286*(a-c)	0.044643*(a-c)	0.022321*(a-c)
2.9	0.25*a+0.75*c	0.172414*(a-c)	0.086207*(a-c)	0.043103*(a-c)	0.021552*(a-c)
3.0	0.25*a+0.75*c	0.166667*(a-c)	0.083333*(a-c)	0.041667*(a-c)	0.020833*(a-c)

Source: own processing.