Magnetic mirrors. Radiation belts.

Adiabatic invariants.

([8], p.28-47; F.Chen, Introduction to plasma physics and controlled fusion, 1984, v.1, p.30-34, 41-49]

Consider an axisymmetric magnetic field:

$$\vec{B} = (B_r, 0, B_z), \quad \partial\vec{B}/\partial\theta = 0$$

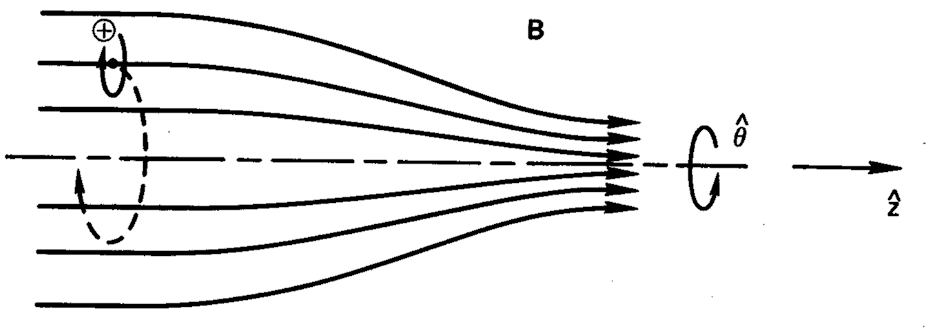

Figure 1:

and make use of equation

$$\mathrm{div}\vec{B} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$

If $\partial B_z/\partial z$ is given at $r=0$ and does not change much with r, then

$$rB_r = \int_0^r r\frac{\partial B_z}{\partial z}dr \approx -\frac{1}{2}r^2\left(\frac{\partial B_z}{\partial z}\right)_0$$

$$B_r = -\frac{1}{2}r\left(\frac{\partial B_z}{\partial z}\right)_0$$

Variation of $|B|$ with r causes $\nabla\ B$ drift perpendicular to ∇B and $\vec{B}$, that is in θ direction.

Calculate components of the Lorentz force $\vec{F} = q/c(\vec{v}\times\vec{B})$:

$$F_r = \frac{q}{c}(\underbrace{v_\theta B_z}_{①} - v_z \not{B}_\theta)$$

$$F_\theta = \frac{q}{c}(\underbrace{-v_r B_z}_{②} + \underbrace{v_z B_r}_{③})$$

$$F_z = \frac{q}{c}(v_r \not{B}_\theta - \underbrace{v_\theta B_r}_{④})$$

Terms 1 and 2 describe Larmor gyration. Term 3 describes drift in radial direction following the lines of force.

Consider term 4

$$F_z = -\frac{q}{c}(v_\theta B_r) = \frac{q}{2c} v_\theta r \left(\frac{\partial B}{\partial z} \right)_0$$

$$v_\theta = -\mathrm{sign}(q) \cdot v_\perp$$

is a constant during a gyration. Therefore, the average force is:

$$\overline{F_z} = -\frac{\mathrm{sign}(q) \cdot q}{2c} v_\perp r_L \left(\frac{\partial B_z}{\partial z} \right)_0$$

Here,

$$r_L = \frac{v_\perp}{\omega_c} = \frac{v_\perp mc}{|q|B}$$

Then,

$$\overline{F_z} = -\frac{1}{2} \frac{m v_\perp^2}{B} \left(\frac{\partial B_z}{\partial z} \right)_0 \equiv -\mu \frac{\partial B_z}{\partial z},$$

where

$$\mu = \frac{1}{2} \frac{m v_\perp^2}{B}$$

is the *magnetic moment* of the gyrating particle.

Compare this with the standard definition of magnetic moment of a moving particle with a circular orbit:

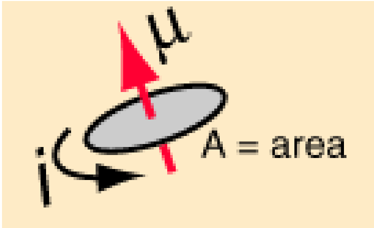

Figure 2: Magnetic moment $\mu = \frac{1}{c} I \cdot A$.

The effective current is $I = dQ/dt = q/P$, where P is the period of rotation; $P = 2\pi/\omega_c$, $A = \pi r_L^2$. Then,

$$\mu = \frac{q}{c}\frac{2\pi}{\omega_c}\pi r_L^2 = \frac{1}{2}\frac{mv_\perp^2}{B}$$

In general, the force on a gyrating particle is:

$$\vec{F}_\parallel = -\mu \nabla_\parallel \vec{B}$$

The important property of the magnetic moment is that it remains invariant. When the particle moves in regions of stronger field the Larmor radius changes but μ remains constant.

Let's proof this. Multiply the equation of motion of the particle by $v_\parallel$:

$$\frac{m}{2}\frac{dv_\parallel^2}{dt} = -\mu \frac{\partial B}{\partial z} v_\parallel = -\mu \frac{dB}{dt}$$

because $dz/dt = v_{\|}$. Here, dB/dt is the variation of B as seen by the particle.

The particle kinetic energy is conserved:

$$\frac{d}{dt}\left(\frac{mv_{\|}^2}{2} + \frac{mv_{\perp}^2}{2}\right) = 0$$

Using the definition of the magnetic moment and the equation of motion we get:

$$\frac{d}{dt}\left(\frac{mv_{\|}^2}{2} + \mu B\right) = -\mu\frac{dB}{dt} + \frac{d(\mu B)}{dt} = B\frac{d\mu}{dt} = 0$$

Thus, $\mu = \text{const}$.

If B increases as particle moves then $v_{\perp}$ increases, hence $v_{\|}$ decreases, and may become zero. At this point the particle stops and reverses the direction of motion. This is the effect of *magnetic mirror*.

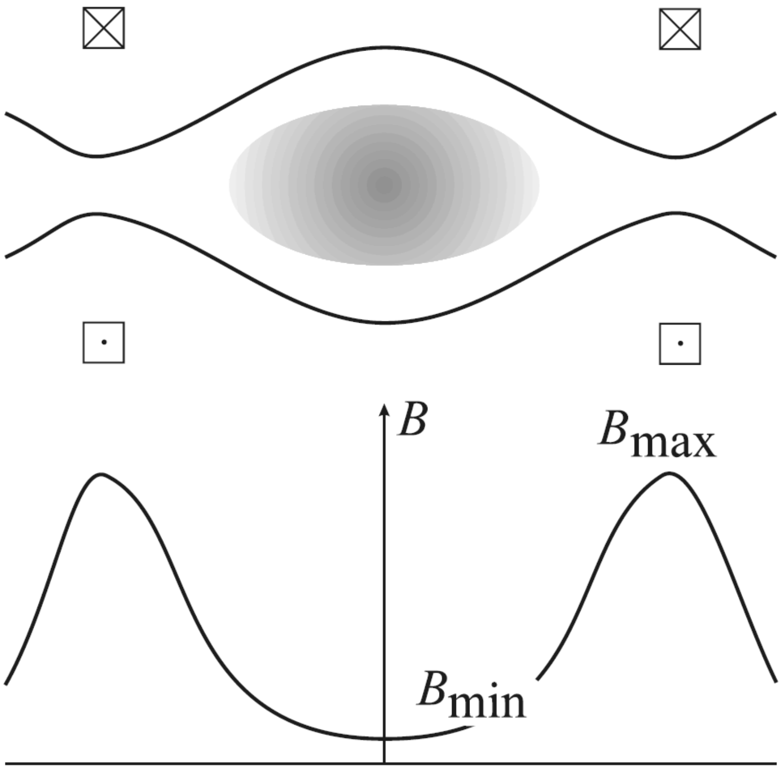

Figure 3: A plasma trapped between magnetic mirrors made of a pair of coils.

The trapping is not perfect. For instance, particles with $v_\perp = 0$ have $\mu = 0$ and do not feel any force along $\vec{B}$. These particles are not trapped. Also, particles that reach the points at the maximum magnetic field strength with non-zero $v_{||}$ escape. These are particles with small $v_\perp/v_{||}$ ratio.

Consider a particle with $v_\perp = v_{\perp 0}$, $v_{||} = v_{||0}$ and the total velocity, v_0, ($v_0^2 = v_{\perp 0}^2 + v_{||0}^2$) in the middle of the trap where $B = B_{\rm min} = B_0$, and with $v_\perp = v_{\perp\rm m}$ and $v_\| = v_{\|\rm m}$ at the turning point where $B = B_{\rm max} = B_{\rm m}$.

Obviously, particles that reach $B_{\rm m}$ with $v_{\|\rm m} \geq 0$ escape. For these particles from conservation of energy:

$$v_{\perp\rm m}^2 + v_{\|\rm m}^2 = v_{\perp0}^2 + v_{\|0}^2 = v_0^2$$

we get

$$v_{\perp\rm m}^2 \leq v_0^2$$

Then because of invariance of μ:

$$\frac{1}{2}\frac{mv_{\perp0}^2}{B_0} = \frac{1}{2}\frac{mv_{\perp\rm m}^2}{B_{\rm m}},$$

$$v_{\perp\rm m}^2 = \frac{B_{\rm m}}{B_0}v_{\perp0}^2.$$

Hence, particles with

$$\frac{v_{\perp0}^2}{v_0^2} \leq \frac{B_0}{B_{\rm m}}$$

escape.

If the angle between the velocity vector and z-axis is α (*pitch angle*):

$$\sin\alpha = \frac{v_{\perp 0}}{v_0}$$

then particles escape when

$$\alpha \leq \alpha_{\mathrm{m}},$$

where

$$\sin^2\alpha_{\mathrm{m}} = \frac{B_0}{B_{\mathrm{m}}} = \frac{1}{R_m}$$

R_m is called the mirror ratio (the ratio of max B to min B).

The cone of $\alpha \leq \alpha_m$ in the velocity space is called *the loss cone.*

The loss cone is independent of q and m. Hence electrons and ions are equally trapped. However, due to collisions the pitch angle may change, and since electrons collide more frequently than ions, thus they are lost more easily.

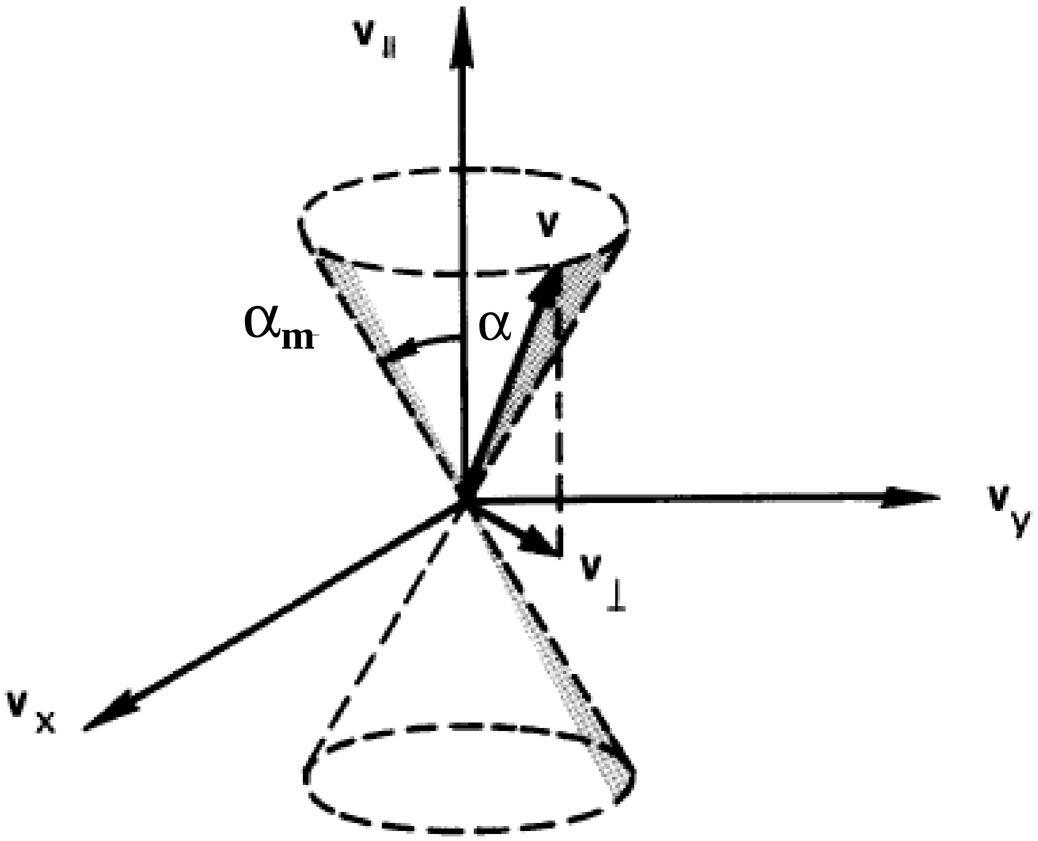

Figure 4: The loss cone $\alpha \leq \alpha_m$.

Fermi suggested that particles trapped between two magnetic mirrors moving towards each other accelerate. This is *Fermi acceleration mechanism.* It is suggested to explain acceleration of cosmic rays.

Time-varying magnetic field

Consider now effects of magnetic field variations with time. We have discussed that magnetic field cannot change the particle energy. However, electric field associated with varying magnetic field can accelerate particles.

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

Let $\vec{v}_\perp = d\vec{l}/dt$ be the transverse velocity along path $\vec{l}$ (with $v_\parallel$ neglected). Taking the scalar product of $\vec{v}_\perp$ and the equation of motion

$$m\frac{d\vec{v}_\perp}{dt} = q\vec{E}$$

we have

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = q\vec{E}\cdot\vec{v}_\perp = q\vec{E}\cdot\frac{d\vec{l}}{dt}.$$

The energy change during one gyration

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \int_0^{2\pi/\omega_c} q\vec{E}\cdot\frac{d\vec{l}}{dt}dt = \oint q\vec{E}\cdot d\vec{l} =$$

$$= q \int_S (\nabla \times \vec{E}) d\vec{s} = -\frac{q}{c} \int_S \dot{\vec{B}} \cdot d\vec{s}.$$

Here $\vec{S}$ is the area of the Larmor orbit, and a direction given by the right-hand rule when the fingers point in the direction of $\vec{v}$.

The direction of gyration is opposite for ions and electrons. Therefore, $\vec{B} \cdot d\vec{s} < 0$ for ions, and $\vec{B} \cdot d\vec{s} > 0$ for electrons. Hence,

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \text{sign}(q) q \frac{1}{c} \pi r_L^2 \dot{B} = \frac{\frac{1}{2}mv_\perp^2}{B} \frac{2\pi \dot{B}}{\omega_c} = \mu \delta B,$$

where δB is the change of δB during one period of gyration. Thus

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \mu \delta B$$

Since

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \delta(\mu B)$$

then

$$\delta\mu = 0.$$

This means that the magnetic moment is invariant in slowly varying magnetic fields.

When the magnetic field strength increases the particle energy also increases. This is used for plasma heating.

Theorem. Magnetic flux through a Larmor orbit is constant.

Proof:

$$\Phi = B\pi r_L^2 = \pi B \frac{v_\perp^2}{\omega_c^2} = \pi B \frac{v_\perp^2 m^2 c^2}{q^2 B^2} = \frac{2\pi c^2 m}{q^2}\mu = \text{const.}$$

The Van Allen radiation belts

The radiation belts were discovered by James Van Allen using data taken from Geiger counters on the first US satellite, Explorer 1. There two radiation belts. The inner belt extending from about 1 to 3 Earth radii in the equatorial region is populated by protons with energies greater than 10 MeV. The protons are produced from decay of neutrons which are emitted from the atmosphere bombarded by cosmic rays. The lifetime of particles in this belt range from few hours to 10 years. The outer belt at 3-9 Earth radii consists mostly of electrons with energies below 10 MeV. The electrons are ejected from the outer magnetosphere. This belt is very dynamic, changing on the time scale of few hours.

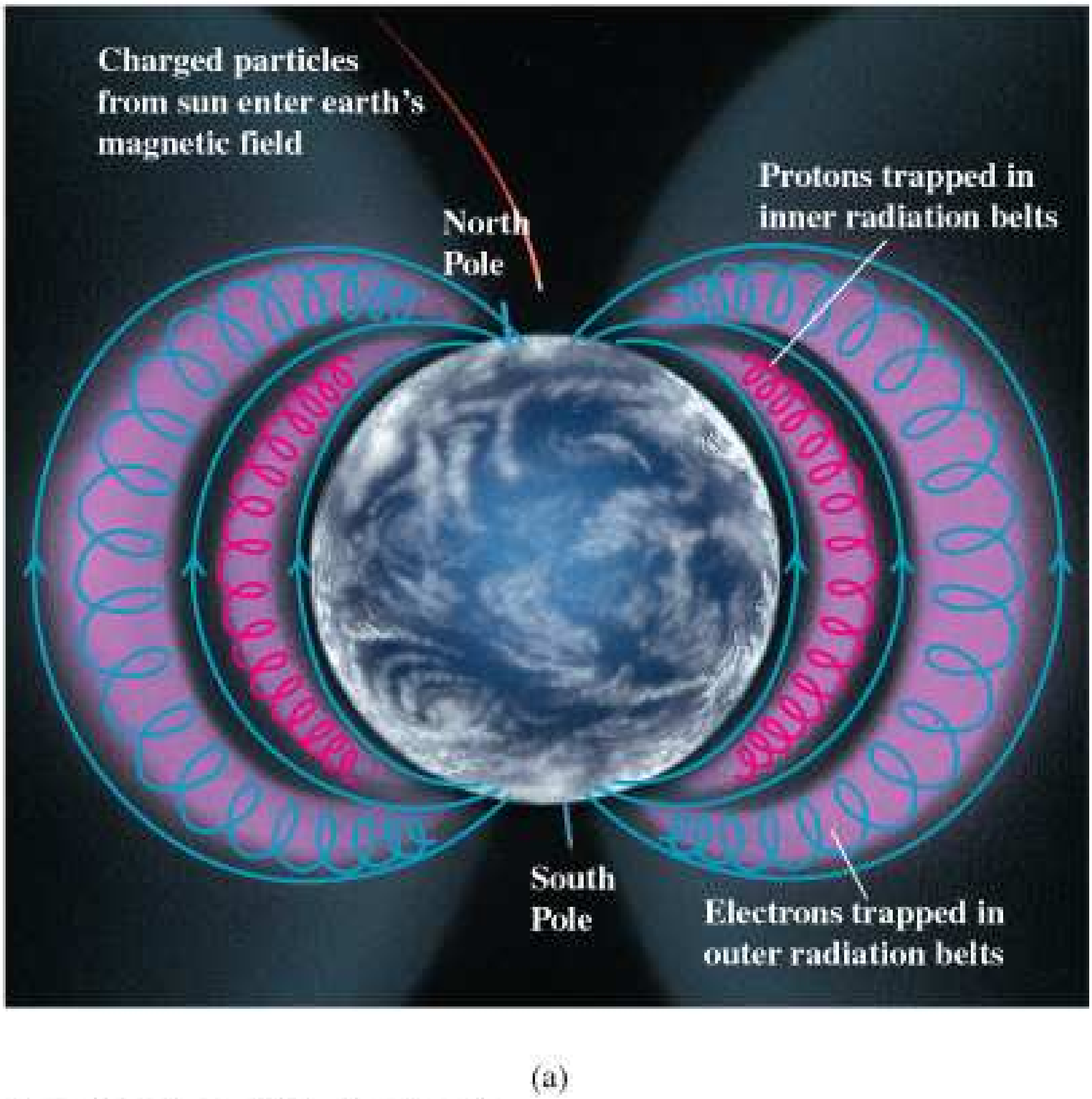

Figure 5: The radiation belts.

The Earth's magnetic field can be approximated as a dipole field in spherical coordinates (r, Θ, Φ) (Θ is colatitude):

$$B_r = \frac{2M_E}{r^3}\cos\Theta$$

$$B_\Theta = \frac{M_E}{r^3}\sin\Theta$$

$$|B| = \frac{M_E}{r^3}(1 + 3\cos^2\Theta)^{1/2}$$

where

$$B_E = \frac{M_E}{R_E^3} \approx 0.3 \;\; \mathrm{G}$$

Equation for magnetic field lines:

$$\frac{dr}{B_r} = \frac{rd\Theta}{B_\Theta}$$

or

$$\frac{dr}{2\cos\Theta} = \frac{rd\Theta}{\sin\Theta}$$

has solution

$$r = \mathrm{const}\cdot\sin^2\Theta$$

Since at the equator, $\Theta = \pi/2$, $r = r_{eq}$ (where r_{eq} is the radial distance to the field lines in the equatorial plane, we have:

$$r = r_{eq}\sin^2\Theta$$

or in terms of latitude $\theta = \pi/2 - \Theta$:

$$r = r_{eq} \cos^2 \theta$$

Hence,

$$B = \frac{M_E}{r_{eq}^3} \frac{(1 + 3\sin^2\theta)^{1/2}}{\cos^6\theta}$$

or in terms of *L-shell parameter*, $L \equiv r_{eq}/R_E$:

$$B = \frac{B_E}{L^3} \frac{(1 + 3\sin^2\theta)^{1/2}}{\cos^6\theta}$$

where $B_E = M_E/R_E^3 = 3.11 \times 10^{-5}$ T is the equatorial magnetic field strength on the Earth's surface.

Practical formulas for particle motion in the radiation belts (for $\theta = 0$):

Cyclotron frequencies:

$$\nu_e = \frac{eB}{2\pi m_e c} = 5.46 \frac{1}{L^3} \text{ MHz}$$

$$\nu_p = 2.98 \frac{1}{L^3} \text{ kHz}$$

Gyroradii:

$$\frac{r_{L,e}}{R_E} = \sqrt{E_{\rm MeV}} \left(\frac{L}{38.9} \right)^3$$

$$\frac{r_{L,p}}{R_E} = \sqrt{E_{\rm MeV}} \left(\frac{L}{11.1} \right)^3$$

For the MeV radiation belt particles at $L < 10$ the cyclotron frequencies are much higher than the typical rates of magnetic field variations (on the scale hours or days). Also, the gyroradii are much smaller than the typical variations of the geomagnetic field (on the scale of R_E). Hence, the particles are trapped in the Earth's magnetic field.

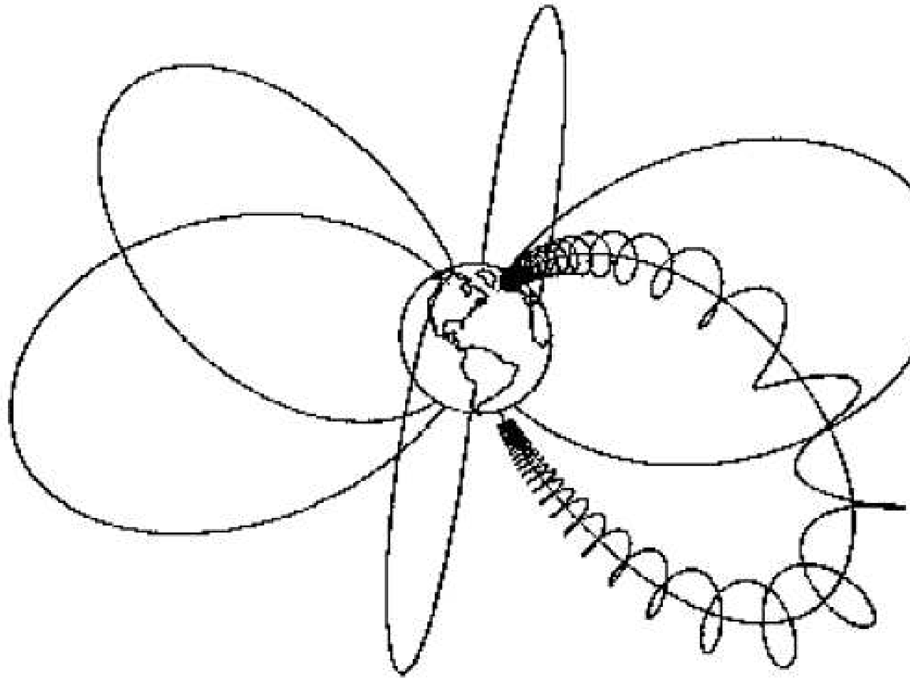

Figure 6: A typical trajectory of a charged particle in the radiation belts.

Define the pitch angle α:

$$\tan\alpha = \frac{v_\perp}{v_\parallel}$$

Because of invariance of the magnetic moment:

$$\frac{\sin^2\alpha}{\sin^2\alpha_e} = \frac{B}{B_{eq}}$$

where $B_{eq} = B_E/L^3$.

The loss cone is:

$$\sin^2 \alpha_{eq,m} = \frac{B_{eq}}{B_m} = \frac{\cos^6 \theta_m}{(1 + 3\sin^2 \theta_m)^{1/2}}$$

where B_m is the magnetic field strength at the mirror points, θ_m is the latitude of the mirror points (which depends only on the equatorial pitch angle).

Particles with large pitch angle are lost at low latitudes. Particles with small pitch angle have large parallel velocities, and their mirror points are at high latitudes. If the pitch angle is too small then the mirror point may be located at very small r, in the atmosphere, or, formally, even below the Earth's surface. Such particles are also lost.

If the magnetic field lines intersect the Earth atmosphere ($r = R_E$) at latitude θ_E then the loss cone:

$$\sin^2 \alpha_l = \frac{\cos^6 \theta_E}{(1 + 3\sin^2 \theta_E)^{1/2}}$$

Since $r = r_{eq} \cos^2 \theta$ and $L = r_{eq}/R_E$

$$\cos^2 \theta_E = 1/L$$

$$\sin^2 \alpha_l = \frac{1}{L^{5/2}(4L-3)^{1/2}}$$

For a typical $L = 6.6$ (a geostationary orbit):
$\alpha_l \approx 3°$.

The loss cone is very small.

The bounce period:

$$\tau_b = 4 \int_0^{\theta_m} \frac{ds}{v_{\|}}$$

where ds is an element of arc length along the field lines.

For protons:

$$\tau_{b,p} \simeq 2.41 \frac{L}{\sqrt{E_{\rm MeV}}} (1 - 0.43 \sin \alpha_{eq,m}) \;\; \text{sec}$$

for electrons:

$$\tau_{b,e} \simeq 5.62 \times 10^{-2} \frac{L}{\sqrt{E_{\rm MeV}}} (1 - 0.43 \sin \alpha_{eq,m}) \;\; \text{sec}$$

Typically, this is less than a second for electrons, and 1-10 sec for protons.

The ring current

Particles in the radiation belts move in non-uniform magnetic field with curved field lines. Hence we have to consider a combined centrifugal and gradient drift.

$$\vec{v}_d = \frac{mc\vec{R}_c \times \vec{B}}{qR_c^2 B^2}\left(v_\parallel^2 + \frac{v_\perp^2}{2}\right).$$

This drift is in the azimuthal direction: ions drift westward, electrons drift eastward. This is so-call ring current.

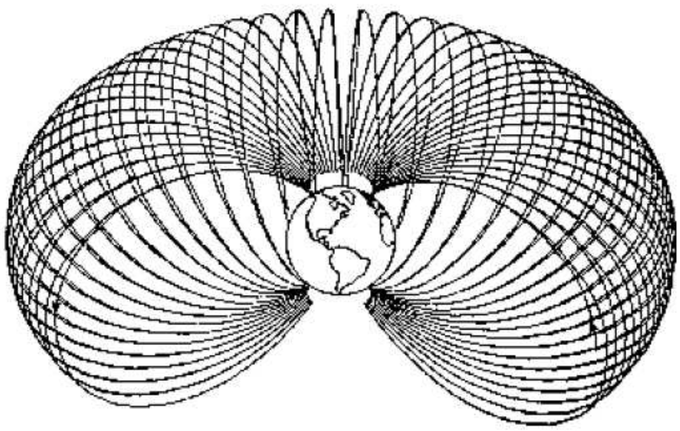

Figure 7: A typical trajectory of a charged particle in the radiation belts including azimuthal drift (ring current).

The drift period for electron and ions is:

$$\tau_d = \frac{2\pi L R_E}{v_d} \simeq \frac{1.05}{E_{\rm MeV} L}(1 + 0.43 \sin\alpha_{eq,m})^{-1} \ \text{hours}$$

The ring current causes a small reduction of the Earth's magnetic field in equatorial regions. The size of this reduction is a good measure of the number of particles in the radiation belts. During geomagnetic storms charged particles are injected into the Van Allen belts from the outer magnetosphere.

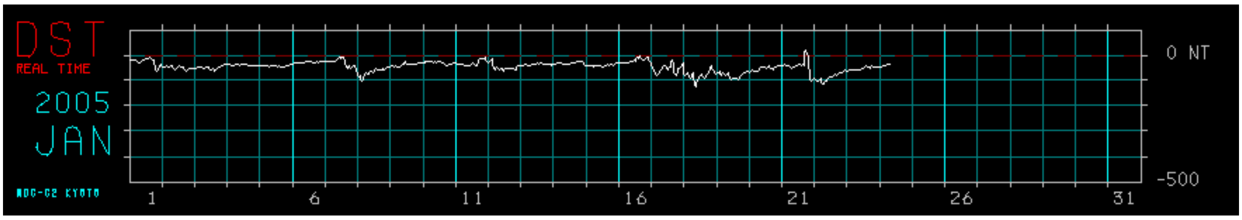

Figure 8: Dst data for January 2005 showing geomagnetic storms. The storm on January 16-17 was caused by a strong solar flare.

Adiabatic invariants

Adiabatic invariant is a lowest order approximation to the more fundamental type of invariant, a Poincare invariant:

$$J = \oint_{C(t)} \vec{p} \cdot d\vec{q},$$

where $\vec{p}$ and $\vec{q}$ are generalized canonical coordinates and momentum, and $C(t)$ is a closed curve in the phase space.

Lagrangian of a charged particle in magnetic field is:

$$L = \frac{m\vec{v}^2}{2} + \frac{e}{c}\vec{v} \cdot \vec{A}$$

where A is the vector potential: $\vec{B} = \nabla \times \vec{A}$.

Canonical momentum:

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + \frac{e}{c}\vec{A}$$

Consider a circular path (gyration):

$$J = \oint \vec{p}_\perp \cdot d\vec{l} = \oint m\vec{v}_\perp \cdot d\vec{l} + \frac{e}{c}\oint \vec{A} \cdot d\vec{l} =$$

$$= \oint m\omega_c r_L^2 d\theta + \frac{e}{c}\int_S \vec{B}\cdot\vec{n}ds =$$

$$= 2\pi \underbrace{m\omega_c}_{eB/c} r_L^2 + \frac{e}{c}B\pi r_L^2 = 3\pi\frac{e}{c}\Phi$$

where $\Phi = \pi r_L^2 B$ is the magnetic through the orbit.

$$J \propto \Phi \propto \frac{mv_\perp^2}{B} \propto \mu$$

Hence, μ is adiabatic invariant. It is called **the first adiabatic invariant**.

The second adiabatic invariant is associated with the particle bouncing motion between the magnetic mirrors:

$$J = \oint p_\parallel ds$$

where ds is an element of arc-length along the field line.

$$\vec{p} = m\vec{v} + \frac{e}{c}\vec{A}$$

$$J = \oint v_\parallel ds + \frac{e}{c}\oint A_\parallel ds = m\oint v_\parallel ds + \frac{e}{c}\Phi$$

where Φ is the total flux enclosed by the curve. In this case, it is zero. Thus, the second adiabatic invariant is:

$$J = \oint v_{\|} ds$$

Invariance of μ means that $|B|$ at the mirror points is the same. Invariance of $\oint v_{\|} ds$ fixes the length of the magnetic line.

Consider a mirror-trapped particle slowly drifting in longitude around the Earth. If the magnetic field were perfectly symmetric the particle would return to the same field line. However, the actual field is distorted, and the particle may return to a different line at different altitude. However, this cannot happen if J is conserved, because it determines the length of the field line between to turning points, and no two lines have the same length between two points with the same $|B|$.

The third adiabatic invariant is associated with the precession of particles around the Earth.

$$K = \oint p_{\phi} ds$$

$$p_{\phi} = mv_{\phi} + \frac{e}{c} A_{\phi}$$

Since the drift velocity is very small:

$$K \approx \frac{e}{c} \oint A_\phi ds = \frac{e}{c} \Phi,$$

where Φ is the total magnetic flux by the orbit of the bounce center around the Earth. Flux Φ is constant only if the magnetic field varies on the time scale much longer than the drift period τ_d, which is of order an hour.

Suppose the strength of the solar wind increases slowly, compressing the Earth's magnetic field. The invariance of Φ would cause particle move closer to the Earth to conserve the flux enclosed by their orbits. This means an increase of the ring current, causing a decrease of the Earth's magnetic field (reduction of Dst index) and an geomagnetic storm because of increased particle precipitation (auroral activity etc).