Topological groups

1 Basic facts

1.1 Subgroup of a topological group is a topological group

1.2 Product of compact subsets

Lemma 1. *If $K, L \subseteq G$ are compact subsets of G, then KL is also compact.*

Proof. The map $\cdot\colon G \times G \to G$ is continuous. This implies that KL is a continuous image of the compact space $K \times L$. $\square$

1.3 Every neighborhood of identity contains "square" of some other neighborhood

Lemma 2. *For any neighborhood W of e there exists a symmetric neighborhood U such that $U^2 \subseteq W$.*

Proof. From the continuity of $\cdot\colon G \times G \to G$ we have that there is a neighborhood $V \ni e$ such that $V^2 \subseteq W$. If we put $U = V \cap V^{-1}$ then U is a neighborhood of e with the desired properties. $\square$

1.4 $\overline{V} \subseteq V^2$ for a neighborhood of e

Lemma 3. *If V is a neighborhood of e, then $V \subseteq \overline{V} \subseteq V^2$.*

Proof. If $x \in \overline{V}$ then every neighborhood of x intersects V. In particular, we have

$$xV^{-1} \cap V \neq \emptyset$$

and $x \in V^2$. $\square$

1.5 Closure in open neighborhood

[AT, Proposition 1.4.4] gives a more general claim for left topological groups with continuous inverse.

Proposition 1. *Let G be a topological group. Then, for every subset $A \subseteq G$ and every open neighborhood U of e we have $\overline{A} \subseteq AU$.*

Proof. Take any open neighborhood V of e such that $V^{-1} \subseteq U$.

For any $x \in \overline{A}$ we have $xV \cap A \neq \emptyset$, which implies $x \in AV^{-1} \subseteq AU$. $\square$

1.6 Every open subgroup is closed

1.7 Closure of a subgroup is a subgroup.

Lemma 4. *If $H \subseteq G$ is a subgroup, then so is $\overline{H}$.*

Dôkaz. TODO $\square$

1.8 Every subgroup is either clopen or has empty interior

2 Products of topological groups

2.1 Product of topological groups is a topological group

2.2 σ-product of topological groups is a topological group

3 P-groups

P-group is a topological group which is a P-space. (Every G_δ-set is open; i.e., intersection of countably many open sets is open.)

Notice that an equivalent condition is that intersection of countably many neighborhoods of e is open.

3.1 Dense P-subgroup implies P-group

[AT, Lemma 4.4.1d]

Lemma 5. *If G has a dense subgroup H and H is a P-group, then G is a P-group.*

Proof. Let U_n, $n \in \omega$, be a sequence of open neighborhoods of e in G. Then we can choose symmetric neighborhoods of e such that $V_{n+1}^2 \subseteq V_n \subseteq U_n$. We put

$$N = \bigcap_{n\in\omega} V_n.$$

The set N is a closed subgroup of G. (To see that N is closed, just notice that $\overline{N} \subseteq \overline{V_{n+1}} \subseteq V_n$, which implies $\overline{N} \subseteq \bigcap\limits_{n\in\omega} V_n = N$.)

Then $P = N \cap H$ is a G_δ-subset of H, so it is an open subgroup of H. So there is an open subset W of G such that $P = W \cap H$.

We have $e \in W$. Since H is dense in G, we get $\operatorname{cl}_G W = \operatorname{cl}_G P \subseteq N$.

This means that $\bigcap\limits_{n\in\omega} U_n \supseteq N \supseteq W$, i.e., we have shown that intersection of the given sequence of neighborhoods of e contains another neighborhood of e. □

3.2 Every P-group is zero-dimensional

[AT, Lemma 4.4.1a]:

Lemma 6. *If G is a P-group, then G has a base at identity consisting of open subgroups, so G is zero-dimensional.*

Proof. TODO □

4 Lindelöf P-groups

4.1 Lindelöf subspace of a Hausdorff P-space is closed

[AT, Lemma 4.4.3]

Lemma 7. *A Lindelöf subset Y of a Hausdorff P-space X is closed in X.*

The proof is very similar to the proof of the fact that a compact subset of a Hausdorff space is closed.

Proof. Let us fix any $a \in X \setminus Y$. We only need to show that a has a neighborhood disjoint from Y.

For any $y \in Y$ there are neighborhoods $U_y \ni y$ and $V_y \ni a$ such that $U_y \cap V_y = \emptyset$.

Since Y is Lindelöf, there is a countable subcover $\{U_{y_k}; k \in \omega\}$.

Then $V = \bigcap_{k\in\omega} V_{y_k}$ is an open neighborhood of a (since X is a P-space). And $V \cap Y = \emptyset$. □

As a corollary we get that any Lindelöf P-group is Raikov complete. (Since it is dense subgroup of the completion. And then Lemma 5 implies that the completion is a P-group.)

5 ω-narrow groups

5.1 Subgroup of an ω-narrow group is ω-narrow

TODO [AT, Proposition 3.4.4]

5.2 Dense ω-narrow subgroup implies ω-narrow

[AT, Theorem 3.4.9]

Theorem 1. *If a topological group G contains dense subgroup H such that H is ω-narrow, then G is also ω-narrow.*

Proof. Let U be a neighborhood of e in G. There exist a symmetric open neighborhood V of e such that $V^2 \subseteq U$. Since H is ω-narrow, there is a countable set $A \subseteq H$ with $H \subseteq AV$. Then by Proposition 1 we get $G = \overline{H} \subseteq AVV \subseteq AU$. □

5.3 Image of ω-narrow P-group

[AT, Lemma 4.4.2]:

Lemma 8. *Let G be an ω-narrow P-group. Then every homomorphic continuous image K of G with $\psi(K) \le \aleph_0$ is countable.*

Proof. Let $\varphi\colon G \to K$ be a continuous homomorphism. Let us denote $N = \operatorname{Ker}\varphi$. Then N is a closed invariant subgroup. (Since $\{e_K\}$ is closed.)

Now since $\psi(K)$ is countable, we get that $\{e_K\}$ is G_δ. Hence N is G_δ-subset of a P-space, so it is open.

Since G is ω-narrow, G/N is countable. We have group isomorphism between G/N and K, therefore $|G| = |G/N| \le \aleph_0$. □

References

[AT] A. V. Arkhangelskii and M. Tkachenko. *Topological Groups and Related Structures.* Atlantis Press/World Scientific, Amsterdam, 2008.