Math 541 Lecture #3
I.13: Metric Spaces, Part II
I.16: On the Structure of a Complete Metric Space

13.1: Separation and axioms of countability. Metric spaces enjoys many nice properties.

Proposition 13.2. A metric space $\{X; d\}$ is separable (i.e., has a countable dense subset) if and only if it satisfies the second axiom of countability.

Proof. If the metric topology satisfies the second axiom of countability, then by Proposition 4.2, it is separable.

So suppose that $\{X; d\}$ is separable.

Then there is a countable dense subset A of X.

The collection of balls centered at points of A having rational radii forms a countable base for the topology. □

16: On the structure of complete metric spaces. For a topological space $\{X; \mathcal{U}\}$, a subset E of X is **nowhere dense** in X if $\overline{E}^c = X - \overline{E}$ is dense in X, i.e., if the closure of $X - \overline{E}$ is X.

Proposition. Here are some basic facts about nowhere dense sets.

(i) If E is nowhere dense in X, then $\overline{E}$ is nowhere dense.

(ii) A closed set E is nowhere dense if and only if E does not contain any nonempty open set.

(iii) If E is nowhere dense and $\mathcal{O}$ is nonempty and open, then $\mathcal{O} - \overline{E}$ contains an nonempty open set.

Proof. (i) If E is nowhere dense, then $X - \overline{E}$ is dense in X.

Since the closure of the closed set $\overline{E}$ is itself, then $X - \overline{\overline{E}} = X - \overline{E}$ is dense, so $\overline{E}$ is nowhere dense.

(ii) For a closed set E, suppose that E is nowhere dense, so $X - E$ is dense in X.

If E did contain an nonempty open set $\mathcal{O}$, then $X - \mathcal{O} \supset X - E$.

Any $x \in \mathcal{O}$ satisfies $x \notin \overline{X - E}$ because $\mathcal{O} \cap (X - E) = \emptyset$.

This contradiction implies that the closed E does not contain an nonempty open set.

Now suppose that closed E does not contain an nonempty open set.

If $E = \overline{E}$ were not nowhere dense, then $X - E$ would not be dense, and so there would be $x \in X$ and an open $\mathcal{O}$ containing x such that $\mathcal{O} \cap (X - E) = \emptyset$.

This implies that $\mathcal{O} \subset E$.

This contradiction implies that E is nowhere dense.

(iii) Suppose E is nowhere dense, and let $\mathcal{O}$ be any open set.

By part (i), the closed set $\overline{E}$ is nowhere dense.

The difference $\mathcal{O} - \overline{E}$ is nonempty, because if it were not, then $\mathcal{O} \subset \overline{E}$ which would imply by part (ii) that $\overline{E}$ is not nowhere dense, a contradiction.

The nonempty $\mathcal{O} - \overline{E} = \mathcal{O} \cap \overline{E}^c$ is open because it is the intersection of finitely many open sets. $\square$

A subset E of a topological space $\{X; \mathcal{U}\}$ is **meager**, or **of first category**, if it is the countable union of nowhere dense sets.

The quintessential example of a meager set is $\mathbb{Q}$ as a subset of $\mathbb{R}$.

A subset E is **of second category** if it is not of first category.

A subset E is **residual** or **nonmeager** if it is the complement of a set of first category.

The meager set $\mathbb{Q}$ with the standard Euclidean metric $d(x, y) = |x - y|$ is not a complete metric space. Its completion $\mathbb{R}$ (the set of all Cauchy sequences in $\mathbb{Q}$ modulo the equivalence relation that two Cauchy sequences are equivalent if their difference converges to 0) is a complete metric space which is of second category and $\mathbb{R} - \mathbb{Q}$ is nonmeager.

That a complete metric space is not meager is a consequence of Baire's Category Theory.

Theorem 16.1 (Baire). A complete metric space is of second category.

Proof. Suppose otherwise that a complete metric space $\{X; d\}$ is of first category.

Then there is a countable collection $\{E_n\}$ of nowhere dense subset of X for which

$$X = \bigcup_{n=1}^{\infty} E_n.$$

For any fixed $x_0 \in X$ consider the open ball $B_{r_0}(x_0)$ for some $r_0 < 1$.

Since E_1 is nowhere dense, the difference $B_{r_0}(x_0) - \overline{E}_1$ contains a nonempty open set, so there $x_1 \in X$ and $r_1 > 0$ such that $B_{r_1}(x_1) \subset B_{r_0}(x_0) - \overline{E}_1$.

Choosing r_1 small enough, say $r_1 < 1/2$, we obtain

$$\overline{B}_{r_1}(x_1) \subset B_{r_0}(x_0) - \overline{E}_1 \subset \overline{B}_{r_0}(x_0).$$

Since E_2 is nowhere dense, the difference $B_{r_1}(x_1) - \overline{E}_2$ contains an open set, so there $x_2 \in X$ and $r_2 > 0$ such that $B_{r_2}(x_2) \subset B_{r_1}(x_1) - \overline{E}_2$.

Choosing r_2 small enough, say $r_2 < 1/3$, we obtain

$$\overline{B}_{r_2}(x_2) \subset B_{r_1}(x_1) - \overline{E}_2 \subset \overline{B}_{r_1}(x_1).$$

Continuing this process gives sequences $\{x_n\}$ of points in X and radii $\{r_n\}$ with $r_n < 1/(n+1)$ such that

$$\overline{B}_{r_{n+1}}(x_{n+1}) \subset \overline{B}_{r_n}(x_n) \text{ for all } n,$$

and

$$\overline{B}_{r_n}(x_n) \bigcap \left(\bigcup_{j=1}^{n} \overline{E}_j \right) = \emptyset \text{ for all } n.$$

The sequence $\{x_n\}$ is Cauchy because $\overline{B}_{r_{n+1}}(x_{n+1}) \subset \overline{B}_{r_n}(x_n)$ for all n and $r_n \to 0$.

Since $\{X; d\}$ is a complete metric space, the Cauchy sequence converges to some $x \in X$.

The limit x belongs to $\overline{B}_{r_n}(x_n)$ for all n, for otherwise it could not be the limit.

Since $\overline{B}_{r_{n+1}}(x_{n+1}) \subset B_{r_n}(x_n) - \overline{E}_{n+1}$ for all n, the limit x does not belong to $\overline{E}_n$ for any n.

Hence

$$x \notin \bigcup_{n=1}^{\infty} \overline{E}_n$$

which implies (because $X \supset \cup\overline{E}_n \supset \cup E_n = X$, and hence $\cup\overline{E}_n = \cup E_n$) that

$$x \notin \bigcup_{n=1}^{\infty} E_n = X.$$

This is a contradiction. □

Corollary 16.2. A complete metric space does not contain nonempty open sets of first category.

Homework problem 3A. Give a proof of Corollary 16.2 (in Ed.1, it is Corollary 16.1 in Ed.2).

Homework problem 3B. Give a proof of Proposition 16.2c (on p.63 in Ed.1; on p.65 in Ed.2).