Chapter 4: Turbulence at Small Scales

Part 2: Consequence of Isotropy

Preliminaries

Statistically stationary: all statistics are invariant under a shift in time.

Statistically homogeneous: all statistics are invariant under a shift in position.

$\overline{\underline{U}} = \langle U_i \rangle =$ constant/uniform and under a shift of reference frame $\overline{\underline{U}} = 0$.

Homogeneous turbulence: $\underline{u}(\underline{x}, t)$ is statistically homogeneous, i.e.,

$$\frac{\partial}{\partial x_j} \overline{fluctuating\ terms} = 0 \rightarrow \frac{\partial \overline{u_i}}{\partial x_j} = \text{ Constant/uniform}$$

Which can be approximated by wind-tunnel experiments.

Homogeneous Turbulence

In homogeneous turbulence, the time-averaged properties of the flow are uniform and independent of position. For example, whereas $\overline{v_x}$, $\overline{v_y}$, *and* $\overline{v_z}$ may differ from each other, each must be constant throughout the system. The same applies to $v'_{x,\text{rms}}$, $v'_{y,\text{rms}}$, and $v'_{z,\text{rms}}$, and to the time-averaged gradients of the fluctuating velocity components, for example, $\overline{\left(\frac{\partial v'_x}{\partial y}\right)^2}$, $\overline{\left(\frac{\partial v'_y}{\partial z}\right)^2}$, and $\overline{\left(\frac{\partial v'_z}{\partial x}\right)^2}$.

Although such a state of motion is not realised readily in experiments, homogeneous turbulence has been given much attention because it greatly simplifies the theoretical treatment of turbulent flow. The assumption of homogeneous turbulence can be justified to a certain extent over small distances somewhat greater than the size of the smallest eddies: at this scale, the mean flow properties are essentially independent of position. However, if turbulence is assumed to be spatially homogeneous it cannot, strictly speaking, also be assumed stationary. From energy balance considerations, a homogeneous turbulent flow field must at the same time be a decaying turbulent flow field; that is, its properties will be changing with time. Fortunately, the rate of decay of the mean flow properties is relatively slow at the smaller scales of turbulence, so that this condition of nonstationarity is not a serious problem in experimental studies that rely on averaging many replicate measurements over time.

Isotropic Tensors

A tensor which has the special property that its components take the same value in all Cartesian coordinate systems is called an *isotropic tensor*. We have already encountered two such tensors: namely, the second-order identity tensor, δ_{ij} , and the third-order permutation tensor, ϵ_{ijk} . Of course, all scalars are isotropic. Moreover, as is easily demonstrated, there are no isotropic vectors (other than the null vector). It turns out that the most general isotropic Cartesian tensors of second-, third-, and fourth-order are $\lambda\,\delta_{ij}$, $\mu\,\epsilon_{ijk}$, and $\alpha\,\delta_{ij}\,\delta_{kl} + \beta\,\delta_{ik}\,\delta_{jl} + \gamma\,\delta_{il}\,\delta_{jk}$, respectively, where λ , μ , α , β , and γ are scalars.

https://farside.ph.utexas.edu/teaching/336L/Fluid/node252.html

Isotropic turbulence: in additions to being homogeneous, also statistically invariant under rotation and reflection of the coordinate system = statistically isotropic.

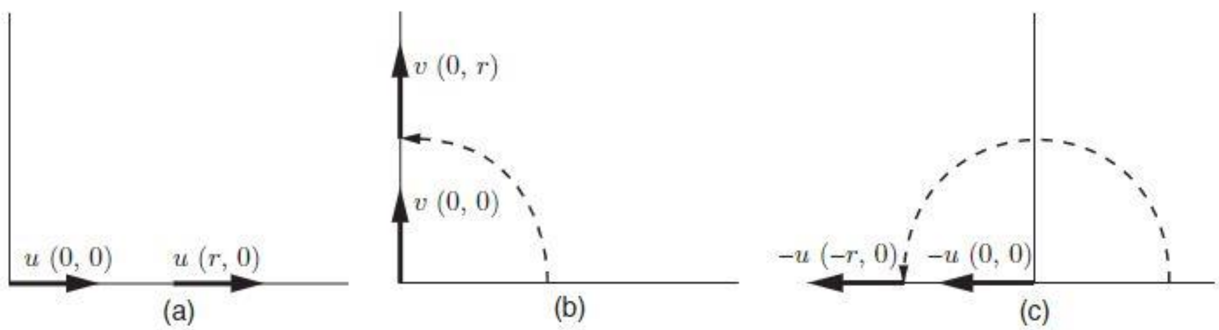


Figure 4.2 Rotational invariance in isotropic turbulence. The two-point correlations based on the velocities in (a), (b), and (c) all yield $f(r)$.

$$\mathcal{R}_{11}(r\hat{e_1}) = \mathcal{R}_{22}(r\hat{e_2})$$

$$\underbrace{\overline{(v(0,0))(v(0,r))}}_{\text{(b)=(a) rotated 90°}} = \underbrace{\overline{(u(0,0))(u(r,0))}}_{\text{(a)}}$$

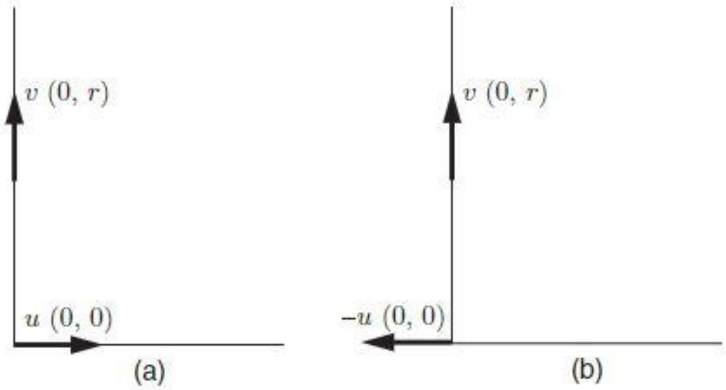
$$\mathcal{R}_{11}(r\hat{e_1}) = \mathcal{R}_{11}(-r\hat{e_1})$$

$$\underbrace{\overline{(-u(0,0))(-u(-r,0))}}_{\text{(c)=(a) rotated 180°}} = \underbrace{\overline{(u(0,0))(u(r,0))}}_{\text{(a)}}$$

$$\mathcal{R}_{11}(r\hat{e_1}) = \mathcal{R}_{11}(-r\hat{e_1}) \Rightarrow f(r) = f(-r), \quad g(r) = g(-r)$$

Also:

Figure 4.3 Antisymmetry of $\mathcal{R}_{12}(r\mathbf{e}_2)$ under reflection. The two-point correlations based on the velocity components in (a) and (b) in isotropic turbulence are equal.



$$\underbrace{\overline{(u(0,0))(v(0,r))}}_{\text{(a)}} = \underbrace{\overline{(-u(0,0))(v(0,r))}}_{\text{(b)}}$$

Since (b) represents (a) reflected by 180° with respect to the z axis and (a) and (b) must be equal in isotropic turbulence, then

$$(a) = (b) = -(a)$$

Which can only be true if (a) and (b) are equal to zero.

Therefore,

$$\mathcal{R}_{12}(r\widehat{e_2}) = -\mathcal{R}_{12}(r\widehat{e_2}) = 0$$

Also, in isotropic turbulence

$$\mathcal{R}_{ij} = 0 \quad i \neq j$$

Rules used for the derivation of the equations in Chapters 4 and 5 in Turbulent Fluid Flow (P. Bernard, 2019)

<u>Tensors and vectors</u>

$R_{ij}(\mathbf{r}),\ S_{ij,k}(\mathbf{r})$

$u_i(\mathbf{x}), u_i(\mathbf{y})$

$\mathbf{r} = \mathbf{y} - \mathbf{x}, \mathbf{r_j} = \mathbf{y_j} - \mathbf{x_j}$

<u>Scalers</u>

$f(r),\ g(r), h(r),\ k(\mathrm{r}),\ \mathrm{q(r)}$

$\mathrm{r}^2 = \mathrm{r_1}^2 + \mathrm{r_3}^2 + \mathrm{r_3}^2, r = |r|, \mathbf{r^2} = \mathbf{r_j}^2, k^2 = \mathbf{k_l}^2, k = |\mathbf{k}|$

<u>Rules</u>

$$\frac{\partial \mathrm{f}(r)}{\partial r_l} = \frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial r_l} \qquad \frac{\partial \mathrm{R_{ij}}(\mathbf{r})}{\partial \mathrm{x}_j} = \frac{\partial \mathrm{R_{ij}}(\mathbf{r})}{\partial \mathrm{r}_l}\frac{\partial r_l}{\partial x_j}$$

$$\frac{\partial r_i}{\partial r_j} = \delta_{ij} \qquad \frac{\partial r}{\partial r_l} = \frac{r_l}{r} \qquad \frac{\partial r_l^2}{\partial r_l} = r_l \qquad \frac{\partial r_l k_l}{\partial r_l} = k_l \qquad \frac{\partial r_j}{\partial r_j} = 3 \qquad \frac{\partial r_j}{\partial x_j} = -1 \qquad \frac{\partial r_j}{\partial y_j} = 1$$

$$\frac{\partial \delta_{ij}}{\partial r_j} = 0$$

$$\frac{\partial}{\partial x_i}\overline{u_j^n} = 0, \quad \overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}, \quad \text{and} \quad \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^n} = \overline{\left(\frac{\partial u_2}{\partial x_2}\right)^n} = \overline{\left(\frac{\partial u_3}{\partial x_3}\right)^n}, \tag{12.36}$$

but relative directions must be respected:

$$\overline{\left(\frac{\partial u_1}{\partial x_2}\right)^n} = \overline{\left(\frac{\partial u_1}{\partial x_3}\right)^n} = \overline{\left(\frac{\partial u_2}{\partial x_1}\right)^n} = \overline{\left(\frac{\partial u_2}{\partial x_3}\right)^n} = \overline{\left(\frac{\partial u_3}{\partial x_1}\right)^n} = \overline{\left(\frac{\partial u_3}{\partial x_2}\right)^n}. \tag{12.37}$$

Note that the continuity equation requires derivative moments in the third set of equalities of (12.36) to be zero when $n = 1$.

Also,

$\overline{u_i u_j} = 0$ for $i \neq j$

Consequence of Isotropy

$$\mathcal{R}_{ij}(\underline{r},t) = \overline{u_i(\underline{x},t)u_j(\underline{x}+\underline{r},t)}$$

$$S_{ijk}(\underline{r},t) = \overline{u_i(\underline{x},t)u_j(\underline{x},t)u_k(\underline{x}+\underline{r},t)}$$

$\underline{r} = \underline{y} - \underline{x}$

Two-point correlation tensors take on special forms for isotropic turbulence, which facilitates simplified analysis of turbulent physics.

$\mathcal{R}_{ij} \neq f(\underline{x})$ i.e., turbulence has no preferred direction

$$\mathcal{R}_{ij}(0,t) = \overline{u_i u_j} = \overline{u^2}\delta_{ij} \;\Rightarrow\; \overline{u_i u_j} = 0 \quad i \neq j$$

Recall that

$$\overline{u^2} f(r) = \mathcal{R}_{11}(r\hat{e_1}) \quad f(0) = 1$$

$$\overline{v^2} g(r) = \mathcal{R}_{22}(r\hat{e_1}) \quad g(0) = 1$$

$$\overline{u_1{}^2} = \overline{u_2{}^2} = \overline{u_3{}^2} = \overline{u^2} = constant$$

And note that

$$\mathcal{R}_{33} = \mathcal{R}_{22}$$

Proof in Appendix A.1

and

$$\mathcal{R}_{ij} = 0 \text{ for } i \neq j$$

As already referenced, formal mathematical theory isotropic tensor provides the general form that $\mathcal{R}_{ij}$ (Pope, 2000) and S_{ijl} (Robertson, 1940) must take.

Pope: To within scalar multiples, the only second-order tensors that can be formed from the vector $\underline{r}$ are δ_{ij} and $r_i r_j$. Consequently, $\mathcal{R}_{ij}$ can be written as

$$\mathcal{R}_{ij} = \overline{u^2}\left[R_1(r) r_i r_j + R_2(r)\delta_{ij}\right] \quad (1)$$

Robertson, H. (1940). *The invariant theory of isotropic turbulence. Mathematical Proceedings of the Cambridge Philosophical Society*, 36(2), 209-223:

$$S_{ijl} = S_1(r) r_i r_j r_l + S_2(r) r_l \delta_{ij} + S_3(r)\left(r_j \delta_{il} + r_i \delta_{jl}\right) \quad (2)$$

Where R_1, R_2, S_1, S_2, S_3 are scalar functions of r.

$f(r)$ and $g(r)$ can be expressed in terms of R_1 and R_2 using their definitions, as follows.

$$f(r) = \frac{\overline{u(x)u(x+r)}}{\overline{u^2(x)}}, \quad g(r) = \frac{\overline{v(x)v(x+r)}}{\overline{v^2(x)}}$$

With Eq. (1), it is possible to show that the two functions assume the following form

$$f(r) = R_1 r^2 + R_2$$
$$g(r) = R_2$$

Proof in Appendix A.2

i.e.,

$$R_1 = \frac{f(r) - g(r)}{r^2}$$

$$R_2 = g(r)$$

Substituting these expressions in Eq. (1) gives

$$\mathcal{R}_{ij}(\underline{r}) = \overline{u^2}\left[(f - g)\frac{r_i r_j}{r} + g\delta_{ij}\right] \quad (3)$$

For S_{ijl} define scalar correlation functions $k(r), h(r), q(r)$, which can be expressed in terms of S_1, S_2, S_3

$$S_{111}(r\hat{e_1}) = u_{rms}^3 k(r) \quad (4)$$

$$S_{221}(r\hat{e_1}) = u_{rms}^3 h(r) \quad (5)$$

$$S_{212}(r\hat{e_1}) = u_{rms}^3 q(r) \quad (6)$$

Where $u_{rms} = \sqrt{\langle u^2 \rangle}$.

Combining Eq. (2) with Eq. (4), (5), (6), the system of equations is obtained

$$\begin{cases} u_{rms}^3 k = S_1 r^3 + S_2 r + 2rS_3 \\ u_{rms}^3 h = S_2 r \\ u_{rms}^3 q = S_3 r \end{cases}$$

Proof in Appendix A.3

Solving this system for S_1, S_2, S_3 and substituting the results into Eq. (2) yields

$$S_{ijl}(\underline{r}) = u_{rms}^3\left[(k - h - 2q)\frac{r_i r_j r_l}{r^3} + \delta_{ij} h \frac{r_l}{r} + q\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\right] \quad (7)$$

Proof in Appendix A.4

Note $\mathcal{R}_{ij}(0) = \overline{u^2}\delta_{ij}$ provides scaling factor for $\mathcal{R}_{ij}$, whereas $S_{ijl}(0) = 0$ such that need select scaling factor for $S_{ijl}(\underline{r})$ for which $u_{rms} = \sqrt{\langle u^2 \rangle}$ is used.

Note that $k(r) = -k(-r)$, i.e., anti-symmetric

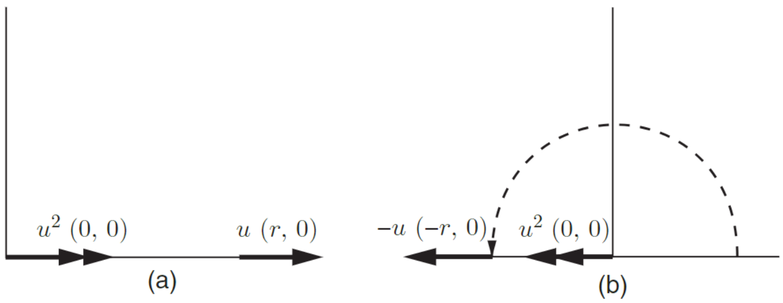


Figure 4.5 Antisymmetry of the two-point longitudinal triple velocity correlation. The correlations in (a) $k(r)$ and (b) $-k(-r)$ are equal.

$$\overline{u(0,0)^2 u(r,0)} = \overline{(-u(0,0))^2(-u(-r,0))} = -\overline{(u(0,0))^2(u(-r,0))}$$

whereas $f(r) = f(-r)$ is symmetric as is $g(r)$ (see Page 3). Since k(r) is antisymmetric, $k(0) = 0$ as are all its even derivatives at $r = 0$

Note that

$$S_{ijl}(\underline{r}, t) = \overline{u_i(\underline{x}, t) u_j(\underline{x}, t) u_l(\underline{x} + \underline{r}, t)}$$

Dropping the time dependence

$$S_{ijl}(\underline{r}) = \overline{u_i(\underline{x}) u_j(\underline{x}) u_l(\underline{x} + \underline{r})}$$

Therefore

$$S_{111}(\underline{r}) = \overline{u_1(\underline{x}) u_1(\underline{x}) u_1(\underline{x} + \underline{r})}$$

And if $\underline{r} = \{r, 0,0\}$,

$$S_{111}(r\hat{e}_1) = \overline{u_1(x)u_1(x)u_1(x+r)}$$

Combining this result with Eq. (4), yields

$$u_{rms}^3 k(r) = \overline{{u_1}^2(x)u_1(x+r)}$$

then

$$u_{rms}^3 \frac{dk}{dr}(0) = \overline{{u_1}^2(x)\frac{\partial u_1}{\partial x}} = \frac{1}{3}\frac{\partial \overline{{u_1}^3}}{\partial x} = 0$$

Thus, $\frac{dk}{dr}(0) = 0$ and $k(r)$ Taylor series leading term $\sim r^3 \frac{d^3k}{dr^3}(0)$ for small r.

$\mathcal{R}_{ij}$ and S_{ijl} can be simplified for incompressible flow using $\nabla \cdot \underline{u} = 0$.

$$\frac{\partial \mathcal{R}_{ij}}{\partial r_j} = \overline{u_i(\underline{x})\frac{\partial u_j}{\partial y_j}(\underline{y})} = 0$$

Proof in Appendix A.5

Evaluating $\mathcal{R}_{ij,j}$ and using the relations $\frac{\partial}{\partial r_j}(r) = \frac{r_j}{r}$ and $\frac{\partial}{\partial r_j}(r_i) = \delta_{ij}$ shows that

$$g = f + \frac{r}{2}\frac{df}{dr} \quad (8)$$

Proof in Appendix A.6

Combining Eq. (8) and Eq. (3) yields

$$\mathcal{R}_{ij}(\underline{r}) = \overline{u^2}\left[\left(f - f - \frac{r}{2}\frac{df}{dr}\right)\frac{r_i r_j}{r} + \left(f + \frac{r}{2}\frac{df}{dr}\right)\delta_{ij}\right]$$

$$\mathcal{R}_{ij}(\underline{r}) = \overline{u^2}\left[\left(f + \frac{r}{2}\frac{df}{dr}\right)\delta_{ij} - \frac{r_i r_j}{r^2}\frac{r}{2}\frac{df}{dr}\right]$$

Which shows that $\mathcal{R}_{ij}$ only depends on the scalar function $f(r)$.

Similarly, using the continuity equation on S_{ijl}

$$\frac{\partial S_{ijl}}{\partial r_j}(\underline{r}) = 0 \quad (9)$$

Proof in Appendix A.7

And combining Eq. (9) and Eq. (7), after a long number of calculations, yields

$$q = \frac{1}{4r}\frac{d(kr^2)}{dr}$$

$$h = -\frac{k}{2}$$

Proof in Appendix A.8

Such that Eq. (7) becomes

$$S_{ijl}(\underline{r}) = u_{rms}^3\left[\left(k - r\frac{dk}{dr}\right)\frac{r_i r_j r_l}{2r^3} - \frac{k}{2}\delta_{ij}\frac{r_l}{r} + \frac{1}{4r}\frac{d(kr^2)}{dr}\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\right]$$

And S_{ijl} depends only on the scalar function $k(r)$.

Confirmation

$$g = f + \frac{r}{2}\frac{df}{dr} \quad (8)$$

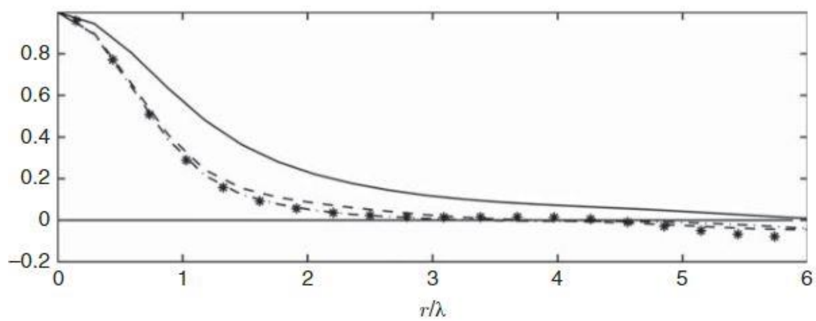


Figure 4.6 Confirmation of the isotropic identity Eq. (4.33) from a numerical simulation of isotropic turbulence using a vortex filament scheme [6]. $f(r)$, —; $g(r)$ based on v velocity, −−; $g(r)$ based on w velocity, − · −; ∗, evaluation of Eq. (4.33). Used with permission. Copyright (2006) National Academy of Sciences, USA.

B5.3.2: Similarity solution final period isotropic decay: $f(r,t) = e^{-\frac{r^2}{2\lambda_g{}^2}}$

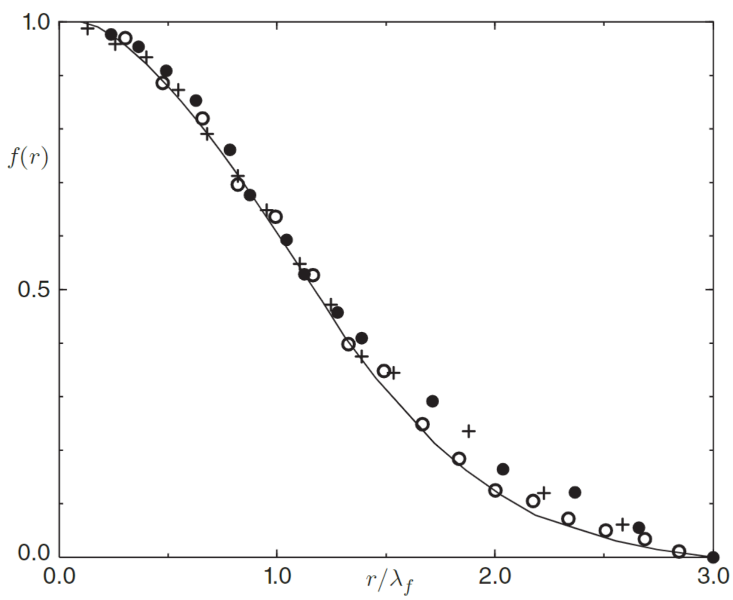


Figure 5.1 Measured and predicted $f(r/\lambda_f)$ in the final period [2]. With permission of the Royal Society.

Appendix A

A.1

Definition of longitudinal and transverse coefficients

$$f(r) = \frac{\overline{u(x)u(x+r)}}{\overline{u^2(x)}}, \quad g(r) = \frac{\overline{v(x)v(x+r)}}{\overline{v^2(x)}}$$

Definition of two-point correlation tensor according to isotropic tensor theory

$$\mathcal{R}_{ij}(\underline{r}) = \overline{u^2}\left[R_1(r)r_ir_j + R_2(r)\delta_{ij}\right] \quad (1A)$$

Therefore

$$\overline{u^2}f(r) = \mathcal{R}_{11}(r\widehat{e_1})$$

$$\overline{v^2}g(r) = \mathcal{R}_{22}(r\widehat{e_1})$$

We can show that $\mathcal{R}_{22}(r\widehat{e_1}) = \mathcal{R}_{33}(r\widehat{e_1})$ using Eq. (1A) for $i,j = 2$ and $i,j = 3$

$$\mathcal{R}_{22}(r\widehat{e_1}) = \overline{u^2}[R_1(r)r_2r_2 + R_2(r)\delta_{22}]$$

$$\mathcal{R}_{33}(r\widehat{e_1}) = \overline{u^2}[R_1(r)r_3r_3 + R_2(r)\delta_{33}]$$

But $\underline{r} = \{r, 0,0\}$, therefore

$$\mathcal{R}_{22}(r\widehat{e_1}) = \overline{u^2}[R_1(r)\cdot 0\cdot 0 + R_2(r)] = \overline{u^2}R_2(r)$$

$$\mathcal{R}_{33}(r\widehat{e_1}) = \overline{u^2}[R_1(r)\cdot 0\cdot 0 + R_2(r)] = \overline{u^2}R_2(r)$$

Which proves that $\mathcal{R}_{22}(r\widehat{e_1}) = \mathcal{R}_{33}(r\widehat{e_1})$.

A.2 (Kundu et al. Ex. 12.18)

Combining the definitions of longitudinal and transverse coefficients with Eq. (1A)

$$\overline{u^2}f(r) = \overline{u^2}[R_1(r)r_1r_1 + R_2(r)\delta_{11}]$$

$$\overline{v^2}g(r) = \overline{u^2}[R_1(r)r_2r_2 + R_2(r)\delta_{22}]$$

Where $r_1 = r$ and $r_2 = 0$

$$\overline{u^2}f(r) = \overline{u^2}[R_1(r)r^2 + R_2(r)]$$

$$\overline{v^2}g(r) = \overline{u^2}[R_2(r)]$$

In isotropic turbulence $\overline{u^2} = \overline{v^2} = \overline{w^2}$

$$f(r) = R_1(r)r^2 + R_2(r)$$

$$g(r) = R_2(r)$$

A.3

$$S_{ijl} = S_1(r)r_ir_jr_l + S_2(r)r_l\delta_{ij} + S_3(r)\left(r_j\delta_{il} + r_i\delta_{jl}\right) \quad (2A)$$

For S_{ijl} define scalar correlation functions $k(r), h(r), q(r)$

$$S_{111}(r\widehat{e_1}) = u_{rms}^3 k(r)$$

$$S_{221}(r\widehat{e_1}) = u_{rms}^3 h(r)$$

$$S_{212}(r\widehat{e_1}) = u_{rms}^3 q(r)$$

Substitute into Eq. (2A)

$$S_{111} = S_1r_1r_1r_1 + S_2r_1\delta_{11} + S_3(r_1\delta_{11} + r_1\delta_{11})$$

$$S_{221} = S_1r_2r_2r_1 + S_2r_1\delta_{22} + S_3(r_2\delta_{21} + r_2\delta_{21})$$

$$S_{212} = S_1r_2r_1r_2 + S_2r_2\delta_{21} + S_3(r_1\delta_{22} + r_2\delta_{12})$$

Where $\underline{r} = \{r, 0{,}0\}$, therefore

$$S_{111} = u_{rms}^3 k(r) = S_1r^3 + S_2r + 2S_3r$$

$$S_{221} = u_{rms}^3 h(r) = S_2r$$

$$S_{212} = u_{rms}^3 q(r) = S_3r$$

A.4

$$S_{ijl} = S_1(r)r_ir_jr_l + S_2(r)r_l\delta_{ij} + S_3(r)\left(r_j\delta_{il} + r_i\delta_{jl}\right) \quad (3A)$$

$$\begin{cases} u_{rms}^3 k = S_1 r^3 + S_2 r + 2rS_3 \\ u_{rms}^3 h = S_2 r \\ u_{rms}^3 q = S_3 r \end{cases}$$

Solve for S_1, S_2, S_3:

$$\begin{cases} S_2 = \dfrac{u_{rms}^3 h}{r} \\ S_3 = \dfrac{u_{rms}^3 q}{r} \\ S_1 = \dfrac{u_{rms}^3 k - S_2 r - 2rS_3}{r^3} \end{cases}$$

$$\begin{cases} S_2 = \dfrac{u_{rms}^3 h}{r} \\ S_3 = \dfrac{u_{rms}^3 q}{r} \\ S_1 = \dfrac{u_{rms}^3 k - u_{rms}^3 h - 2u_{rms}^3 q}{r^3} = \dfrac{u_{rms}^3}{r^3}(k - h - 2q) \end{cases}$$

Substitute S_1, S_2, S_3 in Eq. (3A)

$$S_{ijl} = \frac{u_{rms}^3}{r^3}(k - h - 2q)r_ir_jr_l + \frac{u_{rms}^3 h}{r} r_l\delta_{ij} + \frac{u_{rms}^3 q}{r}\left(r_j\delta_{il} + r_i\delta_{jl}\right)$$

$$S_{ijl}(\underline{r}) = u_{rms}^3\left[(k - h - 2q)\frac{r_ir_jr_l}{r^3} + \delta_{ij}h\frac{r_l}{r} + q\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\right]$$

A.5 (Pope Ex. 3.35)

$$\frac{\partial \mathcal{R}_{ij}}{\partial r_j} = \frac{\partial}{\partial r_j} \overline{u_i(\underline{x}) u_j(\underline{x} + \underline{r})}$$

$$\frac{\partial \mathcal{R}_{ij}}{\partial r_j} = \overline{u_i(\underline{x}) \frac{\partial u_j(\underline{x} + \underline{r})}{\partial r_j}} \qquad (4A)$$

Define

$$x_j{}' = x_j + r_j$$

Therefore

$$\frac{\partial x_j{}'}{\partial r_j} = \cancel{\frac{\partial x_j}{\partial r_j}} + \frac{\partial r_j}{\partial r_j} = 1$$

Substitute into Eq. (4A)

$$\frac{\partial \mathcal{R}_{ij}}{\partial r_j} = \overline{u_i(\underline{x}) \underbrace{\frac{\partial u_j(x_j{}')}{\partial x_j{}'}}_{\nabla \cdot \underline{u}} \underbrace{\frac{\partial x_j{}'}{\partial r_j}}_{1}} = 0$$

A.6 (Kundu et al. Ex. 12.19, Pope Ex. 6.4, Bernard Prob. 4.3)

$$\frac{\partial \mathcal{R}_{ij}}{\partial r_j} = \overline{u^2}\frac{\partial}{\partial r_j}\left[(f-g)\frac{r_i r_j}{r^2} + g\delta_{ij}\right] = 0$$

$$= \overline{u^2}\left[(f-g)\frac{\partial}{\partial r_j}\left(\frac{r_i r_j}{r^2}\right) + \left(\frac{r_i r_j}{r^2}\right)\frac{\partial}{\partial r_j}(f-g) + g\frac{\partial \delta_{ij}}{\partial r_j} + \frac{\partial g}{\partial r_j}\delta_{ij}\right] = 0$$

Using the identities

$$\frac{\partial}{\partial r_j}(r) = \frac{r_j}{r}$$

And

$$\frac{\partial}{\partial r_j}(r_i) = \delta_{ij}$$

We obtain

$$(f-g)\left[r_i r_j\frac{\partial}{\partial r_j}\left(\frac{1}{r^2}\right) + \frac{r_i}{r^2}\frac{\partial r_j}{\partial r_j} + \frac{r_j}{r^2}\frac{\partial r_i}{\partial r_j}\right] + \left(\frac{r_i r_j}{r^2}\right)\frac{\partial}{\partial r}(f-g)\frac{\partial r}{\partial r_j} + \frac{\partial g}{\partial r}\frac{\partial r}{\partial r_j}\delta_{ij} = 0$$

$$(f-g)\left[-2r_i r_j\frac{1}{r^3}\frac{\partial}{\partial r_j}(r) + \frac{r_i}{r^2}\delta_{jj} + \frac{r_j}{r^2}\delta_{ij}\right] + + \left(\frac{r_i r_j}{r^2}\right)\frac{r_j}{r}(f'-g') + g'\delta_{ij}\frac{r_j}{r} = 0$$

$$(f-g)\left[-2\frac{r_i r_j r_j}{r^4} + 3\frac{r_i}{r^2} + \frac{r_i}{r^2}\right] + \frac{r_i}{r}(f'-g') + \frac{r_i}{r}g' = 0$$

$$(f-g)\left[-2\frac{r_i}{r^2} + 4\frac{r_i}{r^2}\right] + \frac{r_i}{r}(f') = 0$$

$$2(f-g)\frac{r_i}{r^2} + \frac{r_i}{r}f' = 0$$

$$f - g + \frac{r}{2}f' = 0$$

$$g = f + \frac{r}{2}f'$$

A.7

$$S_{ijl}(\underline{r},t) = \overline{u_i(\underline{x},t)u_j(\underline{x},t)u_l(\underline{x}+\underline{r},t)}$$

$$\frac{\partial S_{ijl}}{\partial r_l} = \frac{\partial}{\partial r_l}\overline{u_i(\underline{x})u_j(\underline{x})u_l(\underline{x}+\underline{r})} \quad (5A)$$

Define

$$x_l{}' = x_l + r_l$$

Therefore

$$\frac{\partial x_l{}'}{\partial r_l} = \cancel{\frac{\partial x_l}{\partial r_l}} + \frac{\partial r_l}{\partial r_l} = 1$$

Substitute into Eq. (5A)

$$\frac{\partial S_{ijl}}{\partial r_l} = \overline{u_i(x_j)u_j(x_j)\underbrace{\frac{\partial u_l(x_j{}')}{\partial x_l{}'}}_{\nabla\cdot\underline{u}}\underbrace{\frac{\partial x_l{}'}{\partial r_l}}_{1}} = 0$$

A.8

P53

Eq.25 → Eq.35 → Eq.4.30. 4.37.

$$\frac{\partial S_{ijl}}{\partial r_l} = 0 = \frac{\partial}{\partial r_l}\left[(k-h-2q)\frac{r_i r_j r_l}{r^3} + \delta_{ij} h \frac{r_l}{r} + q\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\right]$$

$$\Rightarrow \frac{\partial}{\partial r_l}\left[(k-h-2q)\frac{r_i r_j r_l}{r^3}\right] + \frac{\partial}{\partial r_l}\left[\delta_{ij} h \frac{r_l}{r}\right] + \frac{\partial}{\partial r_l}\left[q\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\right] = 0$$

$$\Rightarrow \underbrace{\frac{r_i r_j r_l}{r^3}\frac{\partial}{\partial r_l}(k-h-2q) + (k-h-2q)\frac{\partial}{\partial r_l}\left(\frac{r_i r_j r_l}{r^3}\right)}_{1} +$$

$$\underbrace{\delta_{ij}\frac{\partial}{\partial r_l}\left(h\frac{r_l}{r}\right) + \frac{h r_l}{r}\frac{\partial}{\partial r_l}(\delta_{ij})}_{2} + \underbrace{\left(\delta_{il}\frac{r_j}{r} + \delta_{jl}\frac{r_i}{r}\right)\frac{\partial q}{\partial r}}_{3} + \underbrace{q\frac{\partial}{\partial r_l}\left(\frac{\delta_{il} r_j}{r}\right)}_{4} + \underbrace{q\frac{\partial}{\partial r_l}\left(\frac{\delta_{jl} r_i}{r}\right)}_{5}$$

① $$\frac{r_i r_j r_l}{r^3}\frac{\partial}{\partial r}(k-h-2q)\frac{\partial r}{\partial r_l} + (k-h-2q)\left[\frac{1}{r^3}\frac{\partial (r_i r_j r_l)}{\partial r_l} + r_i r_j r_l \frac{\partial}{\partial r_l}\left(\frac{1}{r^3}\right)\right] \quad \left(\frac{\partial r}{\partial r_l} = \frac{r_l}{r}\right)$$

$$= \frac{r_i r_j r_l}{r^3}\frac{r_l}{r}(k'-h'-2q') + (k-h-2q)\left[\frac{r_i r_j}{r^3}\frac{\partial r_l}{\partial r_l} + \frac{r_i r_l}{r^3}\frac{\partial r_j}{\partial r_l} + \frac{r_j r_l}{r^3}\frac{\partial r_i}{\partial r_l} - \frac{3 r_i r_j r_l}{r^4}\frac{\partial r}{\partial r_l}\right]$$

$$= \frac{r_i r_j}{r^2}(k'-h'-2q') + (k-h-2q)\left[\frac{3 r_i r_j}{r^3} - \frac{3 r_i r_j r_l r_l}{r^5} + \frac{r_i r_l}{r^3}\delta_{jl} + \frac{r_j r_l}{r^3}\delta_{il}\right]$$

$$= \frac{r_i r_j}{r^2}(k'-h'-2q') + (k-h-2q)\left(\frac{2 r_i r_j}{r^3}\right)$$

② $$\delta_{ij}\left[h\frac{\partial}{\partial r_l}\left(\frac{r_l}{r}\right) + \frac{r_l}{r}\frac{\partial h}{\partial r}\frac{\partial r}{\partial r_l} + 0\right] = \delta_{ij}\left[\frac{h}{r}\frac{\partial r_l}{\partial r_l} + h r_l\frac{\partial}{\partial r_l}\left(\frac{1}{r}\right) + \frac{r_l}{r}\frac{r_l}{r}h'\right]$$

$$= \delta_{ij}\left[\frac{3h}{r} + h' - \frac{h r_l}{r^2}\frac{\partial r}{\partial r_l}\right] = \left[\frac{3h}{r} + h' - \frac{h r_l}{r^2}\frac{r_l}{r}\right]$$

$$= \delta_{ij}\left[\frac{3h}{r} + h' - \frac{h}{r}\right] = \delta_{ij}\left[h' + \frac{2h}{r}\right]$$

③ $\left(\delta_{il}\frac{r_j}{r}+\delta_{jl}\frac{r_i}{r}\right)\frac{\partial q}{\partial r}\frac{\partial r}{\partial r_l}$

$$=\left(\delta_{il}\frac{r_j}{r}+\delta_{jl}\frac{r_i}{r}\right)q'\frac{r_l}{r}=\frac{q'}{r^2}\left(\delta_{il}r_jr_l+\delta_{jl}r_ir_l\right)$$

$$=\frac{2q'}{r^2}r_ir_j$$

④ $q\frac{\partial}{\partial r_l}\left(\delta_{il}\frac{r_j}{r}\right)=q\left[\frac{r_j}{r}\frac{\partial}{\partial r_l}(\delta_{il})+\delta_{il}\frac{1}{r}\frac{\partial r_j}{\partial r_l}+\delta_{il}r_j\frac{\partial\left(\frac{1}{r}\right)}{\partial r_l}\right]$

$$=q\left[0+\delta_{il}\frac{1}{r}\delta_{jl}+\delta_{il}r_j\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial r_l}\right]$$

$$=q\left[\delta_{il}\delta_{jl}\frac{1}{r}-\frac{1}{r^2}\delta_{il}r_jr_l\frac{1}{r}\right]=q\left(\frac{1}{r}\delta_{ij}-\frac{\delta_{il}r_jr_l}{r^3}\right)$$

$$=q\left(\frac{1}{r}\delta_{ij}-\frac{r_ir_j}{r^3}\right)$$

⑤ $q\frac{\partial}{\partial r_l}\left(\delta_{jl}\frac{r_i}{r}\right)=q\left[\delta_{jl}\frac{\partial}{\partial r_l}\left(\frac{r_i}{r}\right)+\frac{r_i}{r}\frac{\partial}{\partial r_l}(\delta_{jl})\right]$

$$=q\left[\delta_{jl}\frac{1}{r}\frac{\partial r_i}{\partial r_l}+\delta_{jl}r_i\frac{\partial}{\partial r_l}\left(\frac{1}{r}\right)+0\right]$$

$$=q\left[\frac{1}{r}\delta_{jl}\delta_{il}+\left(-\frac{\delta_{jl}r_i}{r^2}\frac{\partial r}{\partial r_l}\right)\right]=q\left[\frac{1}{r}\delta_{ij}-\frac{\delta_{jl}r_i}{r^2}\frac{r_l}{r}\right]$$

$$=q\left[\frac{1}{r}\delta_{ij}-\frac{r_ir_j}{r^3}\right]$$

①+②+③+④+⑤

$$=(k'-h'-2q')\frac{r_ir_j}{r^2}+(k-h-2q)\frac{2r_ir_j}{r^3}+\delta_{ij}\left(h'+\frac{2h}{r}\right)+\frac{2q'}{r^2}r_ir_j$$
$$+2q\left(\frac{1}{r}\delta_{ij}-\frac{r_ir_j}{r^3}\right)=0$$

$$\Rightarrow r_ir_j\left(\frac{k'-h'-2q'}{r^2}+\frac{2k-2h-4q}{r^3}+\frac{2q'}{r^2}-\frac{2q}{r^3}\right)+\delta_{ij}\left(h'+\frac{2h}{r}+\frac{2q}{r}\right)=0$$

$$\Rightarrow \begin{cases} h' + \frac{2h}{r} + \frac{2q}{r} = 0 \\ \frac{k'-h'}{r^2} - \frac{2q}{r^3} + \frac{2k-2h-4q}{r^3} = 0 \end{cases} \Rightarrow$$

$$\begin{cases} rh' + 2h + 2q = 0 \\ rk' - rh' - 2q + 2k - 2h - 4q \end{cases} \Rightarrow \begin{cases} 2q + rh' + 2h = 0 \\ rk' - 6q - (rh' + 2h) = 0 \\ +2k \end{cases} \Rightarrow$$

$$rk' - 4q + 2k = 0 \Rightarrow 4q = rk' + 2k \Rightarrow q = \frac{1}{4} rk' + \frac{1}{2} k$$

$$\boxed{\Rightarrow q = \frac{1}{4r} \frac{d(kr^2)}{dr}}$$

$$r(k' + 2h') = -(2k + 4h)$$

$$rk' + 2k + (2h'r + 4h) = 0$$

$$r^2k' + 2rk + r^2 2h' + 4rh = 0$$

$$\frac{d(r^2k)}{dr} + \frac{d(2r^2h)}{dr} = 0$$

$$\frac{d(r^2k + 2r^2h)}{dr} = 0$$

$$r^2k + 2r^2h = \text{Constant}$$

$$k + 2h = \frac{\text{Constant}}{r^2}$$

$$r = 0,\ k(0) = 0$$

$$\text{Constant} = 0$$

$$\boxed{k = -2h \Rightarrow h = -\frac{k}{2}}$$