Chapter 6: Turbulent Transport and its Modeling

Part 3: Homogeneous Shear Flow

(1) Pope 5.4.5; also recall discussion Chapter 4 Part 7, shear-stress spectrum)

In homogeneous turbulence[1] $\underline{u}(\underline{x}, t)$ and $p'(\underline{x}, t)$ are statistically homogeneous and $\overline{U_{i,j}}$ must be uniform, although it may be $f(t)$ (Pope Ex. 5.41). In homogeneous shear flow $S = \overline{U_{i,j}}$ = constant, which can be realized in wind tunnel experiments by using screens controlling the inflow velocity profile.

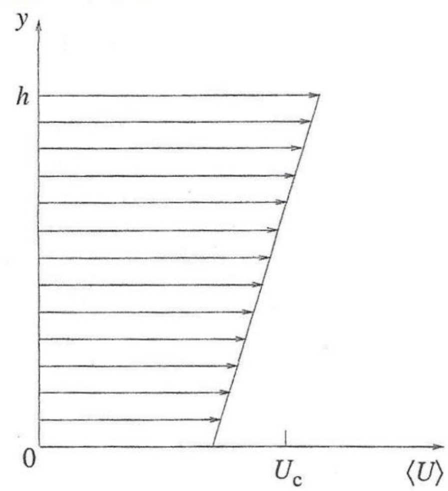

Fig. 5.30. A sketch of the mean velocity profile in homogeneous shear flow.

$$\underline{\overline{U}} = f(y)$$

$$\underline{\overline{V}} = \underline{\overline{W}} = 0$$

$$S = \frac{\partial \overline{U}}{\partial y} = \text{constant}$$

At x/h = 0, the RS are nearly uniform normal to the flow direction, which persists downstream. However, they show increasing values in the axial direction, which can be removed using a refence frame moving with the mean velocity $\overline{U}$ such that the turbulence is approximately homogeneous, as per Fig. 5.31.

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[1] Homogeneous turbulence: the time-averaged properties of the flow are uniform and independent of position → invariant under translation, i.e., shift in the origin of the coordinate system. For example, whereas $\overline{u^2}, \overline{v^2}, \overline{w^2}$ may differ from each other, each must be constant throughout the system. The time-averaged gradients of the fluctuating components, i.e., $\overline{\left(\frac{\partial u}{\partial y}\right)^2}$ are equal to zero.

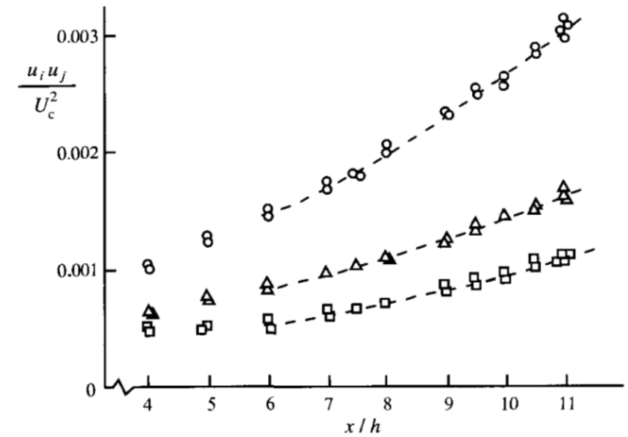

Fig. 5.31. Reynolds stresses against axial distance in the homogeneous-shear-flow experiment of Tavoularis and Corrsin (1981): ○, $\langle u^2 \rangle$; □, $\langle v^2 \rangle$; △, $\langle w^2 \rangle$.

Despite axial variation, in frame of reference moving at U_c,

$$\overline{u_i u_j} \approx \begin{cases} \text{const} & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

An important conclusion is that after the initial development time the flow becomes self-similar when statistics are normalized by S and k, as shown in Table below.

Table 5.4. *Statistics in homogeneous turbulent shear flow from the experiments of Tavoularis and Corrsin (1981) and the DNS of Rogers and Moin (1987)*

	Tavoularis and Corrsin		Rogers and Moin
	$x/h = 7.5$	$x/h = 11.0$	$\mathcal{S}t = 8.0$
$\langle u^2 \rangle / k$	1.04	1.07	1.06
$\langle v^2 \rangle / k$	0.37	0.37	0.32
$\langle w^2 \rangle / k$	0.58	0.56	0.62
$-\langle uv \rangle / k$	0.28	0.28	0.33
$-\rho_{uv}$	0.45	0.45	0.57
$\mathcal{S}k/\varepsilon$	6.5	6.1	4.3
$\mathcal{P}/\varepsilon$	1.8	1.7	1.4
$L_{11}\mathcal{S}/k^{1/2}$	4.0	4.0	3.7
$L_{11}/(k^{3/2}/\varepsilon)$	0.62	0.66	0.86

Between $x/h = 7.5$ and 11, $k(t)$ increases by 65%, yet normalized Reynolds stresses nearly constant.

$\tau = k/\varepsilon =$ turbulent time scale nearly constant such that $Sk/\varepsilon \sim$ constant.

L_{11} increases by 30%, but when normalized nearly constant.

The TKE equation for homogeneous shear flow is (Pope Ex. 5.40):

$$\frac{dk}{dt} = P - \varepsilon$$

Such that

$$\frac{\tau}{k}\frac{dk}{dt} = \frac{P}{\varepsilon} - 1$$

Which has solution:

$$k(t) = k(0)\exp\left[\frac{t}{\tau}\left(\frac{P}{\varepsilon} - 1\right)\right]$$

Since $P/\varepsilon \approx 1.7$, $k(t)$ grows exponentially in time and both ε and $L = k^{3/2}/\varepsilon = k^{1/2}/\tau$ also grow exponentially. Recall for grid turbulence P = 0, and $k(t)$, $\varepsilon(t)$, and $L(t)$ decay with time with decay exponent n between 1.15 and 1.45, as per Chapter 5 Part 2.

(2) Bernard 6.6

Idealized flow such that $S = d\overline{U}/dy > 0 =$constant superimposed on homogeneous/isotropic turbulence.

k equation in homogeneous shear flow:

$$\frac{dk}{dt} = P - \varepsilon$$

Where the production term

$$P = -\overline{uv}\frac{d\overline{U}}{dy}$$

Is positive, since $\overline{uv} < 0$ associated with $S > 0$.

ε equation in homogeneous shear flow:

Same as isotropic decay due homogeneous/isotropic turbulence assumption.

$$\frac{d\varepsilon}{dt} = P_\varepsilon^1 + P_\varepsilon^2 + \underbrace{P_\varepsilon^4 - \Upsilon_\varepsilon}$$

For homogeneous isotropic turbulence

$$P_\varepsilon^1 + P_\varepsilon^2 = -\left(\varepsilon_{ij}^c + \varepsilon_{ij}\right)S = -2\varepsilon S = 2\varepsilon\frac{P}{\overline{uv}}$$

For homogeneous shear flow

$$-\frac{\overline{uv}}{k} \approx constant \approx 0.3$$

Such that:

$$P_\varepsilon^1 + P_\varepsilon^2 = C_{\varepsilon_1} P\frac{\varepsilon}{k}$$

For P_ε^4 and Υ_ε same expressions as isotropic turbulence:

$$P_\varepsilon^4 - \Upsilon_\varepsilon = S_k^* R_T^{\frac{1}{2}}\frac{\varepsilon^2}{k} - G^*\frac{\varepsilon^2}{k}$$

And for the palenstrophy, assuming vortex stretching not preempted by dissipation, it is assumed that:

$$G^* = \left(S_k^* - C_{\varepsilon_3}\right)\sqrt{R_T} + C_{\varepsilon_2}$$

Where $C_{\varepsilon_3} = 0$ produces standard model for RANS. Therefore, the ε equation becomes:

$$\frac{d\varepsilon}{dt} = C_{\varepsilon_1} P\frac{\varepsilon}{k} + C_{\varepsilon_3} R_T^{\frac{1}{2}}\frac{\varepsilon^2}{k} - C_{\varepsilon_2}\frac{\varepsilon^2}{k}$$

And it needs to be solved in conjunction with

$$\frac{dk}{dt} = P - \varepsilon$$

Once a model is introduced for

$$P = -\overline{uv}\frac{d\overline{U}}{dy}$$

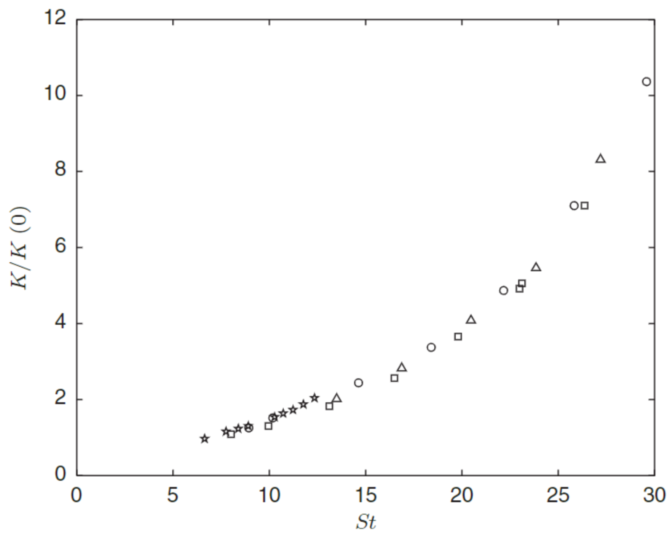

Figure 6.14 Measured $K/K(0)$ in homogeneous shear flow for $St < 30$ from [15]. Reprinted with permission of Cambridge University Press.

EFD and DNS show exponential growth K and ε, but $St < 30$; and following asymptotic relationships:

$$\frac{Sk}{\varepsilon} \approx 6 \quad (1)$$

$$\frac{P}{\varepsilon} \approx 1.8 \quad (2)$$

although in some cases LES statistics still not converged at $St = 30$.

Using $t^* = St$ and $k^*(St) = k(t)/k(0)$, the equation for k becomes

$$\frac{dk^*}{dt^*} = \frac{\varepsilon}{Sk}\left(\frac{P}{\varepsilon} - 1\right)k^* \quad (3)$$

And for $St < 30$, substituting Eqs. (1) and (2), the solution to Eq. (3) becomes

$$k^*(t^*) = e^{0.13t^*}$$

With similar analysis ε equation, with $C_{\varepsilon_3} = 0$, (i.e., neglecting vortex stretching term) results in an exponential growth for $\varepsilon^*(t^*)$.

Appendix A.1, Prob. 6.2 Bernard

Long time EFD and simulations not achievable. Two hypotheses put forward:

1) $P = \varepsilon$ such that k and ε asymptote to constant values (Townsend 1956)

2) k and ε continue exponential growth, which is not physical as unlimited growth k unrealistic → only pertains to ideal case.

To solve k and ε equations for long time growth $\overline{uv}$ model needed.

$$\overline{uv} = -\mathcal{T}_{22}\overline{v^2}S$$

Where for isotropic turbulence $\overline{v^2} = 2k/3$.

Assume $\mathcal{T}_{22} \propto$ eddy turnover time k/ε, same as $k - \varepsilon$ model approach:

$$\mathcal{T}_{22} = \frac{3}{2}C_\mu\frac{k}{\varepsilon}$$

Such that the set of equations becomes:

$$\frac{dk}{dt} = C_\mu \frac{k^2}{\varepsilon} S^2 - \varepsilon$$
$$\frac{d\varepsilon}{dt} = C_{\varepsilon_1} C_\mu k S^2 + C_{\varepsilon_3} R_T^{\frac{1}{2}} \frac{\varepsilon^2}{k} - C_{\varepsilon_2} \frac{\varepsilon^2}{k}$$

And solution to these equations can be used to illustrate physics of homogeneous shear flow at long times.

$C_{\varepsilon_2} = 1.45$ same value as isotropic turbulence self-similarity solutions.

$C_\mu = 0.09$ and $C_{\varepsilon_1} = 1.9$ same values used for near wall BL.

$C_{\varepsilon_3} = 0 \rightarrow$ no vortex stretching.

$C_{\varepsilon_3} = 0.1 \rightarrow$ used to investigate of vortex stretching in homogeneous shear flow.

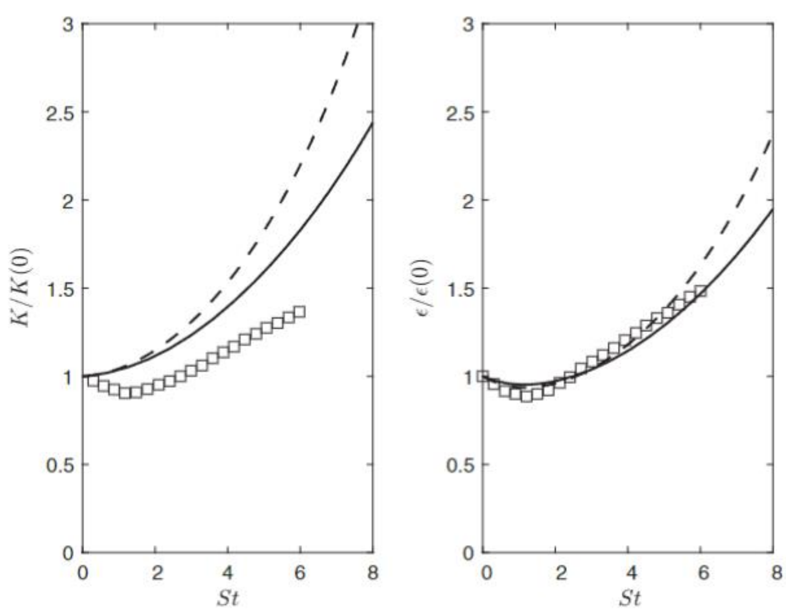

Figure 6.15 Computed solution for $K/K(0)$ (left) and $\epsilon/\epsilon(0)$ (right) in homogeneous shear flow: —, with vortex stretching; – –, without vortex stretching; o, LES calculation [17].

Fig. 6.15 shows EFD solutions for short time ($St < 10$), trends look similar.

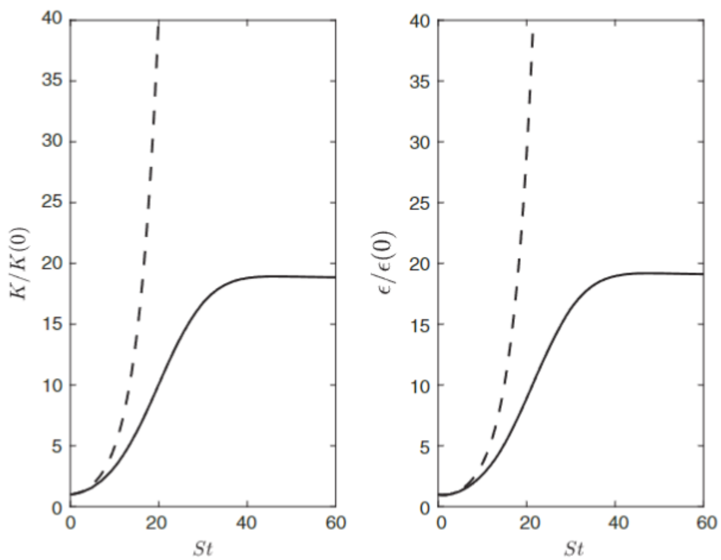

Figure 6.16 Computed solutions for $K/K(0)$ (left) and $\epsilon/\epsilon(0)$ (right) in homogeneous shear flow: —, with vortex stretching; – –, without vortex stretching.

Fig. 6.16 shows long time solutions ($St < 60$):

- $C_{\varepsilon_3} = 0 \rightarrow$ exponential growth
- $C_{\varepsilon_3} \neq 0 \rightarrow$ growth plateaus and $P = \varepsilon$ equilibrium achieved, as predicted by Townsend.

Asymptotic values for k and ε are found by setting $dk/dt = d\varepsilon/dt = 0$, resulting in:

$$k_\infty = \frac{\sqrt{C_\mu}\left(C_{\varepsilon_2} - C_{\varepsilon_1}\right)^2}{C_{\varepsilon_3}^2}\,\nu S$$

$$\varepsilon_\infty = \frac{C_\mu\left(C_{\varepsilon_2} - C_{\varepsilon_1}\right)^2}{C_{\varepsilon_3}^2}\,\nu S^2$$

The magnitude of the asymptotic values increases with the inverse square of C_{ε_3}, which highlights importance vortex stretching as an additional source of dissipation.

$C_{\varepsilon_3} \neq 0$ likely most realistic physics, i.e., vortex stretching maintains independent physical process. Related to considering high Re equilibrium solution for self-similar flows.

(3) The Spectral view of the energy cascade (Pope 6.6)

For large Re, energy-containing and dissipative motions have clear separation of scales $L_{11}/\eta \sim Re^{3/4} \gg 1$ and bulk of TKE is contained in motions of length scale l, where $6L_{11} > l > \frac{1}{6}L_{11} = l_{EI}$, with characteristic velocity $k^{1/2}$.

Since $l \sim \mathcal{L}$ large-scale motions anisotropic and f(geometry). Timescale $\tau = L_{11}/k^{1/2}$ is large compared to mean-flow time scale $\mathcal{L}/\overline{U}$ and f(flow history), i.e., smaller eddies turn over at a higher rate than the larger eddies.

∴ energy-containing motions do not have universal form arising from statistical equilibrium.

Anisotropy and production of turbulence confined to energy-containing motions and viscous dissipation is negligible.

Initial steps energy cascade, energy removed by inviscid processes (Production) and transferred to smaller scales $l < l_{EI}$ at rate $\mathcal{T}_{EI} \sim \frac{u_0^3}{l_0}$ which scales with $u_{rms}^3/L_{11} = k^{3/2}/L_{11}$. $\mathcal{T}_{EI}$ is not universal ∴ non-dimensional ratio $\mathcal{T}_{EI}/(k^{3/2}/L_{11})$ is not universal.

Energy spectrum balance for homogeneous shear flow (Hinze Chapter 4):[2]

$$\underbrace{\frac{\partial}{\partial t}E(\kappa,t)}_{\boxed{\text{Rate of change energy spectrum}}} = \underbrace{P_k(\kappa,t)}_{\boxed{\text{Production due to shear}}} \underbrace{-\frac{\partial}{\partial k}T_k(\kappa,t)}_{\boxed{\text{Spectral transfer}}} \underbrace{-2\nu\kappa^2 E(\kappa,t)}_{\boxed{\text{Dissipation}}} \quad (1)$$

$P_k =$ product of the mean velocity gradient $\partial\overline{U_i}/\partial x_j$ and anisotropic part of the spectrum tensor.

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[2] Can be compared with Chapter 5 Part 3 $\mathcal{R}_{ij}$ equation for homogeneous turbulent flow, whereas Eq. (1) is for homogeneous shear flow.

$$P_k = \mathcal{E}(\kappa, t)\frac{d\overline{U}}{dy} = 2\pi\kappa^2 \left[2E_{12} - \kappa_1 \frac{\partial E_{i,i}}{\partial \kappa_2}\right]$$

Hinze 4-4

$$P_{(\kappa_a,\kappa_b)} = \int_{\kappa_a}^{\kappa_b} P_k \, d\kappa$$

Represents contribution to the production from wave number range (κ_a, κ_b).

$$P = P_{(0,\infty)} \approx P_{(0,\kappa_{EI})}$$

i.e.,

$$\frac{P_{(\kappa_{EI},\infty)}}{P} \ll 1$$

Most of the anisotropy contained in energy-containing range.

$T_k(\kappa, t)$ represents the spectral energy transfer rate, i.e., net rate at which energy is transferred from modes of lower wave number than κ to those with wave numbers higher than κ.

Rate of gain of energy in (κ_a, κ_b) due to spectral transfer is:

$$\int_{\kappa_a}^{\kappa_b} -\frac{\partial}{\partial k} T_k(\kappa, t)\, d\kappa = T_k(\kappa_a) - T_k(\kappa_b)$$

Since for $T_k(\kappa_a = 0) = T_k(\kappa_b = \infty) = 0$, this term makes no contribution to the balance of TKE.

$$T_k = T_k^1 + T_k^2$$

Interaction triads of wave numbers

$\underline{\kappa} + \underline{\kappa}' + \underline{\kappa}'' = 0$

Kinematic effect of mean velocity gradients has on the spectrum.

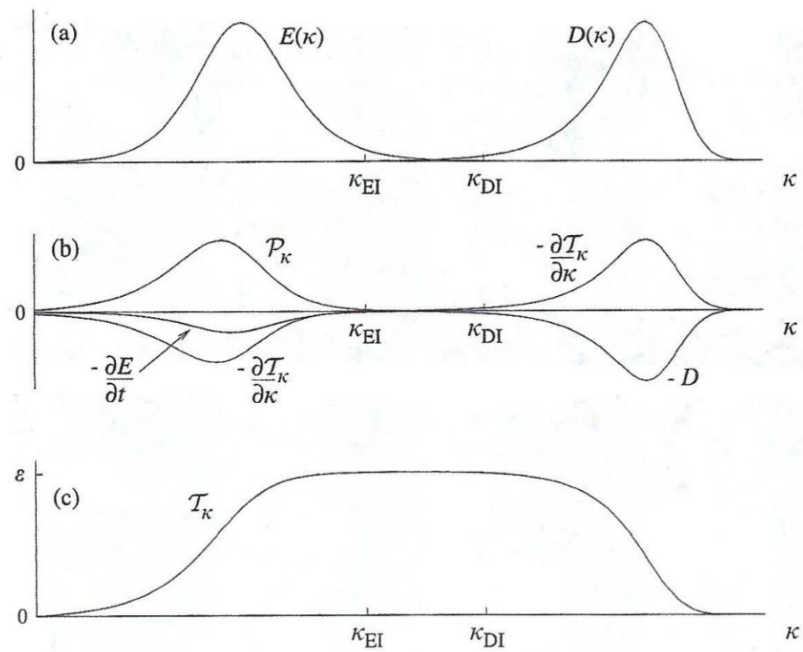

Fig. 6.28. For homogeneous turbulence at very high Reynolds number, sketches of (a) the energy and dissipation spectra, (b) the contributions to the balance equation for $E(\kappa, t)$ (Eq. (6.284)), and (c) the spectral energy-transfer rate.

1) For $\kappa < \kappa_{EI}$ in energy-containing range all terms significant except for dissipation. Assuming $k_{(0,\kappa_{EI})} \approx k$, $\varepsilon_{(0,\kappa_{EI})} \approx 0$ and $P_{(0,\kappa_{EI})} \approx P$, integration of Eq. (1) over the energy-containing range gives

$$\frac{dk}{dt} \approx P - T_{EI} \quad (2)$$

Where $T_{EI} = T_k(\kappa_{EI})$. Energy is generated by P and transferred to T_{EI}.

2) In the inertial subrange, $\kappa_{EI} < \kappa < \kappa_{DI}$, and spectral transfer only significant process so that integration of Eq. (1) from κ_{EI} to κ_{DI} gives:

$$0 \approx T_{EI} - T_{DI} \quad (3)$$

Where $T_{DI} = T_k(\kappa_{DI})$. Energy cascades without change in the inertial subrange.

3) In the dissipation range $\kappa > \kappa_{DI}$, integration of Eq. (1) from κ_{DI} to ∞ gives:

$$0 \approx T_{DI} - \varepsilon \quad (4)$$

Energy dissipates such that $T_{DI} = \varepsilon$.

Eqs. (2), (3) and (4) highlight the essential characteristics of the energy cascade and adding them together gives:

$$\frac{dk}{dt} = P - \varepsilon$$

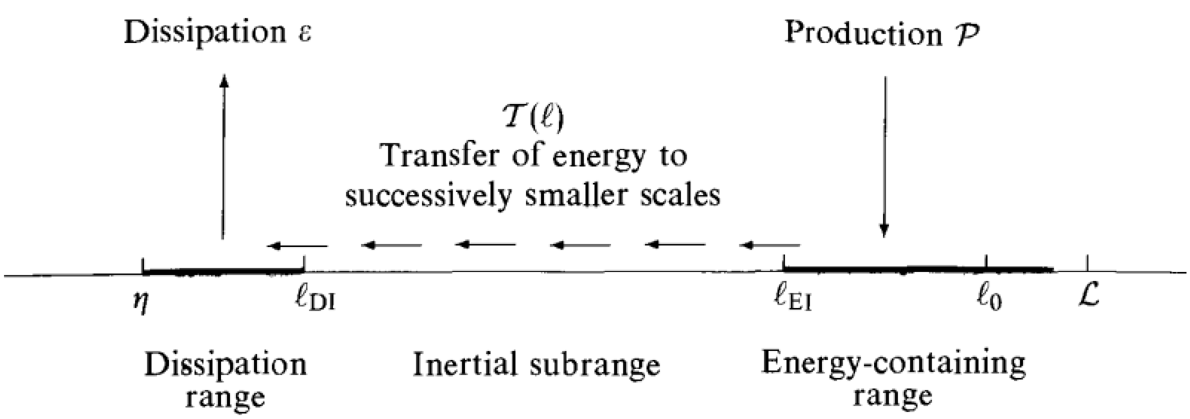

Fig. 6.2. A schematic diagram of the energy cascade at very high Reynolds number.

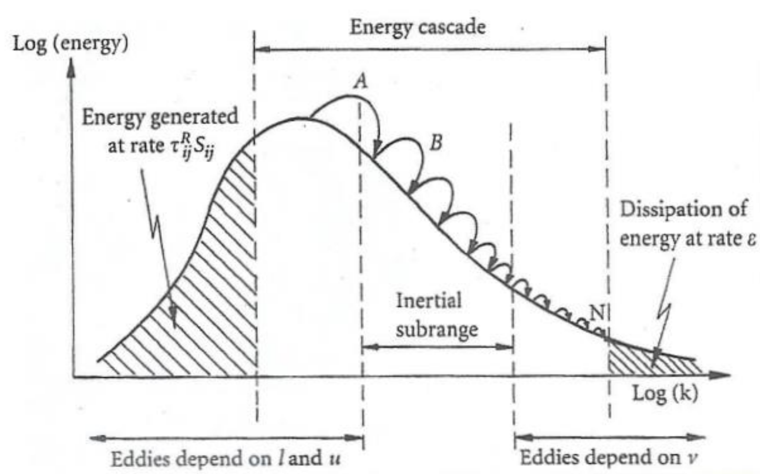

Figure 5.3 Schematic representation of the energy cascade.

The cascade timescale

Flow of energy in inertial subrange analogous to incompressible continuity equation

$$A_1V_1 = A_2V_2$$

Through variable area stream tube.

$T_{EI} = \frac{u_0^3}{l_0} = \left[\frac{m^2}{s^3}\right] = [\frac{m^2}{s^2}/s]$ = rate of change of energy, analogue to $Q = AV =$ flowrate = constant and A analogue to $E(\kappa) =$ energy per wave number = m^2/s^2/1/m = $\left[\frac{m^3}{s^2}\right]$. So, the speed (in units of wave number per time $[1/ms]$) at which energy travels through the cascade is:

$$\dot{\kappa}(\kappa) = \frac{T_{EI}}{E(\kappa)} = \frac{\varepsilon}{C\kappa^{-5/3}\varepsilon^{2/3}} = \frac{\kappa^{5/3}\varepsilon^{1/3}}{C}$$

And it can be noted that this speed increases rapidly with increasing κ.

It follows from the solution of (with speed in units of wave number per unit time):

$$\frac{d\kappa}{dt} = \dot{\kappa}$$

Integrated from κ_a to κ_b

$$\int_{\kappa_a}^{\kappa_b} \frac{C}{\kappa^{5/3}\varepsilon^{1/3}} d\kappa = t_{(\kappa_a,\kappa_b)}$$

Thus, the time needed for the energy to flow from κ_a to κ_b is:

$$t_{(\kappa_a,\kappa_b)} = \frac{3}{2}C\varepsilon^{-1/3}(\kappa_a^{-2/3} - \kappa_b^{-2/3})$$

And substituting for $\varepsilon^{-1/3}$:

$$\varepsilon^{-1/3} = \frac{1}{\left(\frac{u^3}{L}\right)^{1/3}} = \frac{L^{1/3}}{u} = \frac{L}{u}L^{-2/3} = \tau L^{-2/3}$$

$$t_{(\kappa_a,\kappa_b)} = \tau\frac{3}{2}C((\kappa_a L)^{-2/3} - (\kappa_b L)^{-2/3}) \qquad (5)$$

Using the relations

$$\kappa_{EI} = \frac{2\pi}{l_{EI}}, \qquad l_{EI} = \frac{1}{6}L_{11}, \qquad \frac{L_{11}}{L} \approx 0.4$$

Into Eq. (5) gives

$$\begin{aligned} t_{(\kappa_{EI},\infty)} &= \tau\frac{3}{2}C\left((\kappa_{EI}L)^{-\frac{2}{3}} - (\infty L)^{-\frac{2}{3}}\right) \\ &= \tau\frac{3}{2}C(\kappa_{EI}L)^{-2/3} \\ &\approx \tau\frac{3}{2}C\left(\frac{2\pi}{l_{EI}}\frac{L_{11}}{0.4}\right)^{-\frac{2}{3}} \\ &\approx \tau\frac{3}{2}C\left(\frac{12\pi}{L_{11}}\frac{L_{11}}{0.4}\right)^{-\frac{2}{3}} \\ &\approx \tau\frac{3}{2}C\left(\frac{12\pi}{0.4}\right)^{-\frac{2}{3}} \\ &\approx 0.0725\tau C \end{aligned}$$

Substituting C~1.5 = Kolmogorov universal constant

$$t_{(\kappa_{EI},\infty)} \approx 0.109\tau$$

$$t_{(\kappa_{EI},\infty)} \approx \frac{1}{10}\tau = \frac{1}{10}\frac{k}{\varepsilon}$$

i.e., lifetime of energy from $t_{EI} \to \infty = 1/10$ total lifetime.

Spectral energy-transfer models

For $\kappa > \kappa_{EI}$,

$$0 = -\frac{d}{d\kappa} T_k(\kappa) - 2\nu\kappa^2 E(\kappa) \quad (6)$$

From 1940-1970 many models for $T_k(\kappa)$ to obtain $E(\kappa)$ using Eq. (6) → most of these models are non-local due to interaction of wave number triads in energy transfer.

Simplest local model Pao (1965):

$$\dot{k} \equiv \frac{T_k(\kappa)}{E(\kappa)} = f(\varepsilon, \kappa)$$

Using dimensional analysis

$$T_k(\kappa) = \dot{k}E(\kappa) = E(\kappa)\alpha^{-1}\varepsilon^{1/3}\kappa^{5/3}$$

Where $\alpha =$ constant.

Substituting this expression into Eq. (6) and integrating gives (Pope Ex. 6.36)

$$E(\kappa) = C\varepsilon^{2/3}\kappa^{-5/3} \exp[-\frac{3}{2}C(\kappa\eta)^{4/3}]$$

i.e., Pao energy spectrum for the dissipation range.

Appendix A

A.1

$$t^* = St$$

$$k^*(t^*) = k\left(\frac{t^*}{S}\right)$$

Similarly

$$\varepsilon^*(t^*) = \varepsilon\left(\frac{t^*}{S}\right)$$

Evolution equation for ε:

$$\frac{d\varepsilon}{dt} = C_{\varepsilon_1} P \frac{\varepsilon}{k} + C_{\varepsilon_3} R_T^{\frac{1}{2}} \frac{\varepsilon^2}{k} - C_{\varepsilon_2} \frac{\varepsilon^2}{k}$$

Written in non-dimensional form and assuming $C_{\varepsilon_3} = 0$:

$$S\frac{d\varepsilon^*}{dt^*} = C_{\varepsilon_1} P \frac{\varepsilon^*}{k^*} - C_{\varepsilon_2} \frac{(\varepsilon^*)^2}{k^*} \quad (1A)$$

Using

$$\frac{Sk}{\varepsilon} \approx 6$$

And

$$\frac{P}{\varepsilon} \approx 1.8$$

Eq. (1A) becomes:

$$\frac{d\varepsilon^*}{dt^*} = \left(C_{\varepsilon_1}\left(\frac{P}{\varepsilon^*}\frac{\varepsilon^*}{Sk^*}\right) - C_{\varepsilon_2}\left(\frac{\varepsilon^*}{Sk^*}\right)\right)\varepsilon^*$$

And substituting $C_{\varepsilon_1} = 1.45$ and $C_{\varepsilon_2} = 1.9$:

$$\frac{d\varepsilon^*}{dt^*} = \left(C_{\varepsilon_1}\frac{1.8}{6} - \frac{C_{\varepsilon_2}}{6}\right)\varepsilon^* = 0.1183\varepsilon^* \rightarrow \varepsilon^* = e^{0.1183t^*}$$